No 10-Element Subset of the First 100 Counting Numbers is Subset-Sum-Distinct

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Abstract: *The purpose of this article is to present a solution to an interesting problem suggested by Richard Little of Baldwin-Wallace College involving "subset-sum-distinctness." Paul Erdös introduced the notion of "subset-sum-distinctness;" however, the focus of this article is instead on the problem and solution rather than Erdös. A set of counting numbers is subset-sum-distinct if and only if there is a one-to-one correspondence between the power set and the set of subset sums. In this article, we establish that no 10-element subset of the first 100 counting numbers is subset-sum-distinct by demonstrating the impossibility of a bijection from the power set of a 10-element subset to the set of sums. In order to prove a bijection between the sets is not possible, it is clearly sufficient to prove that any mapping from the power set to the set of sums is not injective. Lastly, we list several results about subset-sum-distinct sets in instances in which one is trying to find k-element subsets of the first n counting numbers.*

Keywords. *Algebra, proof, set, subset, power set, mapping, one-to-one correspondence*

1 Introduction

At the October 2011 MAA Ohio Section Meeting held at The University of Findlay, Baldwin-Wallace College Professor of Mathematics Richard Little alluded to an interesting problem during his talk entitled "How I Escaped the Peter Principle!!." A paraphrased version of the problem is to find a set of ten numbers from the first one hundred counting numbers so that no two subsets of the subset of ten have the same sum for their elements. That is, find a 10-element subset of the first 100 counting numbers that is subset-sum-distinct. The concept of "subset-sum-distinctness" according to Bae (2002, p. 215) was introduced by Paul Erdös (1913-1996).

2 Preliminaries

2.1 Subset-Sum-Distinct (SSD) Sets

A subset-sum-distinct set is defined as a set of positive integers such that no two finite elements of the power set have the same sum (Bae, 2002, p. 215). A notation convention for "subset-sum-distinct set" is "SSD-set." An example of a SSD-set is $\{1, 2, 5\}$. The elements of the power set are $\{\}$, $\{1\}$, $\{2\}$, $\{5\}$, $\{1, 2\}$, $\{1, 5\}$, $\{2, 5\}$, $\{1, 2, 5\}$, which have subset sums $0, 1, 2, 5, 3, 6, 7$, and 8, respectively, with the agreement the empty set is assigned 0 as the sum of its elements. Another example is $\{1, 2, 4\}$. The elements of the power set are $\{\}$, $\{1\}$, $\{2\}$, $\{4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, $\{1, 2, 4\}$, which have subset sums (listed in ascending order) $0, 1, 2, 3, 4, 5, 6$, and 7. Interestingly, each set has distinct subset sums, and the power set is in one-to-one correspondence with the set of sums. This observation is consistent with the assertion that a k-element set of counting numbers has distinct subset sums if and only if the set of sums has 2^k distinct elements (Bohman, 1997, p. 1). An example of a set that does not have the property of "subset-sum-distinctness" is $\{1, 2, 4, 8, 16, 32, 64, 75, 99\}$, since elements of the power set, $\{4, 32, 64\}$ and $\{1, 99\}$ have the same sum.

2.2 Conjecture and Query

After examination of several 10-element geometric SSD-sets such as $\{2^n\}$, $\{3^n\}$, $n = 0, 1, \ldots, 9$, in addition to nine 10-element SSD-sets obtained using a construction algorithm outlined by Bohman (1997, pp. 4-5), for example, $\{16, 20, 21, 22, 24, 32, 64, 192, 704, 2752\}$, $\{64, 80, 84, 85, 86, 88, 96, 128, 256, 768\}$, $\{150, 228, 267, 289, 300, 306, 310, 311, 312, 314\}$ and so forth, none of which is contained in the first 100 counting numbers, the possibility that such a 10-element set did not exist was certainly plausible.

It is reasonable to ask how an n-element set of positive integers can fail to be subset-sum-distinct. The answer, of course, is the existence of a surjection from the power set of the n-element set to the set of subset sums, which is not an injection.

3 Proof

3.1 More Pigeons than Holes

Consider any 10-element subset of the first 100 counting numbers. The number of distinct subsets of this 10-element set is $2^{10} = 1024$. Of the 1024 subsets, the smallest sum a subset can have is 0, and the largest is $91+92+\ldots+99+100 = 955$. Evidently then the set of subset sums is bounded between 0 and 955, inclusive, for a maximum of 956 different possible sums. Moreover, any correspondence pairing each of the 1024 subsets with the sum of its elements; a nonnegative integer between 0 and 955, cannot be one-to-one (since $1024 > 956$), hence at least two subsets have the same sum. Said differently, a flock of 1024 pigeons roosting in a house containing fewer than 1024 pigeonholes will by necessity have at least two pigeons roosted in the same hole (i.e., The Pigeon Hole Principle).

3.2 A Mapping Approach

A different approach to the SSD problem is to let f be a function defined by $f(A) = s$, where A is one of the 1024 subsets and s is the sum of its elements; $0 \leq s \leq 955$ or equivalently, $0 \leq f(A) \leq 955$. Evidently, f is a well-defined mapping from the set of 1024 subsets into the set of the first 956 nonnegative integers. Because both the domain and codomain of the function are finite, the domain having more elements than the codomain, an injection from the domain onto the codomain, or to any proper subset is not possible. The inescapable conclusion is at least two subsets are assigned the same nonnegative integer (sum) between 0 and 955, inclusive.

To summarize thus far, we have demonstrated that the power set of a 10-element subset of the first 100 counting numbers is not in one-to-one correspondence with the set of subset sums, since a bijection between the sets is not possible because the number of subsets is strictly greater than the maximal possible sum. Therefore, any mapping from the power set to the set of sums must assign at least two subsets the same sum for their elements. We conclude that no 10-element subset of the first 100 counting numbers is subset-sum-distinct.

Moreover, in instances in which one is trying to find k-element subsets of the first n counting numbers, even if 2^k were to be less than the maximal possible sum $(n+(n-1)+(n-2)+...+(n-k+1))$, that still is not a sufficient condition for the existence of a k-element SSD-set. For example, the set consisting of the first six counting numbers has fifteen 4-element subsets, none of which has distinct subset sums even though the number of subsets $(2^4 = 16)$ is less than the maximal possible sum $(6+5+4+3 = 18)$, whereas $\{1, 2, 5\}$ has distinct subset sums, and the number of subsets $(2^3 = 8)$ is less than the maximal possible sum $(6+5+4 = 15)$. According to Bohman (1997, p. 2), a set fails to have distinct subset sums if and only if there exist disjoint subsets I and J with equal sums. We summarize the results for $k = 4$ and $n = 6$ in the table below where X is a 4-element subset of the first six counting numbers. (For a set of counting numbers Y we write ΣY for the sum of its elements.)

Table 1: *The fifteen four-element subsets of the first six counting numbers.*

4-Element Subsets	$I, J \subset X$, $I \cap J = \emptyset$ and $\Sigma I = \Sigma J$
$\{1,2,3,4\}$	$\{1,4\}$, $\{2,3\}$
$\{1,2,3,5\}$	$\{5\}$, $\{2,3\}$
$\{1,2,3,6\}$	$\{1,2,3\}$, $\{6\}$
$\{1,2,4,5\}$	$\{5\}$, $\{1,4\}$
$\{1,2,4,6\}$	$\{6\}$, $\{2,4\}$
$\{1,2,5,6\}$	$\{1,6\}$, $\{2,5\}$
$\{1,3,4,5\}$	$\{1,3\}$, $\{4\}$
$\{1,3,4,6\}$	$\{1,6\}$, $\{3,4\}$
$\{1,3,5,6\}$	$\{6\}$, $\{1,5\}$
$\{1,4,5,6\}$	$\{5\}$, $\{1,4\}$
$\{2,3,4,5\}$	$\{2,5\}$, $\{3,4\}$
$\{2,3,4,6\}$	$\{6\}$, $\{2,4\}$
$\{2,3,5,6\}$	$\{2,6\}$, $\{3,5\}$
$\{2,4,5,6\}$	$\{6\}$, $\{2,4\}$
$\{3,4,5,6\}$	$\{3,6\}$, $\{4,5\}$

4 Conclusions

In summary, a k-element subset of the first n counting numbers is subset-sum-distinct if and only if the power set is in one-to-one correspondence with the set of sums. But in the event that the number of subsets is strictly greater than the maximal possible sum, it is impossible to have a k-element SSD-set, as this article demonstrates for $k = 10$ and $n = 100$. However, if the number of subsets is strictly less than the maximal possible sum, there is no guarantee of a k-element SSD-set, as this article demonstrates for $k = 4$ and $n = 6$, there is no 4-element SSD-set, whereas for $k = 3$ and $n = 6$, there is at least one 3-element SSD-set.

References

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