CREATING CONTEST PROBLEMS

Leo J. Schneider
John Carroll University
University Heights, OH 44118

As chair of the committee which writes American High School Mathematics Examination [AHSME], I am frequently asked, "How do your committee members create the problems used on the AHSME?" As a partial answer I would like to hypothesize the steps which might have been used in creating the following problem from the 1988 AHSME:

26. Suppose that p and q are positive numbers for which

$$\log_9(p) = \log_{12}(q) = \log_{16}(p+q).$$

What is the value of $\frac{q}{p}$?

(A) $\frac{4}{3}$ (B) $\frac{1+\sqrt{3}}{2}$ (C) $\frac{8}{5}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) $\frac{16}{9}$

I was not the author of this problem. I will use it to illustrate typical steps in problem creation. (The actual process was probably more akin to a random walk, with these steps seen later as the significant forward moves.)

I. Start with a familiar mathematical idea. This is important if the final problem is to be valuable in the mathematical development of those who study past exams to increase their problem-solving ability.

Start with $2^t \cdot 4^t = (2 \cdot 4)^t = 8^t$.

II. Restate the idea. After all, we want our problems to differ from any the student may have encountered in some text.

Convert the identity to logarithms in various bases:

$$\text{If } p = 2^t \text{ and } q = 4^t \text{ then } pq = 8^t \text{ and}$$
$$t = \log_2(p) = \log_4(q) = \log_8(pq).$$

The set of ordered pairs (p,q) for which the last two equalities hold is not especially interesting.

III. Alter the result to obtain something more interesting. You will find that over 99% of such tries yield impossible or uninteresting results.

In this case, substitution of p+q for pq[1] yields

$$t = \log_2(p) = \log_4(q) = \log_8(p+q)$$

which we can rework like this:

$$2^t + 4^t = p + q = 8^t,$$

$$x(x^2 - x - 1) = 0 \quad \text{if } x = 2^t,$$

$$x = \frac{1+\sqrt{1+4}}{2} \qquad \text{since } x = 2^t > 0.$$

We may have found something interesting because the solution is unique and is the golden ratio!

IV. Generalize the interesting result. Generalizations can lead to better ways of phrasing the contest problem.

The 2 could be any valid logarithm base, b:

$$t = \log_b(p) = \log_{b^2}(q) = \log_{b^3}(p+q) \Rightarrow b^t = \frac{1+\sqrt{5}}{2}.$$

Noting the geometric progression of bases we try

$$t = \log_a(p) = \log_{ar}(q) = \log_{ar^2}(p+q)$$

$$a^t = p, \;\; (ar)^t = q, \;\; (ar^2)^t = p + q = a^t + (ar)^t$$

$$(r^t)^2 - r^t - 1 = 0$$

$$r^t = \frac{1+\sqrt{5}}{2}.$$

At this point we have invented a new (at least for us) mathematical fact:

For any a and r for which a, ar, and ar^2 are valid bases of logarithms, the only solutions (p,q) to

$$\log_a(p) = \log_{ar}(q) = \log_{ar^2}(p+q)$$

are $p = a^t$ and $q = (ar)^t$ where $r^t = \frac{1+\sqrt{5}}{2}$.

V. Particularize the fact for use on a multiple choice test. Decide to ask for just one result (rather than an ordered pair). The use of specific numbers rather than variables can lead to answer choices which are less apt to give hints about the solution in a multiple choice test format.

Do not ask for a, r, and t. Ask for $r^t = \frac{q}{p}$. Find three terms in geometric sequence which have enough other properties that they do not immediately reveal the method of solution. The author of the problem chose the sequence 9, 12, 16.

VI. Word the problem as precisely and concisely as possible. This will make the test as fair as possible to all participants.

VII. Write a good solution. The solution should be concise, correct, and one a student might invent based on the final form of the problem.

This best solution may bear little relationship to the thought processes which led to the problem. Compare the above with the following solution published for problem 26 in the 1988 AHSME Solutions Pamphlet[2] to verify this:

26. (D) Let t be the common value of $\log_9(p)$, $\log_{12}(q)$ and $\log_{16}(p+q)$. Then

$$p = 9^t,\ \ q = 12^t,\ \text{and}\ \ 16^t = p + q = 9^t + 12^t.$$

Divide the last equation by 9^t and note that

$$\frac{16^t}{9^t} = \left(\frac{4^t}{3^t}\right)^2 = \left(\frac{12^t}{9^t}\right)^2 = \left[\frac{q}{p}\right]^2.$$

Now let x stand for the unknown ratio $\frac{q}{p}$. From the division referred to above we obtain $x^2 = 1 + x$, which leads easily to $x = \frac{1}{2}(1+\sqrt{5})$ since x must be a positive root.

Steps I–IV are valid mathematical research and VI and VII are important in any mathematical exposition. Only in V do we apply the "art" of test construction.

High school teachers who wish to help their students study problem-solving using our AHSME problems might first help their students understand the solutions to the problems as stated, and then invent similar practice problems to see if this help was effective. I will use the above problem to illustrate how this can be done.

It was not necessary to stop our analysis at step IV. One way to proceed might be as follows:

a. What is your favorite quadratic polynomial which has both a positive and a negative root? Suppose it is $x^2 + 2x - 15$ which has roots $x = 3, -5$.

b. Note that if

$$t = \log_a(p) = \log_{ar}(q) = \log_{ar^2}(15p - 2q)$$

then

$a^t = p, \ (ar)^t = q, \ (ar^2)^t = 15p - 2q = 15a^t - 2(ar)^t$

$$(r^t)^2 + 2r^t - 15 = 0,$$

so $r^t = 3$ which is the only positive root.

c. Vary the original problem by asking for $\frac{p}{q} = r^{-t} = \frac{1}{3}$.

d. We now need a distracting geometric sequence of three positive integers. The 2 and 15 in the polynomial suggest that we might relate the sequence to 2, 3, and 5. Non-monic fractional common ratios can disguise such sequences, so we choose $r = \frac{2}{3}$ and $a = 45$ since 3^2 must evenly divide a if the logarithm bases are to be integers.

e. We now phrase our problem:

26′. Suppose that p and q are positive numbers for which

$$\log_{45}(p) = \log_{30}(q) = \log_{20}(15p - 2q).$$

What is the value of $\frac{p}{q}$?

f. A popular student solution might be:

Let t be the common value of $\log_{45}(p)$, $\log_{30}(q)$, and $\log_{20}(15p - 2q)$.

Then

$p = 45^t$, $q = 30^t$, and $20^t = 15p - 2q = 15 \cdot 45^t - 2 \cdot 30^t$.

Divide the last equation by 30^t to obtain

$$\frac{20^t}{30^t} = 15 \cdot \frac{45^t}{30^t} - 2.$$

Let $x = \frac{p}{q} = \frac{45^t}{30^t}$ and note that $\frac{20^t}{30^t} = \frac{30^t}{45^t} = \frac{1}{x}$. Thus

$$\frac{1}{x} = 15x - 2 \text{ or } 15x^2 - 2x - 1 = 0.$$

Either by factoring or the quadratic formula the last equation leads to $x = \frac{1}{3}, -\frac{1}{5}$. Since $x > 0$, $x = \frac{1}{3}$.

We can all be very proud of the accomplishments of our Ohio students on the

AHSME. They show the high quality of secondary mathematics education throughout the state.

There is a wealth of used contest problems already available from the OCTM tests, from the AMC, and in the new NCTM publication on the ARML/NYSML mathematics meets. I have tried to indicate how teachers who are interested can invent still more problems in analogy to these published problems.

---

[1] This is certainly not the first time that a $(+ \Leftrightarrow \cdot)$ replacement led to interesting results. For example, Leonhard Euler saw that the formula $1 + 2 + ... + n = \frac{n(n+1)}{2}$ could be 'extended' from positive integers to the curve $y = \frac{x(x+1)}{2}$. His work to 'extend' $1 \cdot 2 \cdot ... \cdot n = n!$ to a curve defined for real numbers led to his invention of the gamma function.

[2] All problems, with solutions, used on the AHSME 1950–present are available from the office of the American Mathematics Competitions; University of Nebraska; Lincoln, NE 68588. Write for a price list.

THE DATING GAME

Pick a day, any day, say February 14, 1989. Use the digits 2, 1, 4, 1, 9, 8, 9, and operation signs to generate integers, like so:

$$1 = [9 - (2 + 1 + 4 + 1)] \times (9 - 8)$$
$$2 = (9 \times 8) - (9 + 1)(2 + 1 + 4)$$
$$3 = 99 \div [4(8 + 1) - (2 + 1)]$$

You and your students can make the rules to suit yourselves. For the above example only 2, 1, 4, 8 and 9 could be allowed. Use less than all four operations, include roots and exponents, even factorials and greatest integer. Dots are used when nothing else works: $.\dot{9}$ means .99999... and is equal to 1. A once-a-year version appears on the first page of this issue.

Prettiest little calculation drill you ever saw and every day is a new game!