Exploring Irrational Numbers with *TI-Nspire™*

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This article describes some activities for approximating irrational numbers with partial sums of inifinite series that have been used with preservice elementary teachers and adapted for the TI-Nspire™. *The idea for the article came from an 8th grader's question about how people knew the correct digits of π. Before discussing irrational numbers, some examples of rational numbers approximated by an infinite series are given. Information using the* TI-Nspire™ *to facilitate student understanding, including keystrokes is presented. This is followed by a discussion of approximating π using one of Euler's well-known series. Finally, there is also a discussion of the limitations of technology in displaying decimal approximations beyond the number of significant digits stored in the* Nspire.

Background

The first author presented methods of exploring irrational numbers using calculators (both four-function and graphers) and spreadsheets in 1999 at the 12th *International Conference on Technology in Collegiate Mathematics (ICTCM)*. With the addition of the *TI-Nspire™* handheld computer to the tools available to teachers and students, we were inspired to revisit this topic and demonstrate the linked representations within the handheld device. Below, the first author recounts the origin of his ideas for this topic:

In 1980, I was in my second year of teaching mathematics at Anthony High School in Anthony, Texas – a small town north of El Paso, situated on the border of Texas and New Mexico. I was the sole mathematics teacher for grades 8 through 12. In order to brighten up a very drab classroom, I used some money to buy math posters – one of which showed π calculated to 2,000 decimal places. Most students were suitably awed by this "big" number until I mentioned it was just a little bit bigger than three and that most of the digits were to the right of the decimal. I reminded students π was an irrational number and thus had a non-repeating and non-terminating decimal representation. One of my eighth graders asked, "If there's no pattern to the decimals, how do they know what number comes next?"

This is a fairly deep question for a junior high school student to ask. The remainder of this paper describes some activities that have been used over the years with preservice elementary teachers so that they could give a meaningful answer to this question, should one of their students ever ask it.

First Steps

The key to developing a knowledge base to answer this question lies with the idea that irrational numbers can be approximated by partial sums of an infinite series of rational numbers. And before preservice teachers can be convinced of that, they first need to be convinced that it is possible that rational numbers can be represented that way. More importantly, by having the students use technology (the *TI-Nspire™* in this case) to do the computations, they also need to understand the inherent limitations of their tools to develop a complete understanding of how subsequent digits in π can appear.

The following is an excerpt from a

student activity book (Klespis, 1997) edited by the first author and modified for use with the *TI-Nspire*™. Students are asked to examine the following sequence of fractions, to describe the pattern in the denominators, and to calculate the decimal equivalent of each fraction in the sequence.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048} (*)$$

They are then asked to compute the sum of the following related series:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\frac{1}{2048}$$

Once this is done, we ask, "What simple, well-known number does the sum of this series approach?" Our past students have had little difficulty determining that the sum gets close to 1. Students are then asked to find the well-known limits of additional series such as

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\frac{1}{2187}+\ldots$$

or

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}+\frac{1}{4096}+\frac{1}{16384}+\ldots$$

as well as those that start with $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and $\frac{1}{10}$. They are to look for patterns and generalize their results.

One of my eighth graders asked, "If there's no pattern to the decimals, how do they know what number comes next?"

On the *TI-Nspire*™, a Lists & Spreadsheet page can facilitate this process. When placed in the formula area of column A, the command

$$= seqn\left(\frac{u(n-1)}{2}, \frac{1}{2}, 11\right)$$

places the first 11 terms of the sequence (*) in that column (See Figure 1).

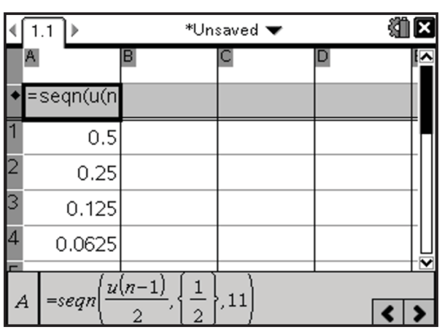

Fig 1 The *TI-Nspire*™ seqn command

To obtain the partial sums of the series in column B, type =A1 in cell B1 and the formula = B1+A2 in cell B2 (See Figure 2).

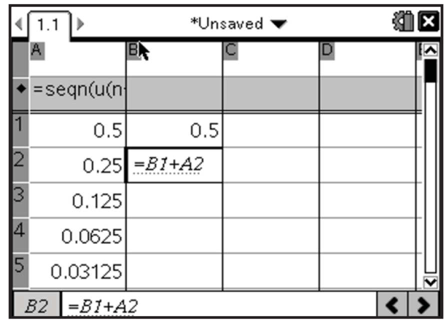

Fig 2 Calculating partial sums

While on cell B2, press menu, *3: Data, 3: Fill Down*. Move the dashed rectangle down to cell B11, and press Enter. This will copy the formula down the column, but keep the cell references relative so that cell B11 contains the formula =B10+A11.It will display a value very close to 1, as shown in Figure 3.

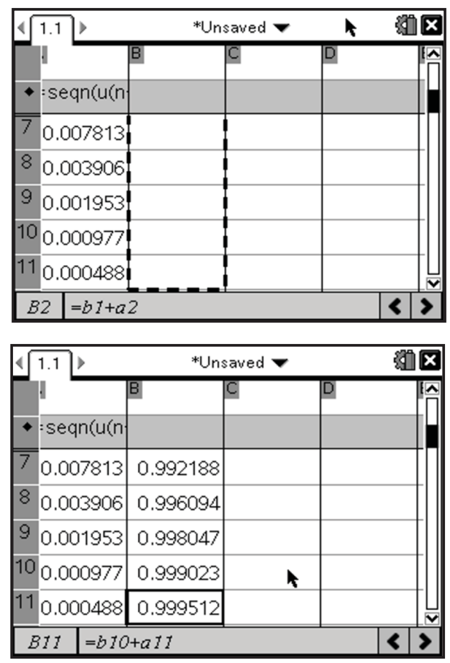

Fig 3 Fill down feature of *TI-Nspire*™

By editing the formula in column A, prospective teachers can easily investigate other series. The example shown on the left in Figure 4 is based on a geometric series with initial term and ratio both equal to $\frac{1}{9}$. On a calculator page, we can find an equivalent fraction by pressing menu, *2: Number, 2: Approximate to Fraction* (See Figure 4).

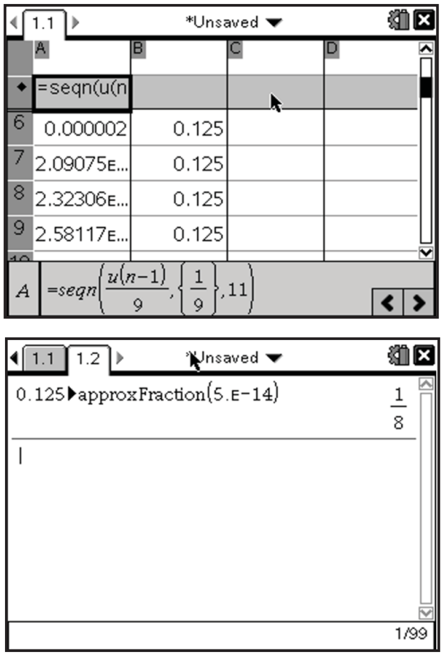

Fig 4 Investigating other series

This activity has worked well with prospective elementary teachers. It serves to reinforce their connections between fraction and decimal representations as well as convincing them that rational numbers can be written as a sum of an infinite series of fractions.

It is important to stress here that each new partial sum will cause a change in the decimal representation of the sum of the series. However, there will be a point where these values do not change. This is due to the fact that many calculators (the *TI-Nspire*™ included) have memory limitations and for this reason, only store about 13 significant digits in a decimal. Thus, as the number of terms in the series increases, we do not gain any additional precision.

This technological limitation and how it impacts the question originally posed on the first page of this article will be explored after the next section.

Back to π

Returning to the original question of how you can determine the digits of π if there is no pattern, we know from Euler that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots + \frac{(-1)^n}{2n+1} + \ldots \text{ for } n = 0, 1, 2, 3, \ldots$$

As with rational numbers, we can use the *TI-Nspire*™ to investigate this process. We'll begin by using column A as an index column, and insert =seqn(u(n-1)+1,{0},50) in the formula line for that column. Column B will hold the terms of the sequence if we type = (-1)A/(2A+1) in the formula line (See Figure 5).

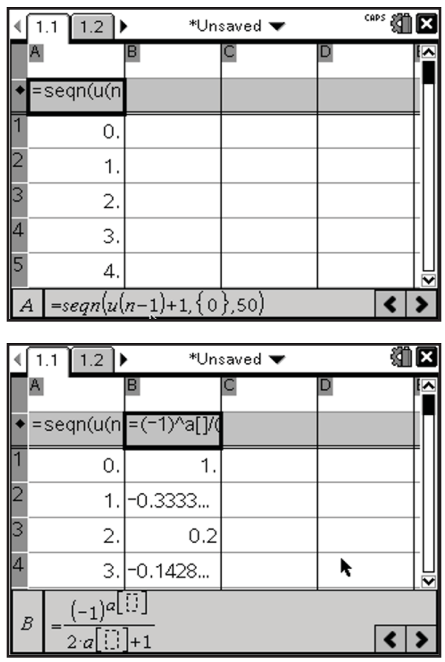

Fig 5 Terms of Euler's sequence

The partial sums for the series are placed in column C utilizing the Fill Down command. Finally, we use the formula =4C in column D to display the approximation for π. Portions of this series are shown in Figure 6. By giving column D a name, such as piapprox (as shown on the second screenshot in Figure 6), we can construct a visual representation of this series.

On a Graphs page, press menu, *3: Graph Type, 4: Scatter Plot.* Use index as the *x*-variable and piapprox as the *y*-variable, and adjust the window using menu, *4: Window/Zoom, 9: Zoom – Data* (See Figure 7).

This activity has worked well with prospective elementary teachers. It serves to reinforce their connections between fraction and decimal representations.

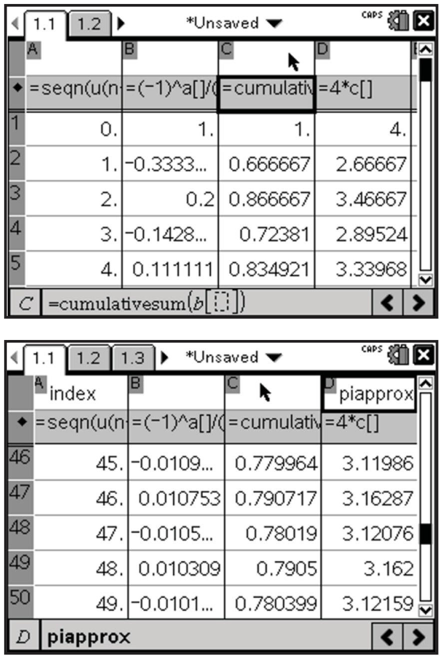

Fig 6 Partial sums approximating pi

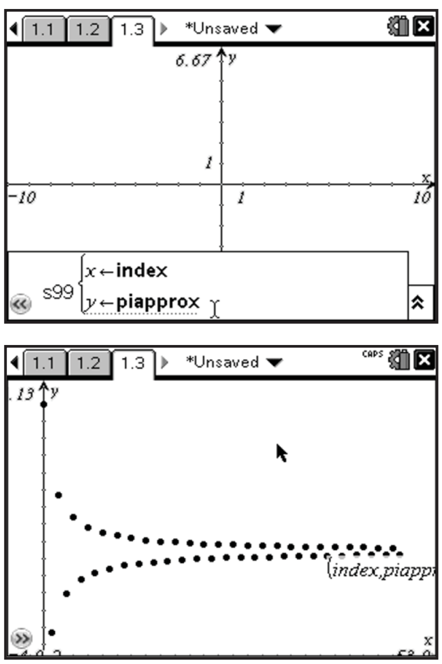

Fig 7 Partial sums represented graphically

By plotting the function f1(x) = π as in Figure 8, students note that this series does appear to approach the value of π.

Digits of π and Decimal Accuracy

The previous section shows a decimal approximation of π. The question then

The question then becomes, "Just how accurate is the representation?" Most hand-held calculators only store decimals to about thirteen significant digits.

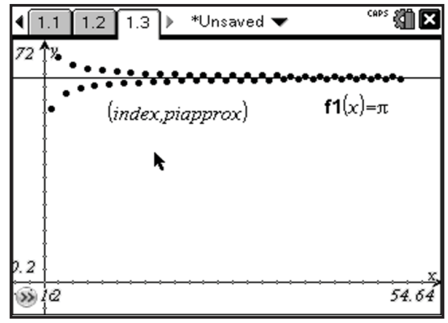

Fig 8 Illustrating convergence to pi

becomes, "Just how accurate is the representation?" As was mentioned earlier, most hand-held calculators only store decimals to about 13 significant digits. Certainly computers used by mathematicians or scientists may have greater internal capacity. Since the key question at the beginning of the paper relates to how one can determine any of the digits in π, it is important for the students to discern that there are changes in the decimal approximation up to *Nspire*'s accuracy. The more powerful the computer, the farther out in the decimal expansion one can determine the digits of π. The same idea can be generalized to other irrational numbers.

Other Irrational Numbers

Well-known irrational numbers such as ϕ, e, or $\sqrt{2}$ can be represented in a similar manner. In Figures 9 and 10, we investigate the Golden Ratio ϕ. On a new spreadsheet page, we begin with the seed numbers 1 and 1 in cells A1 and A2, and the formula =A1+A2 in cell A3. By filling down to row 20, we get the first 20 Fibonacci numbers. Placing the formula =A2/A1 in cell B2, and then filling down, gives us the ratios of successive Fibonacci numbers.

By giving column B a name, such as fibratio in Figure 9, we can now create a scatterplot (index, fibratio) (See Figure 10). What is even more interesting is that the ratio will converge to ϕ for any non-zero seed numbers entered in A1 and A2! Try it!

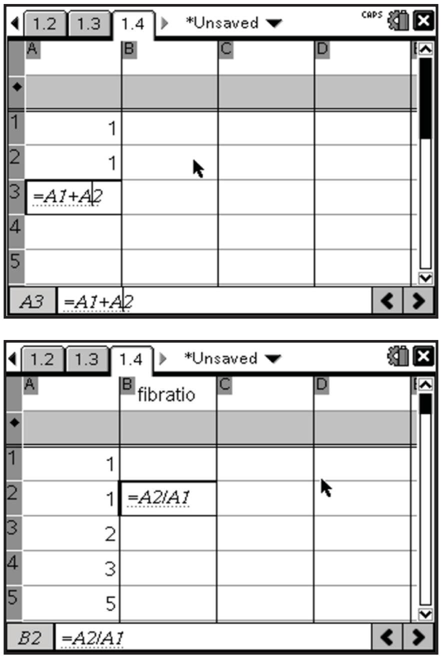

Fig 9 Exploring ratios of Fibonacci numbers

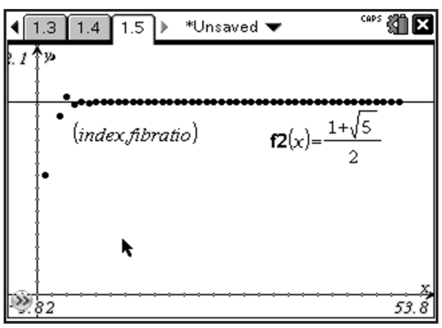

Fig 10 Illustrating convergence to ϕ

Summary

In these few short pages, we have tried to indicate how the decimal representations of irrational numbers can be approximated using the *TI-Nspire*™. The procedures and programs have been easy enough for preservice teachers to use and there has been the added benefit of helping many (though not all) preservice elementary and middle school teachers gain more familiarity with decimal representations of both rational and irrational numbers.

References

Klespis, M. L. (ed.) (1997). *Student Activities Manual for Mathematics Courses for Preservice Elementary Teachers*. Harcourt, Brace: St. Louis.

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Keep your face always toward the sunshine - and shadows will fall behind you.
- Walt Whitman