RECTANGLES, DIAGONALS, AND LATTICE POINTS: AN APPLICATION OF THE GCD

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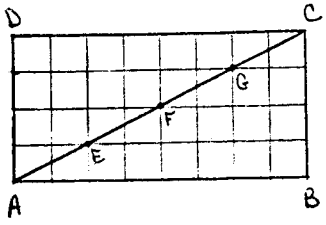
A common occurrence in mathematics is that a problem whose original statement appears to have no relation to number theory often depends for its solution on a simple number theory concept. One such example is the following: If a rectangle is drawn on a lattice, how many lattice points are on each diagonal? For example, consider the 8 by 4 rectangle with the diagonal as shown in Figure I.

Observe that the diagonal contains five points--the two vertices of the rectangle and three interior points.

Figure I



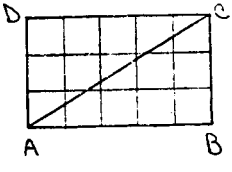
Also consider the 5 by 3 rectangle in Figure II. The diagonal contains only two points, the vertices of the rectangle or the endpoints of the diagonal.

The reader is invited to verify that Table I is correct.

Table I

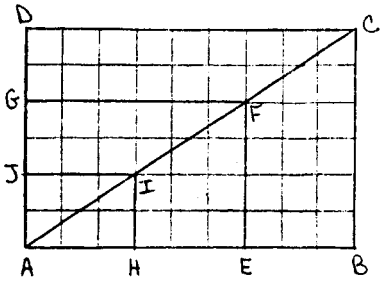
Size of Rectangle	Number of Lattice Points on Each Diagonal
10 by 6	3
6 by 3	4
6 by 4	3
8 by 4	5
5 by 3	2
12 by 6	7
6 by 5	2
7 by 7	8
9 by 6	4

Figure II



To help find a pattern, consider a 9 by 6 rectangle with the positive slope diagonal as shown in Figure III.

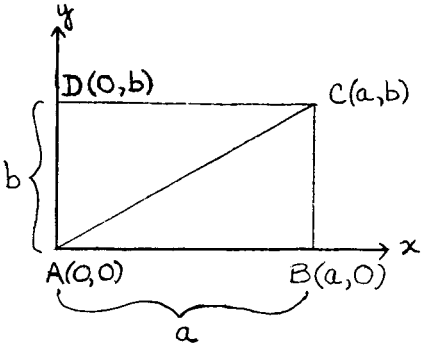
Figure III



Observe that AC has slope $\frac{6}{9}$. If $\overleftrightarrow{AB}$ is considered as the positive x-axis while $\overrightarrow{AD}$ is considered as the positive y-axis, then a lattice point in the first quadrant will be on $\overrightarrow{AC}$ if and only if its (x,y) coordinates have ratio $\frac{y}{x} = \frac{6}{9}$ or x = y = 0. Thus, a lattice point (x,y) is on $\overline{AC}$ if and only if $\frac{y}{x} = \frac{6}{9}$ where $x \leq 9$, or x = y = 0. Four points satisfy these conditions. They are A(0,0); I(3,2); F(6,4); C(9,6).

In general, let the original rectangle ABCD have horizontal and vertical dimensions of a and b (Figure IV). Impose a coordinate system as shown.

Figure IV



A lattice point will lie on the positive slope diagonal if and only if its (x,y) coordinates have ratio $\frac{y}{x} = \frac{b}{a}$ where $x \leq a$, or x = y = 0. The problem is then to count the number of lattice points (x,y) satisfying these conditions.

Consider the fraction $\frac{b}{a}$. Let g be the greatest common divisor (gcd) of a and b; let b = cg and a = dg. Then $\frac{c}{d}$ is irreducible and $\frac{c}{d} = \frac{b}{a}$. Clearly, $\frac{c \times 1}{d \times 1}, \frac{c \times 2}{d \times 2}, \frac{c \times 3}{d \times 3}, \ldots, \frac{c \times g}{d \times g}$ are all equal to $\frac{b}{a}$ and each satisfies the condition that its denominator is less than or equal to a. Consequently, the positive slope diagonal in rectangle ABCD contains each of the (g + 1) points: A(0,0) = (d x 0, c x 0); (d x 1, c x 1); (d x 2, c x 2); (d x 3, c x 3); ...; (d x g, c x g) = (a,b).

Thus (g + 1) lattice points which lie on the positive slope diagonal have been identified. Does this diagonal contain additional points? Suppose that it does contain a point (e,f) not equal to any of the points named in the above paragraph. Then,

$\frac{f}{e} = \frac{b}{a}$ and $e \leq a$. Also, $\frac{f}{e} = \frac{c}{d}$ and $fd = ec$. Therefore d is a divisor of ec. Since c and d are relatively prime, d must be a divisor of e. Consequently, $e = dh$ and $f = ch$ for some positive integer h. But $dh = e \leq a = dg$. Hence $h \leq g$. Since $(e,f) = (dh, ch)$, it may be concluded that (e,f) is one of the points $(d \times 1, c \times 1)$, $(d \times 2, c \times 2)$, ..., $(d \times g, c \times g)$. Therefore, the positive slope diagonal contains exactly the $(g + 1)$ lattice points previously named.

If a rectangle of dimension 270 by 225 were drawn on a lattice, its positive slope diagonal would contain exactly gcd (270,225) + 1 lattice points. The reader should have his/her students name all 46 points!

IN - DIVISIBLE

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Much has been written concerning the concept of divisibility; unfortunately a good deal of this has dealt with specifics and particular cases. Consider, for example, the literature that can be found which focuses on divisibility rules. The author is aware of individual rules for 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 17, etc., as well as variations for products and powers of these numbers. The question that arises is whether or not a more generalized statement concerning divisibility can be formulated.

An article by Steve Meiring which appeared in the February 1979 issue of the Ohio Journal of School Mathematics presented a variety of divisibility tests as well as some of the possible benefits of incorporating these tests into the middle school curriculum. The conclusion of the article hinted at a general divisibility rule in which the divisor is prime ($p > 2$) through use of a recursion formula. A further examination of this "comprehensive" rule proves to be an interesting exercise in number theory.

Suppose we wish to determine (for some outlandish reason) the