Conceptual prerequisites for analogical reasoning: The case of proportions

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Abstract

Analogy plays an important role in cognitive development, but children often need cognitive supports to draw correct ones. Here, we examined the role of conceptual knowledge in proportional analogies, which are often depicted as a simple exercise in pattern completion. In Study 1, adults and children (N = 321) completed 4-term analogy tasks featuring letters, lines, integers, or fractions. Performance was lowest for fractions, and strongly impacted by educational background. In Study 2, we conducted an educational intervention focusing on either conceptual knowledge, procedural knowledge, or both to 3rd-to-5th graders (N = 343) using a pretest-training- posttest design. Children with poor pretest magnitude knowledge were more likely to fail analogical reasoning, and training on conceptual knowledge that fractions denote magnitudes improved children's analogies. Together, these studies indicate that knowledge of fractional magnitudes is important to proportional analogy.

1 Introduction

A central developmental question is how we learn so much from so little. From a finite number of utterances, how do we learn a language? From limited experience with animals, how do we learn taxonomy? From only limited experience with finite numbers, how do we construct a continuous number line with an infinite number of numbers?

Analogy is one learning mechanism that can bootstrap learning far beyond what the input provides (Gentner, 1983). Indeed, analogical reasoning is widely used in our daily life and science (Holyoak, 2012). It allows us to learn from a limited set of experiences, make comparisons based on the surface similarity between the experiences, abstract the common underlying relational structure, and map the structure of the limited experiences to a potentially infinite number of novel situations. Thus, analogy plays an important role in cognitive development, helping children extend their knowledge about the world and acquire novel concepts (Gentner, 2010).

Despite the important role of analogy in cognitive development, young children often fail at making the correct analogy (e.g., Rattermann & Gentner, 1998). In this paper, we examined the development of the ability to solve a very simple, 4-term proportional analogy. We were specifically interested in problems where the terms of the analogy involved a mixture of fractions and whole numbers. This issue is interesting because 1) young children make accurate proportional analogies between magnitudes and integers (Goswami, 1989; Thompson & Opfer, 2010), yet 2) fail to grasp that fractions, like integers, are numbers that denote magnitudes (Vamvakoussi & Vosniadou, 2010). Without this conceptual understanding of fractions, proportional analogies between integers and fractions were hypothesized to be quite difficult and to require special training. This prediction differs from the view that 4-term analogies merely involve pattern-completion (Leech et al., 2008).

1.1 The Development of Knowledge of Fractional Magnitudes

Fractions are commonly used in daily life, the workplace, and science. Knowledge of fractions plays an important role in educational and financial success (Booth & Newton, 2012). Research has shown that knowledge that fractions denote magnitudes and an accurate representation of fractional magnitude at early grades uniquely predict later achievements in college-prep courses, such as algebra. (Siegler et al., 2012).

However, children often lack the knowledge that fractions denote magnitudes. For instance, children often mistakenly overgeneralize the properties of natural numbers (e.g., a natural number has a unique successor) to fractions, and focus on the magnitudes of the integer components of fractions (i.e., numerators and denominators) rather than the holistic fractional magnitudes, which is known as the ”the whole number bias” (Ni & Zhou, 2005). Moreover, even after years of experience with fractions, some children still fail to understand that there are an infinite number of fractions between fractions. (Vamvakoussi & Vosniadou, 2010)

Thus, interventions focused on better understandings of fractional magnitudes are supposed to facilitate fraction learning (Siegler et al., 2011). Indeed, learning to estimate fractions on number lines helps both typically-developing children (Fazio et al., 2016) and at-risk math learners (Dyson et al., 2020) improve their fraction proficiency.

We hypothesized that rational number knowledge–knowing that (1) fractions, like integers, denote magnitudes, and (2) the magnitude of fractions, like integers, can be placed on number lines–can also facilitate children's proportional analogies (e.g, knowing that 3 is to 8 as 3/8 is to 1).

1.2 The Development of Analogical Reasoning

One theory explaining the development of analogical reasoning is the knowledge accretion account (Goswami, 1989; Goswami & Brown, 1990), which emphasizes the role of familiarity in relational reasoning. In support of this view, children are better at analogical reasoning with knowledge of relations in familiar domains rather than unfamiliar domains (Goswami, 1989; Rattermann & Gentner, 1998). For example, familiar labels such as "baby, mommy, daddy" assigned to a triad of objects with monotonically increasing sizes lead to a better understanding of the relation of relative sizes among young children (Rattermann & Gentner, 1998). Moreover, children who better know the relational information on which the relation is based can better complete the classical pictorial 4-term analogy questions (A:B::C:?). For instance, Goswami, 1989 provided correlational evidence that young children who have better knowledge of proportions (e.g., half a diamond: half a square:: half a circle: half a rectangle) can better solve proportional analogy questions with continuous magnitudes (e.g., half a circle: half a rectangle:: a quarter of a circle: a quarter of a rectangle).

Our paper focused on examining the knowledge account. Particularly, we study this using a 4-term analogy production task in the case of proportional analogy. Many previous studies supporting the knowledge account used multiple-choice questions where participants needed to choose among a relational-matching correct answer and perceptual-matching distractors.

Here we designed a 4-term production analogy task where participants produce an answer for A:B::C:? with the base relation (A:B) and the target relation (C:D) depicted on two aligned number lines respectively (Figure 1). By providing spatially-aligned relational structures, we aimed to control for and minimize the effect of executive functions, which are also believed to underlie the development of relational reasoning (Richland et al., 2006).

1.3 Current Study

The main aim of the study is to examine the conceptual prerequisites for proportional analogies. In Study 1, we examined whether the type of terms (letters, magnitudes, integers, or fractions) has an effect on analogical reasoning when the direct alignment between source and target was shown. To do this, we compared both children's and adults' accuracy on 4-term analogy questions with and without fractions.

In Study 2, we provided an experimental test of our hypothesis by training randomly-selected children with conceptual knowledge of fractions and examining the effect of training on their ability to solve the proportional analogies given in Study 1. Specifically, we administered training inventions focusing on either conceptual knowledge of fractions, procedural knowledge on locating the position of fractions on number-lines, or both conceptual knowledge and procedural knowledge. The Conceptual Knowledge Training emphasized the measurement model of fractions (Hecht et al., 2003). In other words, fractions are a special kind of number that denote magnitudes and are used to measure quantities in the world. The Procedural Training emphasized the explicit steps of estimating fractions on number lines as a control group.

2 Study 1

2.1 Methods

2.1.1 Participants

Participants varied in educational background. Ninety-seven were college students (M = 19.01 y, SD = 2.23, 52% females), who participated in the study for course credits. An additional 165 were Mechanical Turkers (M = 39.75 y, SD = 11.33, 41% females) who participated in the study for a small monetary reward. MTurkers' mean educational level was 'Associate degree'. Finally, 59 were 3rd-to-5th graders (M = 9.86y, SD = .98, 66% females) recruited from local elementary schools, a local museum and childrenhelpingscience.com.

2.1.2 Procedures

The study was presented online through Qualtrics. Participants completed the study at their own pace.

Each participant completed 4 types of 4-term analogy questions: letter analogy, magnitude analogy, integer analogy, and fraction analogy (Figure 1), with 4 trials in each task. The order of 4 tasks was counterbalanced across participants. The order of trials was randomized.

In the integer analogy task, on each trial participants see two vertically aligned number lines. For example, the position of 9 on a 0-10 number line, and the position of 90 on a 0-100 number line, and participants needed to fill in the blank '90 is to 100 as 9 is to _____'.

In the fraction analogy task, the aligned number lines were the same as the integer analogy task, except for the labels. For example, participants saw the position of 9 on a 0-10 number line, and the position of 9/10 on a 0-1 number line, and were asked to fill in the blank '9/10 is to 1 as 9 is to _____'.

In the letter analogy task, again the aligned number lines were the same as the integer analogy task, except that the labels for the right-end number and the middle numbers were replaced by letters. For example, participants saw the position of A on a 0-B number line, and the position of C on a 0-D number line. Then the participants filled in the blank 'C is to D as A is to _____'.

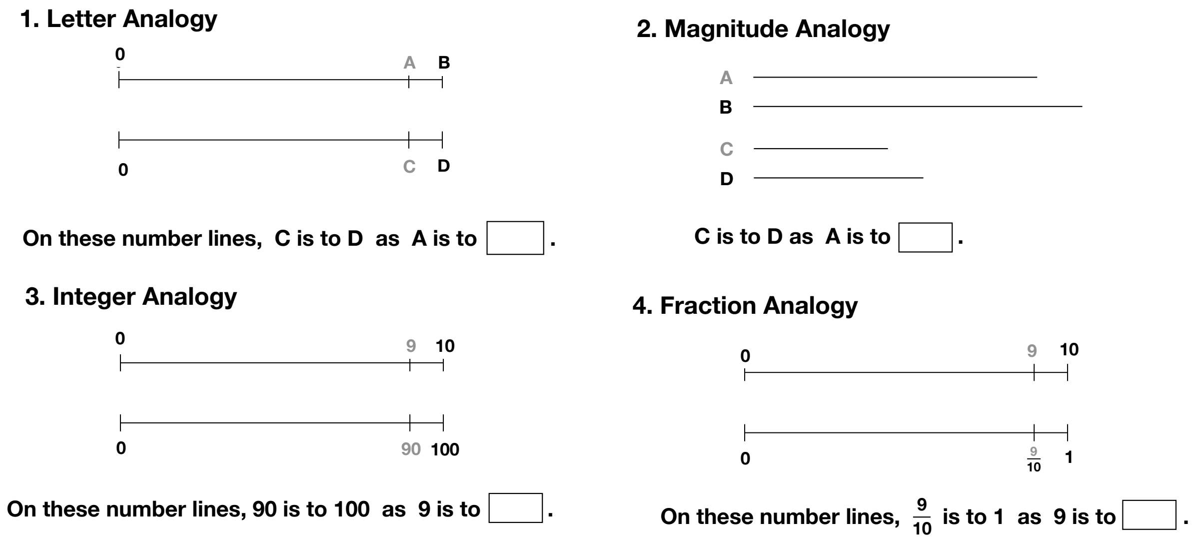

Figure 1: An illustration of the 4-term analogy questions across problem types.

In the magnitude analogy task, on each trial participants saw 4 lines of different lengths labeled with different letters, for example, A, B, C, D. The ratio of the length of line A to line B was the same as the ratio of the length of line C to line D, which was the ratio depicted in the other three tasks (9:10 in our examples).

2.2 Results

To investigate performance on 4-term analogy problems across tasks and participants' groups, we fitted a series of generalized logistical regression model with accuracy as the dependent variable, problem type (letter, magnitude, integer, fraction) as fixed effects, by-task random slopes, and by-participant random intercepts for each participant group (children, MTurkers, college student).

Results showed similar accuracy pattern across problem types among all participants group, as denoted by a lack of interaction effect between age group and problem types (χ^2 (6) = 1.65, p = .949). Fraction analogy problems yielded lowest accuracy across all groups of participants. Specially, among children, the odds of correctly solving the fraction analogy task were 1.89 times lower than the odds of correctly solving the letter task (odds ratio = 2.89, 95% CI = [1.56, 5.36], p <.001), 2.70 times lower than the odds of correctly solving the magnitude task (odds ratio = 3.70, 95% CI = [1.94, 7.06], p <.001), and 1.73 times lower than the odds of

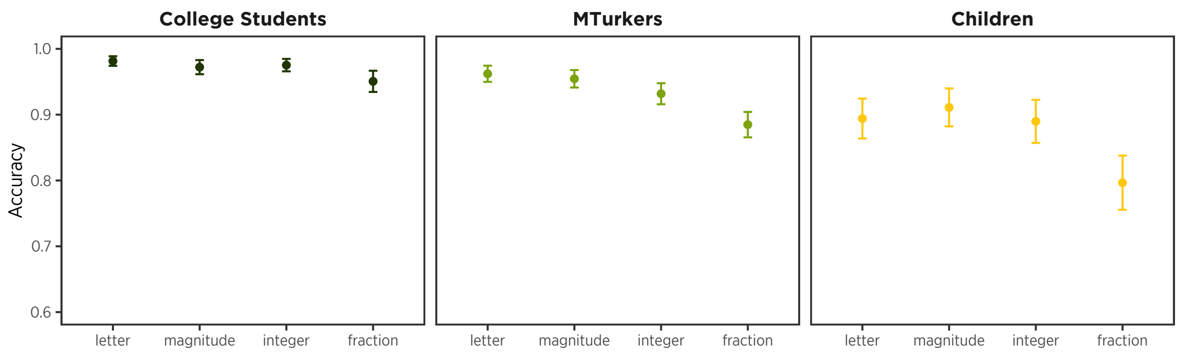

Figure 2: Accuracy of the 4-term analogy questions across problem types and participant groups.

correctly solving the integer task (odds ratio = 2.73, 95% CI = [1.48, 5.04], $p < .001$).

Among MTurkers, the odds of correctly solving the fraction analogy task were 7.74 times lower than the odds of correctly solving the letter task (odds ratio = 8.74, 95% CI = [4.49, 17.00], $p < .001$), 5.23 times lower than the odds of correctly solving the magnitude task (odds ratio = 6.23, 95% CI = [3.36, 11.60], $p < .001$), and 1.88 times lower than the odds of correctly solving the integer task (odds ratio = 2.88, 95% CI = [1.69, 4.91], $p < .001$).

Finally, among college students, the odds of correctly solving the fraction analogy task were not significantly different from the odds of correctly solving the magnitude task (odds ratio = 1.95, 95% CI = [0.86, 4.43], $p = .112$), but they were 2.38 times lower than the odds of correctly solving the letter analogy task (odds ratio = 3.38, 95% CI = [1.29, 8.90], $p < .05$), and 1.48 times lower than the odds of correctly solving the integer analogy task (odds ratio = 2.48, 95% CI = [1.03, 5.98], $p < .05$).

2.3 Discussion

The effect of problem type strongly suggests that proportional analogies require more resources than activation-based priming and pattern completion (Leech et al., 2008). Moreover, even with exact alignment between analogical targets and sources, accuracy on fraction analogy was still lower than accuracy on magnitude and integer analogy, indicating that prior knowledge of fraction magnitude may be critical. To further examine a potential causal link between knowledge

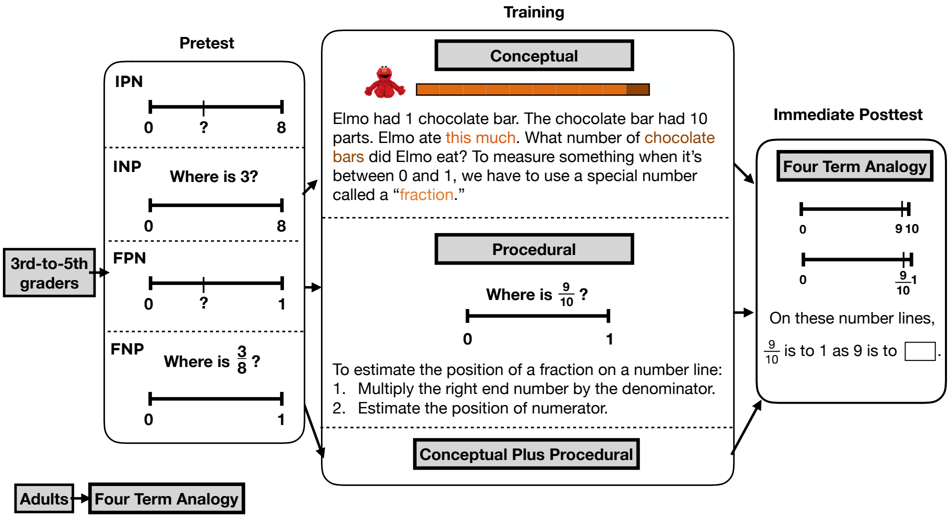

Figure 3: An illustration of the general procedure in Study 2. IPN = integer position-to-number tasks, INP = integer number-to-position tasks, FPN = fraction position-to-number tasks, FNP = fraction number-to-position tasks.

of fraction magnitudes and proportional analogies, we designed an educational intervention with 3rd-to-5th graders to test whether training on conceptual knowledge of fractions would facilitate better proportional analogy.

3 Study 2

3.1 Methods

3.1.1 Participants

Three hundred forty-three 3rd-to-5th graders (*M* = 9.89 y, *SD* = .90, 47% females; 94 3rd graders, *M* = 8.87 y, *SD* = 0.40, 47% females; 135 4th graders, *M* = 9.86 y, *SD* = 0.35, 45% females, and 135 5th graders, *M* = 10.89 y, *SD* = 0.35, 49% females) in Taiwan participated in the study. Twelve percent of the original sample of 391 students failed to complete all items, and they were excluded from further analyses.

In addition, 286 MTurkers (*M* = 38.87, *SD* = 10.74, 36% females) participated in the study. Mean educational level was 'Bachelor's degree'.

3.1.2 Materials

Testing Batteries To assess children's knowledge of fraction and integer magnitudes, the testing batteries consisted of four number line estimation tasks (Figure 2). The order of the tasks was counterbalanced across participants.

Fraction Position-to-number (PN) Task A total of 10 number lines were presented sequentially, with "0" marked on the left end and "1" marked on the right end. On each trial, children were presented with a location indicated by a hatch mark on the number line and were asked to estimate which fraction corresponded to the mark. To-be-estimated fractions were 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10.

Integer Position-to-number Task The integer PN task was identical to the fraction PN task, except for the to-be-estimated and right-end numbers. The to-be-estimated integers and right-end numbers were chosen based on the fractions included in the fraction PN task, such that stimuli were proportionally identical on the number lines to those in the fraction task. For example, for the fraction 3/8, the corresponding integer number line ranged from 0 to 8, and the mark was located at the position of 3. We kept integers in this task the same as the integer components of fractions rather than using a fixed right-end number to control the size of numbers. Ten integers that corresponded to fractions used in the fraction PN task were used: 1 on a 0-11 line, 1 on a 0-7 line, 1 on a 0-4 line, and so on. On each trial, children were asked to estimate which integer corresponded to the mark.

Fraction Number-to-position (NP) Task In this typical fraction estimation task (e.g., Siegler et al., 2011), children estimated the position of a given fraction on a number line flanked by "0" and "1" by dragging a hatch mark. To-be-estimated fractions were the same magnitudes used in the fraction PN task.

Integer Number-to-position Task In this task, children estimated the location of an integer on the number line by dragging a hatch mark. This task differed from typical number line estimation tasks (Siegler & Opfer, 2003) in that the right endpoint differed from trial to trial. Children completed a total of 10 trials in which the magnitudes were the same as those used in the integer PN task.

Training Materials *Conceptual Knowledge Training.* Conceptual Knowledge training emphasized that fractions are a type of number that are used to measure quantities between integers. First, we introduced concepts of fractions using concrete examples, e.g., pictures of a chocolate bar divided into 10 equal parts and 9 parts were shaded. We set up the story as "Elmo had 1 chocolate bar. The chocolate bar had 10 parts. Elmo ate this much."

Next, we introduced the definition of fractions and unit fractions. "To measure something when it's between 0 and 1, we have to use a special number called a fraction. Fractions are numbers with a top and a bottom. The bottom tells us how many equal parts are in the whole. Because the whole chocolate bar has 10 equal parts, one of them is 1/10 of the chocolate bar. When we know the size of each equal part, we can know the fraction Elmo ate by counting the number of parts he ate and counting all the original parts."

Procedural Knowledge Training. Procedural Knowledge training emphasized the steps to estimate fractions on number lines. First, children estimated the position of 9/10 on a 0-1 number line. Then, we taught children a general rule for estimating fractions on number lines: "To estimate the position of a fraction on a number line, follow two steps: 1. Multiply the right end number by the denominator (bottom number). 2. Estimate the position of the numerator (top number)."

Conceptual Plus Procedural Knowledge Training. Conceptual Plus Procedural Knowledge training consisted of both interventions, to examine whether an improvement of both conceptual knowledge and procedural knowledge of fractions has an additive effect on children's ability of proportional analogy reasoning. Children first completed Conceptual Knowledge training and then completed Procedural Knowledge training.

Immediate Posttest Materials After training, children completed two 4-term analogy questions as the fraction analogy questions in Study 1 ("9/10 is to 1 as 9 is to ___." And "4 is to 7 as ____ is to 1.", Figure 2).

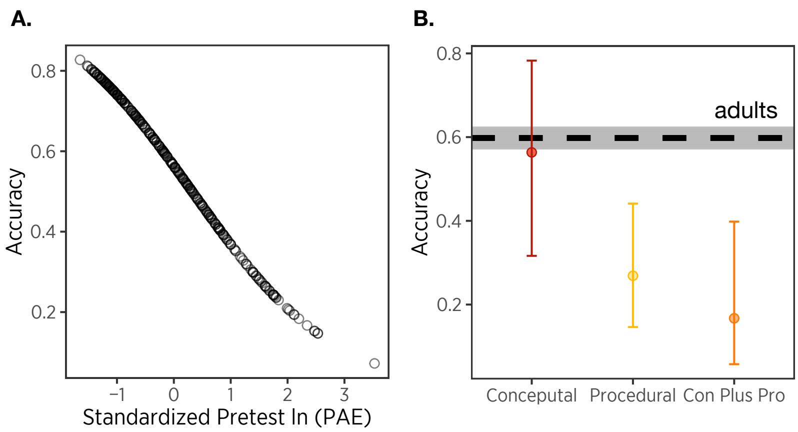

Figure 4: Mixed-linear model predicted accuracy of 4-term analogy questions in Study 2. A. Accuracy on analogy problems as a function of Percent Absolute Error at number-line estimation at pretest. PAE = Percent Absolute Error. B. Accuracy on analogy problems across conditions. Error bars denote 95% confidence interval. The dashed line denotes average accuracy of adults'. The grey area denotes adults' standard errors.

3.1.3 Procedures

The study was presented in Qualtrics. Children first completed the testing battery, and then were randomly assigned to one of three training conditions, Concept Knowledge Training (N = 110), Procedural Knowledge Training (N = 127), and Conceptual Plus Procedural Knowledge Training (N = 106). After training, children completed 4-term analogy questions.

An additional group of 286 adults completed the same 4-term analogy questions as a comparison group.

3.2 Results

We compared children's accuracy on the 4-term analogy questions across different training conditions and pretest magnitude knowledge. To assess children's magnitude knowledge of integers and fractions at the pretest, we calculated Percent Absolute Error (PAE) on each trial. PAE is a measurement of error, defined as the absolute difference between participant's estimate and the true value divided by the total numerical range (i.e., PAE $=|$ participant's estimate – true value $|$ / numerical range). To investigate the effect of training conditions, we fit a mixed-effects logistic regression with accuracy on 4-term analogy questions (the correct answer was coded as 1, the incorrect answer was coded as 0) as dependent variable, gender, grade (3rd, 4th, 5th), pretest performance measured by average PAE across pretest battery tasks, condition (Conceptual Knowledge Training, Procedural Knowledge Training, Conceptual Plus Procedural Knowledge Training), the interaction effect between condition and grade, the interaction effect between condition and pretest performance as fixed effects, by-participant random intercepts, and by-item random intercepts.

We found a main effect of gender (χ^2 (1) = 4.90, p <.05), indicated by a 61% increase in odds of correctly solving the analogy question for girls compared to boys (odds ratio = 1.61, 95% CI = [1.06, 2.45], p .05). Children from different grades did not significantly differ in their accuracy (χ^2 (2) = 3.32, p = .191, 3rd: *Mean* = 29%, *SD* = 0.38; 4th: *Mean* = 43%, *SD* = 0.40; 5th: *Mean* = 49%, *SD* = 0.38). Pretest accuracy with number line estimation significantly predicted accuracy on 4-term analogical questions (χ^2 (1) = 12.29, p <.001, Figure 3A), indicating that magnitude knowledge is positively associated with making correct proportional analogy. There was a main effect of condition (χ^2 (2) = 6.93, p <.05, Conceptual: *Mean* = 45%, *SD* = 0.43; Procedural: *Mean* = 41%, *SD* = 0.37; Conceptual Plus Procedural: *Mean* = 40%, *SD* = 0.38). The odds of a child in the Conceptual Knowledge condition correctly solving the 4-term analogy question is 5.35 times higher than a child in the Procedural Knowledge condition (odds ratio = 3.49, 95% CI = [1.09, 11.10], p <.05), and 5.35 times higher than a child in the Conceptual Plus Procedural condition (odds ratio = 6.35, 95% CI = [1.46, 27.60], <.05), indicating that the conceptual understanding of the analogical source and targets is important for analogical reasoning with integers and fractions. The interaction effect of grade and condition was not significant (χ^2 (4) = 8.34, p = .080), nor was the interaction of pretest performance and condition (χ^2 (2) = .23, p = .890).

To further compare children's accuracy of proportional reasoning after training with adults' accuracy of the same questions, we conducted a mixed-effects logistic models with accuracy as dependent variables, and with condition (Adults, Conceptual Knowledge Training, Procedural Knowledge Training, Conceptual Plus Procedural Knowledge Training) as predictors. We found that children were significantly less accurate than adults in all conditions (Conceptual: odds ratio = 0.42, 95% CI = [0.25, 0.71], p <.01; Procedural: odds ratio = 0.34, 95% CI =

[0.21, 0.56], $p < .001$; Conceptual Plus Procedural: odds ratio = 0.32, 95% CI = [0.19, 0.55], $p < .001$), indicating that there is still room for children to improve their proportional analogy skills.

3.3 Discussion

Study 2 provided both correlational and experimental evidence that proportional analogies depend on magnitude knowledge. Children with poor magnitude knowledge of fractions or integers failed to produce correct analogies, and training children that fractions denote magnitudes facilitated analogical reasoning. Previous research has highlighted the importance of cognitive supports for analogy (e.g., alignment of the source to the target), and the current research suggests that conceptual understanding is another important cognitive support.

However, we did not observe an additive effect of Conceptual Knowledge Training and Procedural Knowledge Training on proportional analogy, indicated by lower accuracy of the 4-term analogies among children in the Conceptual Plus Procedural condition than the Conceptual Knowledge only condition. A very likely explanation of the low performance in the Conceptual Plus Procedural is fatigue; training took twice as long, and children already had difficulty finishing the task. Future studies might examine whether separating Conceptual Plus Procedural Knowledge training into two sessions can improve proportional analogy.

4 General Discussion

The current paper investigated the cognitive supports to facilitate proportional analogy, specifically, whether knowledge of fractional magnitude leads to better analogical reasoning between integers and fractions. Study 1 demonstrated knowledge of fractions may be crucial to solving 4-term proportional analogy questions. Study 2 further provided evidence that training on the conceptual knowledge of fractions facilitated children's analogical reasoning. Together, we found correlational and experimental evidence that the development of proportional analogies is not solely due to pattern completion nor the maturation of executive functions. Consistent with the knowledge accretion account (Goswami, 1989), making correct analogies among the terms requires an understanding of what the terms stand for, even when the relational structures are aligned and readily available.

In conclusion, theoretically, our study supports and extends the knowledge account to explain the development of analogical reasoning in the case of proportions. Practically, our study sheds light on the math curriculum to help children learn rational numbers with analogy.

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References

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253.

Dyson, N. I., Jordan, N. C., Rodrigues, J., Barbieri, C., & Rinne, L. (2020). A fraction sense intervention for sixth graders with or at risk for mathematics difficulties. *Remedial and Special Education*, *41*(4), 244–254.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, *7*(2), 155–170.

Gentner, D. (2010). Bootstrapping the mind: Analogical processes and symbol systems. *Cognitive Science*, *34*(5), 752–775.

Goswami, U. (1989). Relational complexity and the development of analogical reasoning. *Cognitive Development*, *4*(3), 251–268.

Goswami, U., & Brown, A. L. (1990). Higher-order structure and relational reasoning: Contrasting analogical and thematic relations. *Cognition*, *36*(3), 207–226.

Hecht, S. A., Close, L., & Santisi, M. (2003). Sources of individual differences in fraction skills. *Journal of Experimental Child Psychology*, *86*(4), 277–302.

Holyoak, K. (2012). *Analogy and relational reasoning.* Oxford University Press.

Leech, R., Mareschal, D., & Cooper, R. P. (2008). Analogy as relational priming: A developmental and computational perspective on the origins of a complex cognitive skill. *Behavioral and Brain Sciences*, *31*(4), 357–378.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52.

Rattermann, M. J., & Gentner, D. (1998). The effect of language on similarity: The use of relational labels improves young children's performance in a mapping task. *Advances in analogy research: Integration of theory and data from the cognitive, computational, and neural sciences* (pp. 274–282). New Bulgarian University.

Richland, L. E., Morrison, R. G., & Holyoak, K. J. (2006). Children's development of analogical reasoning: Insights from scene analogy problems. *Journal of Experimental Child Psychology*, *94*(3), 249–273.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–697.

Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological Science*, *14*(3), 237–250.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296.

Thompson, C. A., & Opfer, J. E. (2010). How 15 hundred is like 15 cherries: Effect of progressive alignment on representational changes in numerical cognition. *Child Development*, *81*(6), 1768–1786.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209.