STUDENT DISCOVERS AN ORIGINAL THEOREM?

Duane Bollenbacher and Noah Wakeman
Bluffton High School
Bluffton, OH 45817

While reviewing some pre-calculus concepts at the beginning of our AP Calculus course this past year, we derived and then used the formula

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

to find the distance between a point $P(x_1, y_1)$ and a line with equation $ax + by + c = 0$. I later assigned the task of finding the distance between parallel lines $4x - y = 7$ and $4x - y = -17$. Most students did as I had been taught and the way that textbooks tell us: find a point on one line and then use the distance formula above. But one of my students, Noah Wakeman, came up with his own formula for finding the distance between two parallel lines:

$$d = \frac{|b_1 - b_2|}{\sqrt{m^2 + 1}}$$

b_1 and b_2 the y-intercepts, and m the slope

In the example above ($y = 4x - 7$ and $y = 4x + 17$),

$$d = \frac{|b_1 - b_2|}{\sqrt{m^2 + 1}} = \frac{|-7 - 17|}{\sqrt{4^2 + 1}} = \frac{24}{\sqrt{17}},$$

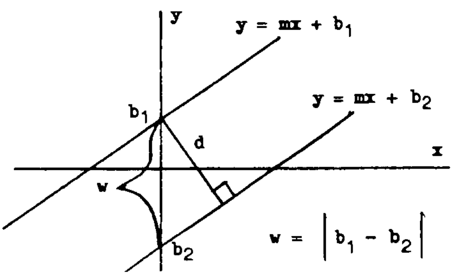
a much easier computation.

I could not find this theorem in any textbooks, so I insisted upon its derivation. At first Noah's proof of this was long, extensive, and extremely hard to follow. I encouraged him to make it simpler, with explanations. Upon completion it looked like this:

Noah's proof:

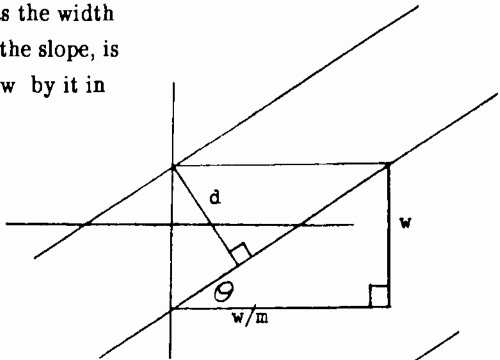
First, I let w equal the difference between the y intercepts.

$$w = |b_1 - b_2|$$



Next, I formed a rectangle with w as the width and w/m as the length. (Since m , the slope, is like the tangent of θ , I must divide w by it in order to get the length.)

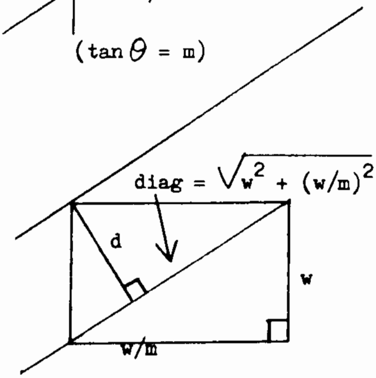
$\tan\theta = m$



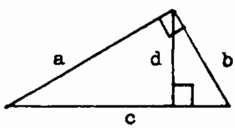
(tan θ = m)

Third, I used the Pythagorean theorem to find the diagonal.

$$\text{length of diagonal} = \sqrt{w^2 + \left(\frac{w}{m}\right)^2}$$

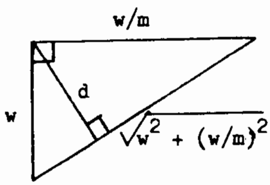


Fourth, I remembered my geometry and knew the altitude to the right angle in a triangle is equal to the product of the legs divided by the hypotenuse



$$d = \frac{a \times b}{c}$$

so



$$d = \frac{w \cdot \frac{w}{m}}{\sqrt{w^2 + \left(\frac{w}{m}\right)^2}}$$

Finally, I used algebra to simplify the equation.

$$d = \frac{w \cdot \frac{w}{m}}{\sqrt{w^2 + (\frac{w}{m})^2}} = \frac{\frac{w^2}{m}}{\sqrt{w^2 + (\frac{w}{m})^2}} = \sqrt{\frac{\frac{w^4}{m^2}}{w^2 + \left(\frac{w^2}{m^2}\right)}}$$

Multiplying numerator and denominator by $\frac{m^2}{w^2}$, we get

$$\sqrt{\frac{w^2}{m^2 + 1}} = \frac{w}{\sqrt{m^2 + 1}} = \frac{|b_1 - b_2|}{\sqrt{m^2 + 1}}$$

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In diligently improving his derivation, Noah also derived another form of the theorem.

Noah's second theorem:

For the second theorem, I knew the y-intercept is equal to the constant divided by the y coefficient and the slope is equal to the opposite of the x coefficient divided by the y coefficient.

$ax + by = c_1 \Rightarrow$
y-intercept $b_1 = \frac{c_1}{b}$
$ax + by = c_2 \Rightarrow$
y-intercept $b_2 = \frac{c_2}{b}$

So, $b_1 = \frac{c_1}{b}$ and $b_2 = \frac{c_2}{b}$ and $m = -\frac{a}{b}$

Then I simplified my original formula:

$$d = \frac{|b_1 - b_2|}{\sqrt{m^2 + 1}} = \frac{\left|\frac{c_1}{b} - \frac{c_2}{b}\right|}{\sqrt{\left(-\frac{a}{b}\right)^2 + 1}} = \frac{\left|\frac{c_1 - c_2}{b}\right|}{\sqrt{\frac{a^2}{b^2} + \frac{b^2}{b^2}}} =$$

$$\sqrt{\frac{\frac{(c_1 - c_2)^2}{b^2}}{\frac{a^2}{b^2} + \frac{b^2}{b^2}}} = \sqrt{\frac{(c_1 - c_2)^2}{a^2 + b^2}} = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

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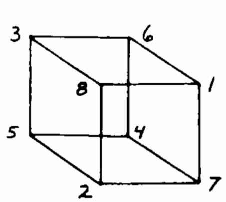
As you can see, this time the distance is given in terms of a , b , c rather than the slope and y-intercept. Again, in the example above,

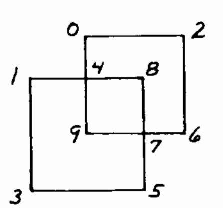
$$d = \frac{|7 - (-17)|}{\sqrt{4^2 + (-1)^2}} = \frac{24}{\sqrt{17}},$$

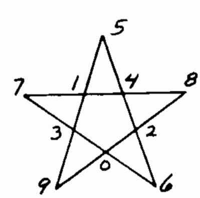
a <u>much</u> easier computation.

First of all, I was amazed to see a formula for this. Upon checking with several authorities and in many textbooks, I could not find such a formula. Perhaps this is a very common bit of knowledge, and I have simply overlooked something right before my eyes. I would, however, be interested in hearing from any of you who have seen either of these theorems, and learning where you have found them. Regardless, I was astounded that in our high school ranks we had a student who could do some very original thinking. The rest of the class watched in awe as Noah painstakingly over a long period of time simplified and shortened his proof so that it is easy to follow. I pointed out that probably many theorems, and concepts, in mathematics were discovered in just such a way: first of all, a need existed; many trials and errors occurred; and then, a proof (possibly understood only by its inventor) was simplified so that many of us are able to follow, and appreciate, it.

Answers to Alaskan magic figures:







Letter to the Editors:

Dear Colleagues,

Why is it that when we talk about functions in general, we use notation like $f(x)$, $g(x)$, and $h(x)$, but when we talk about specific functions we use the notation $\sin x$, $\cos x$, and $\ln x$? I suggest that it is time to become consistent in our notation; in particular, I suggest we start using $\sin(x)$, $\cos(x)$, and $\ln(x)$, and furthermore start verbalizing these as "sine of x" , etc. We all know that once a concept is understood, notation is not important; but while our students are learning that mathematics is not "just a bag of tricks", we should be careful to be consistent in our notation.

Why is it that XY means "X times Y", but AREA does not mean "A times R times E times A"? I suggest, that with the exception of numerical constants, we do away with juxtaposition for multiplication, and adopt the computer scientists' notation of X*Y for multiplication and allow XY to be the name of another variable.

Thirdly, I'd like to suggest that we de-emphasize e^x notation and use $\exp(x)$ in order to emphasize the fact that this is a function.

J. William Friel
University of Dayton

On IQ tests one often finds questions like: "What is the next term in the sequence 1 , 4 , 9 , 16 , ...?" The expected answer is "25", the fifth term in the sequence whose nth term is n^2 . Actually, the fifth term might be any number whatever, say π . Thus, the sequence whose nth term is

$$f(n) = n^2 + (n-1)(n-2)(n-3)(n-4)(\pi-25)/24$$

has 1 , 4 , 9 , 16 , π for its first five terms.

Return to Mathematical Circles
by Howard W. Eves
Prindle, Weber & Schmidt, 1988, p. 95

Solutions to Alaskan cryptarithms:

324	3842	792	65
765	6752	34	85
1089	10594	826	150