A VISIT WITH GEOMETRY

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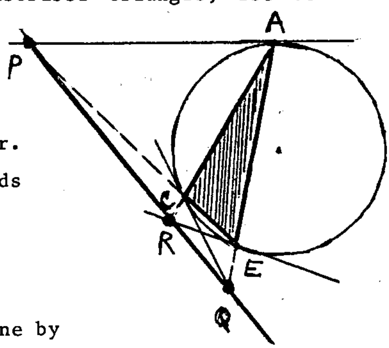
All geometry students know that every triangle can be inscribed in some circle and laboratory studies can be started in the middle and junior high school years to discover some exciting properties.

At the vertices of the inscribed triangle, let us draw or construct tangents and let them intersect the opposite sides produced in points P, Q and R. These points seem to be collinear. Try again and again with all kinds of variations and we have a <u>conjecture</u> to be studied by further examples and a challenge for attempted proof. This can be done by more advanced students even in the secondary school with sufficient background in similar triangles, properties of secants and tangents from an external point and the use of the Theorem of Menelaus converse.*



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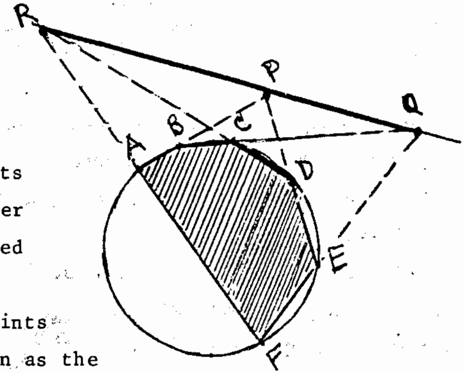
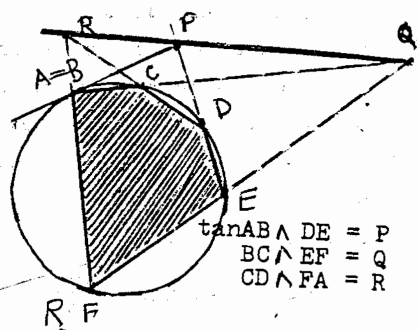
* We must prove that $\frac{AR}{RC} \cdot \frac{CP}{PE} \cdot \frac{EQ}{QA} = -1$. Now PC • PE = $\overline{PA}^2$, AQ • QE = $\overline{QC}^2$, AR • RC = $\overline{ER}^2$, whence $\frac{AR}{RC} \cdot \frac{CP}{PE} \cdot \frac{EQ}{QA} = \frac{\overline{ER}^2}{\overline{RC}^2} \cdot \frac{\overline{PA}^2}{\overline{PE}^2} \cdot \frac{\overline{QC}^2}{\overline{QA}^2}$
But from similar triangles ER/RC = AE/CE, PA/PE = AC/AE and QC/QA = CE/AC and the above expression becomes numerically equal to 1. The theorem follows. All these steps are within the capability of high school students and the Theorem of Menelaus can also be included in secondary school courses.

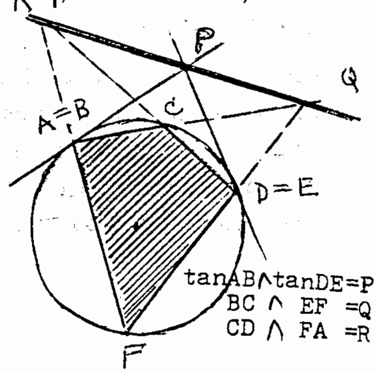
Now let us move to the study of a non-regular hexagon (with no parallel sides) inscribed in a circle and let us produce the opposite sides AB and DE, BC and EF, and CD and FA to meet in points P, Q and R respectively. What do we observe about the three points P, Q and R ? Let us try other non-regular hexagons inscribed in a circle and we have the conjecture that the three points are collinear! This is known as the Theorem of Pascal and is another of the gems through which students can be introduced to beauty in geometry. It is proved in advanced geometry, but the argument makes use of ideas similar to those above.



But now let us "let imagination take over." What happens as A and B approach each other and finally become coincident? The <u>secant</u> AB becomes the <u>tangent</u> AB and we have a <u>pentagon</u> inscribed in a circle with two pairs of opposite sides and a tangent at A=B opposite to side EF. And the three points P, Q and R remain collinear. We have a theorem about inscribed pentagons and a tangent at one of the vertices.

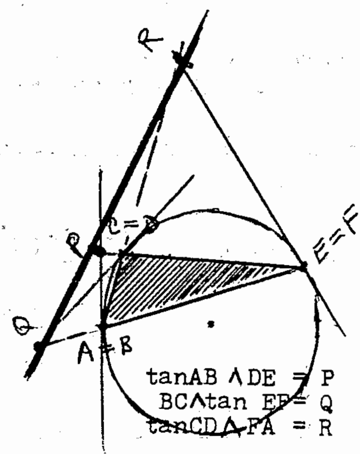


Imagination suggests that we go further. Suppose that D and E merge so that we have a <u>tangent</u> DE at D. We now have the <u>quadrilateral</u> BCDF with a tangent at B (the old secant AB) and a tangent at D (the old secant DE) inscribed in the circle.

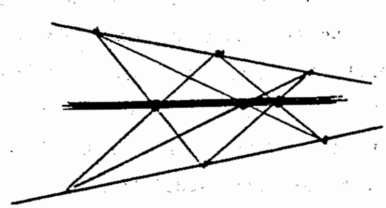


What conjecture can we make as we reduce the Pascal Theorem to a figure concerning a quadrilateral inscribed in a circle and with tangents at opposite vertices? Indeed, perhaps we can note the possibility of <u>four</u> points being collinear. Proofs of the above cases can be pursued in much the same manner as that for the Theorem of Pascal for circles mentioned above.

We let our imagery continue. Suppose that points A and B merge to form the tangent at A=B, C and D coincide to form the tangent C-D and now E and F coalesce to create the tangent at E=F. The Pascal Theorem which says that if AB DE = P, BC EF = Q and CD FA = R then P, Q and R are collinear still holds and we see that our beginning theorem is a <u>special case</u> of the more general Pascal Theorem for circles. What beauty there is in the study of "genealogy" of theorems and we have seen a good example. This should be one of the uppermost aims of students and teachers of geometry.



But this is not all. If <u>any</u> of the figures above are drawn as transparencies and the projector is rotated from its original position so the image is far to the right or to the left of the screen the circles become ellipses, but there is still collinearity of points P, Q and R and we have <u>new</u> possible theorems. There is no end to experimentation and the formulation of conjectures. Moreover, if the ellipse (conic) degenerates to two intersecting straight lines, the Pascal Theorem becomes the Theorem of Pappus and there is still collinearity. Junior high school students love to demonstrate this.



We have outlined a proof for a beautiful theorem concerning a triangle inscribed in a circle and the tangents at the vertices and have suggested that it is also a special case of the more general Pascal Theorem. Along the way were mentioned perhaps less well-known theorems about inscribed pentagons and quadrilaterals. We hope that we have been encouraged to look for a certain "genealogy" of theorems elsewhere.

A REFERENCE

Watson, Emery Ernest and Watson, Margaret Marie. Elements of Projective Geometry. Boston: D.C. Heath and Co., 1935.

HOW MANY BEANS IN THE JAR? NON-COMPUTATIONAL ESTIMATION ACTIVITIES FOR ELEMENTARY TEACHERS

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Non-computational estimation can help children to solve problems at very early ages, especially because there are many problems in life which do not require computation or where computation is less efficient or inappropriate. It closely parallels the problem solving strategy of Guess and Check. Students estimate and then verify their guess by counting concrete objects or measuring with standard unit devices (rulers, scales, stopwatches, etc.).

Proficiency in non-computational estimation is related to a technique used in behavior modification. After all, learning can be defined as a change in student behavior. With each repetition, the student makes successively closer approximations of the desired behavior. For example, a child makes an educated guess. Then, upon checking the guess against a standard, the child is able to form a benchmark in her mind, so that she will be able to come closer to the actual measure on her next try at a similar task.