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Green's Functions in Physical Geodesy and the Computation of the External Gravity Field and the Geodetic Boundary Value Problem

by

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Prepared for

Air Force Cambridge Research Laboratories
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Scientific Report No. 15

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and

THE COMPUTATION OF THE EXTERNAL GRAVITY FIELD AND THE GEODETIC BOUNDARY VALUE PROBLEM

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ABSTRACTS

Green's Functions in Physical Geodesy. Of the two main approaches to boundary value problems--the use of Green's functions and of integral equations--the latter approach has been chosen almost exclusively for the geodetic boundary value problem. The present report shows how Green's functions may be used for re-deriving known formulas and also for obtaining new results.

Formulas for the third boundary value problem for the sphere are developed and then specialized to Stokes' problem. As a limiting case, the boundary value problem for the plane is briefly considered. Then a formula for the variation of Green's function with the boundary surface is developed and applied to the problem of Molodensky.

The Computation of the External Gravity Field and the Geodetic Boundary Value Problem. In the usual way of computing the external gravity field, the earth is considered as a level surface although, strictly speaking, the free-air gravity anomalies refer to the non-level physical surface of the earth.

The main purpose of the present paper is to give formulas and, as an appendix, some estimates for the effect of topographic height on these computations. An application of Green's identities yields direct, but complicated, formulas for the effect of the disturbing potential T and the gravity anomaly Δg outside the earth. A simpler solution for T is obtained through the use of a fictitious surface layer, a coating, on the earth's physical surface. A third method, which seems to be optimal for practical computations, is a free-air reduction to sea level. The accurate performance of this reduction is a problem of downward continuation of Δg, which may be handled by iterative solution of a simpler integral equation. After this reduction, however, the conventional spherical formulas can be applier.

In addition, the paper presents connections between the determination of the external gravity field from surface data, which is related to the conventional boundary value problems of potential theory, and the determination of the earth's physical surface itself, which is specifically geodetic boundary value problem.

GREEN'S FUNCTIONS IN PHYSICAL GEODESY

By

Helmut Moritz

1. Introduction

Stokes' problem is the determination of the (regularized) geoid from gravity anomalies at sea level, the reference ellipsoid being approximated by a sphere. It is thus the boundary value problem of physical geodesy in its simplest form.

It may be formulated mathematically as the problem of determining a function T which satisfies Laplace's equation

$$\Delta T = 0 \tag{1}$$

outside a sphere $r = R$ and the boundary condition

$$\frac{\partial T}{\partial n} + \frac{2T}{R} = - \Delta g \tag{2}$$

on the sphere itself, Δg being a given function. T is the disturbing potential, Δg is the gravity anomaly, and the sphere $r = R$ represents the reference ellipsoid, R being the mean radius of the earth.

Since a linear combination of T and its normal derivative $\partial T/\partial n$ is given on the boundary $r = R$, Stokes' problem is a special case of the <u>third</u>

boundary value problem of potential theory (problem of heat conduction, Robin's problem).

The boundary condition in the general case may be written

$$\frac{\partial T}{\partial n} + \alpha T = - f; \tag{3}$$

the boundary itself is an arbitrary closed surface S, and α and f are functions given on S.

It is desirable to represent the solution T as a functional of the data f :

$$T(P) = \iint_S G(P,Q)\ f(Q)\ dS. \tag{4}$$

P is an arbitrary, but fixed point outside or on S ; Q is a variable point of the surface S ; the surface element dS is situated at Q.

$G(P,Q)$ is Green's function for the third boundary value problem; it is defined for arbitrary points P and Q by the following requirements:

(I) $$G(P,Q) = \frac{1}{4\pi l} + V(P,Q)\ , \tag{5}$$

where l is the distance PQ and $V(P,Q)$, for fixed P, is a regular harmonic function of Q everywhere outside S.

(II) On S it satisfies the boundary condition

$$\frac{\partial G}{\partial n} + \alpha\ G = 0. \tag{6}$$

To prove (4), we start from Green's identities (Moritz, 1964)

$$T = -\frac{1}{4\pi} \iint_S \left(\frac{1}{\ell} \frac{\partial T}{\partial n} - T \frac{\partial}{\partial n} \frac{1}{\ell}\right) dS \tag{7}$$

(third identity applied to the exterior of S) and

$$0 = -\iint_S \left(V \frac{\partial T}{\partial n} - T \frac{\partial V}{\partial n}\right) dS \tag{8}$$

(second identity applied to the exterior of S). By adding these two equations we obtain

$$T = -\iint_S \left(G \frac{\partial T}{\partial n} - T \frac{\partial G}{\partial n}\right) dS,$$

according to the definition (5) of Green's function. The derivative $\partial T/\partial n$ is expressed in terms of f and T by means of (3); we find

$$T = \iint_S G\, f\, dS + \iint_S \left(\frac{\partial G}{\partial n} + \alpha\, G\right) T\, dS. \tag{9}$$

If G is subject to the boundary condition (6), then the second term on the right-hand side of (9) vanishes and there remains (4).

It may be shown that Green's function is symmetric, i.e.,

$$G(Q,P) = G(P,Q) \; . \tag{10}$$

For an excellent introduction to boundary values problems and Green's functions see (Schiffer, 1956).

2. The Spherical Problem

We shall now determine Green's function for the case that the surface S is a shpere $r = R$. Then the normal derivative $\partial/\partial n$ is identical with the radial derivative $\partial/\partial r$.

The boundary condition (6) may then be written

$$\frac{\partial G}{\partial r} + \frac{K}{R} G = 0, \tag{11}$$

where we have put $\alpha = K/R$. (For Stokes' problem we have $\alpha = 2/R$ and hence

$$k = 2). \tag{12}$$

By means of (5) we obtain from (11) a boundary condition for V,

$$\frac{\partial V}{\partial r} + \frac{k}{R} V = - \frac{1}{4\pi} \left(\frac{\partial}{\partial r} \frac{1}{\ell} + \frac{k}{R} \frac{1}{\ell}\right) , \tag{13}$$

from which V can be determined.

For this purpose we develop the reciprocal distance $1/\ell$ in the usual series of zonal spherical harmonics:

$$\frac{1}{\ell} = \sum_{n=0}^{\infty} \frac{r^n}{r_o^{n+1}} P_n (\cos \psi), \tag{14}$$

which is valid for $r < r_o$; the notations are those of Fig. 1. The functions

$P_n(\cos\psi)$ are Legendre's polynomials. The radial derivative of this expression is

$$\frac{\partial}{\partial r}\frac{1}{\ell} = \sum_{n=0}^{\infty} \frac{nr^{n-1}}{r_o^{n+1}} P_n \quad . \tag{15}$$

On inserting (15) and (14), with $r = R$, the boundary condition (13) becomes

$$\frac{\partial V}{\partial r} + \frac{k}{R} V = -\frac{1}{4\pi} \sum_{n=0}^{\infty} (n+k) \frac{R^{n-1}}{r_o^{n+1}} P_n(\cos\psi). \tag{16}$$

As V is a regular harmonic function outside the sphere S, it may be developed in a series of spherical harmonics. Since (16) contains only zonal harmonics, the series for V will also consist exclusively of zonal harmonics and may therefore be written

$$V = \sum_{n=0}^{\infty} A_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi). \tag{17}$$

The coefficients A_n are to be determined.

The radial derivative of this expression is

$$\frac{\partial V}{\partial r} = -\sum_{n=0}^{\infty} (n+1)\, A_n \frac{R^{n+1}}{r^{n+2}} P_n(\cos\psi). \tag{18}$$

We set $r = R$ in (17) and (18) and insert these equations in the boundary condition (16). We obtain

$$-\frac{1}{R}\sum_{n=0}^{\infty} (n+1-k)\, A_n\, P_n\, (\cos\psi) = -\frac{1}{4\pi} \sum_{n=0}^{\infty} (n+k) \frac{R^{n-1}}{r_o^{n+1}} P_n\, (\cos\psi). \tag{19}$$

Comparing the coefficients of P_n on either side of this equation yields

$$A_n = \frac{1}{4\pi} \frac{n+k}{n+1-k} \frac{R^n}{r_o^{n+1}} . \tag{20}$$

Hence (17) becomes

$$V = \frac{1}{4\pi R} \sum_{n=0}^{\infty} \frac{n+k}{n+1-k} \left(\frac{R^2}{r_o r}\right)^{n+1} P_n(\cos \psi). \tag{21}$$

It is now convenient to introduce a point P_1 lying on the radius vector of P at such a distance r_1 from the center of the sphere that

$$r_1 r_o = R^2 \tag{22}$$

or

$$r_1 = \frac{R^2}{r_o} , \; r_o = \frac{R^2}{r_1} \tag{22'}$$

(Fig. 1). P_1 is realted to P by <u>inversion in the sphere</u> $r = R$ (transformation by reciprocal radii; Kelvin transformation) (Kellogg, 1929, p. 231). Then (21) is considerably simplified, becoming

$$V = \frac{1}{4\pi R} \sum_{n=0}^{\infty} \frac{n+k}{n+1-k} \left(\frac{r_1}{r}\right)^{n+1} P_n(\cos \psi). \tag{23}$$

3. Eigenvalues

These formulas break down if k is one of the positive integers

$$1, 2, 3, \ldots \quad .$$

These values are called eigenvalues. Then for

$$n = k - 1 \tag{24}$$

we have

$$n+1-k = 0 \ , \tag{25}$$

so that the denominator of the $(k-1)$th term in (23) becomes zero. This is impossible.

We may trace this difficulty down to eq. (19), which cannot be satisfied if $n = k - 1$ because the $(k-1)$th term is zero on the left-hand side and non-zero on the right-hand side. From this we conclude that the boundary condition (6) cannot be strictly satisfied if $k = 1, 2, 3, \ldots$.

However, it is easy to modify (6) accordingly. We may write (19) in the form

$$-\frac{1}{R} \sum_{n=0}^{\infty}{}' (n+1-k) A_n P_n (\cos \psi) =$$

$$= -\frac{1}{4\pi} \sum_{n=0}^{\infty}{}' (n+k) \frac{R^{n-1}}{r_o^{n+1}} P_n (\cos \psi) - \frac{2k-1}{4\pi} \frac{R^{k-2}}{r_o^{K}} P_{k-1} (\cos \psi) \quad , \tag{26}$$

where the prime in Σ' indicates that the critical term with $n = k - 1$

is omitted from the sum. This boundary condition cannot be satisfied in the present form. We must add to the right-hand side of (26) the quantity

$$\frac{2k-1}{4\pi} \frac{R^{k-2}}{r_o^{\,k}} P_{k-1} (\cos \psi) , \tag{27}$$

which then cancels with the last term on this side, and hence we must add this quantity also to the right-hand side of the original equation (6) or (11). So we obtain

$$\frac{\partial G}{\partial r} + \frac{k}{R} G = \frac{2k-1}{4\pi R^2} \left(\frac{R}{r_o}\right)^k P_{k-1}(\cos \psi) \tag{28}$$

as boundary condition for G, if k happens to be a positive integer, i.e., an eigenvalue.

This new boundary condition (27) yields the same formula (20) for the coefficients A_n as before, except when $n = k - 1$. The coefficient A_{k-1}, however, is entirely arbitrary, because any value of it, being multiplied by zero, is compatible with (19) or (26) as modified by the addition of (27) on the right-hand side.

Hence (17) becomes

$$V = V' + \frac{A'_{k-1}}{4\pi R} \left(\frac{r_1}{r}\right)^k P_{k-1} (\cos \psi), \tag{29}$$

where we have put $A_{k-1} = A'_{k-1} r_1^{\,k}/4\pi R^{k+1}$ and

$$V' = \frac{1}{4\pi R} \sum_{n=0}^{\infty}{}' \frac{n+k}{n+1-k} \left(\frac{r_1}{r}\right)^{n+1} P_n (\cos \psi). \tag{30}$$

Instead of (4) we must now use (9), the second term on the right-hand side being non-zero because of (28). We obtain

$$T = \iint_S G\ f\ dS + \frac{2k-1}{4\pi R^2} \left(\frac{R}{r_o}\right)^k \iint_S T P_{k-1}(\cos\psi)\ dS. \quad (31)$$

Let the development of T in spherical harmonics be

$$T(r,\varphi,\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\varphi,\lambda), \quad (32)$$

where T_n is the surface spherical harmonic of degree n, and φ, λ are geocentric latitude and longitude. Then it is well known that T_n is expressed in terms of the values of T on the sphere S by

$$T_n = \frac{2n+1}{4\pi R^2} \iint_S T\ P_n\ (\cos\psi)\ dS. \quad (33)$$

Hence the second term on the right-hand side of (31) is nothing else than $(R/r_o)^k\ T_{k-1}$, so that this equation becomes simply

$$T = \iint_S G\ f\ dS + \left(\frac{R}{r_o}\right)^k T_{k-1}, \quad (34)$$

which is the necessary modification of (4).

From this equation we draw an important conclusion. Eq. (34) may be written, considering (32),

$$\iint_S G\ f\ dS = T - \left(\frac{R}{r_o}\right)^k T_{k-1} = \sum_{n=0}^{\infty} \left(\frac{R}{r_o}\right)^{n+1} T_n - \left(\frac{R}{r_o}\right)^k T_{k-1}$$

or

$$\iint_S G\ f\ dS = \sum_{n=0}^{\infty}{}' \left(\frac{R}{r_o}\right)^{n+1} T_n \quad ; \tag{35}$$

that is, the spherical harmonic of degree $n = k - 1$ is missing in this equation. Since $G = 1/4\pi\ell + V$ does in general contain this term, according to (29), this is only possible if <u>f does not contain a harmonic of degree k - 1</u>, that is, if

$$\iint_S f\ P_{k-1}\ (\cos\ \psi)\ dS = 0. \tag{36}$$

This condition must of necessity be satisfied by the boundary values f if k is an eigenvalue.

However, if this condition is satisfied, then (36) shows that we may attribute to the constant A'_{k-1} in (29) any value whatever without changing the result (34).

4. Stokes' Problem

According to (2) or (12), Stokes' problem corresponds to $k = 2$, which is indeed an eigenvalue. We put $f = \Delta g$, so that (34) becomes

$$T(r_o, \varphi, \lambda) = \iint_S G\ \Delta g\ dS + \left(\frac{R}{r_o}\right)^2 T_1\ (\varphi, \lambda). \tag{37}$$

The first degree harmonic $T_1(\varphi,\lambda)$ is arbitrary and is usually omitted.

The condition (36) is now

$$\iint_S \Delta g \; P_1 \; (\cos \psi) \; dS = 0 \; , \tag{38}$$

that is, Δg <u>must contain no first degree harmonic</u>.

Green's function becomes

$$G = \frac{1}{4\pi \ell} + V = \frac{1}{4\pi \ell} + V' + \frac{A'_1}{4\pi R} \left(\frac{r_1}{r}\right)^2 P_1 \; (\cos \psi)$$

or

$$G = \frac{1}{4\pi \ell} + \frac{1}{4\pi R} \sum_{n=0}^{\infty}{}' \; \frac{n+2}{n-1} \left(\frac{r_1}{r}\right)^{n+1} P_n \; (\cos \psi) + \frac{A_1}{4\pi R} \left(\frac{r_1}{r}\right)^2 P_1 \; (\cos \psi) \; , \tag{39}$$

according to (29) and (30).

The series in (39) may be summed to give a closed expression. We have

$$\begin{aligned} &\sum_{n=0}^{\infty}{}' \; \frac{n+2}{n-1} \left(\frac{r_1}{r}\right)^{n+1} P_n \; (\cos \psi) = \\ &= \sum_{n=0}^{\infty}{}' \; \frac{(n-1)+3}{n-1} \left(\frac{r_1}{r}\right)^{n+1} P_n = \\ &= \sum_{n=0}^{\infty}{}' \left(\frac{r_1}{r}\right)^{n+1} P_n + 3 \sum_{n=0}^{\infty}{}' \frac{1}{n-1} \left(\frac{r_1}{r}\right)^{n+1} P_n \; . \end{aligned} \tag{40}$$

Denoting the distance QP_1 by ℓ_1 (Fig. 1), we have corresponding to (14)

$$\frac{1}{\ell_1} = \sum_{n=0}^{\infty} \frac{r_1^{\,n}}{r^{n+1}} P_n \; (\text{os} \; \psi) \; , \tag{41}$$

because $r > r_1$ outside the sphere S. Since Σ' omits the term with $n = k - 1 = 1$, we have

$$\frac{r_1}{\ell_1} = \sum_{n=0}^{\infty} \left(\frac{r_1}{r}\right)^{n+1} P_n = \sum_{n=0}^{\infty}{}' \left(\frac{r_1}{r}\right)^{n+1} P_n + \left(\frac{r_1}{r}\right)^2 \; P_1 \; .$$

As

$$P_1 (\cos \psi) = \cos \psi , \tag{42}$$

we find

$$\sum_{n=0}^{\infty}{}' \left(\frac{r_1}{r}\right)^{n+1} P_n = \frac{r_1}{\ell_1} - \left(\frac{r_1}{r}\right)^2 \cos \psi , \tag{43}$$

where

$$\ell_1 = \sqrt{r^2 + r_1^2 - 2r_1 r \cos \psi} \quad . \tag{44}$$

Putting

$$\frac{r_1}{r} = u \tag{45}$$

we may write (43) as

$$\sum_{n=0}^{\infty}{}' u^{n+1} P_n = \frac{u}{\sqrt{1 + u^2 - 2u \cos \psi}} - u^2 \cos \psi .$$

On dividing by u^3 this becomes

$$\sum_{n=0}^{\infty}{}' u^{n-2} P_n = \frac{1}{u^2\sqrt{1+u^2-2u \cos \psi}} - \frac{1}{u} \cos \psi ,$$

and on integrating with respect to u we find

$$\begin{aligned} \sum_{n=0}^{\infty}{}' \frac{u^{n-1}}{n-1} P_n &= - \cos \psi - \frac{1}{u}\sqrt{1+u^2-2u \cos \psi} - \\ &- \cos \psi \, \ell n \, \frac{1}{2} \left(1-u \cos \psi +\sqrt{1 + u^2 - 2u \cos \psi}\right) \qquad (46) \\ &= - \cos \psi - \frac{\ell_1}{r_1} - \cos \psi \, \ell n \, \frac{r-r_1\cos\psi+\ell_1}{2r} \end{aligned}$$

so that

$$\sum_{n=0}^{\infty}{}' \frac{1}{n-1}\left(\frac{r_1}{r}\right)^{n+1} P_n = -\frac{r_1^{\,2}}{r^2}\cos\psi - \frac{\ell_1 r_1}{r^2} - \frac{r_1^{\,2}}{r^2}\cos\psi \, \ln \frac{r - r_1\cos\psi + \ell_1}{2r}. \quad (47)$$

Now (43) and (47) are inserted in (40), and (40) is inserted in (39). The result is

$$4\pi G = \frac{1}{\ell} + \frac{r_1}{R\ell_1} - (4 - A'_1)\frac{r_1^2}{Rr^2}\cos\psi - \frac{3\ell_1 r_1}{Rr^2} - \frac{3r_1^{\,2}}{Rr^2}\cos\psi \, \ln \frac{r - r_1\cos\psi + \ell_1}{2r} \quad (48)$$

This is the general expression for Green's function for arbitrary points P and Q (arbitrary r and r_o ; note that $r_1 = R^2/r_o$!).

In (37) we need only the values G_S of G for Q on S, that is, for $r = R$. If Q lies on S (Fig. 2), then

$$\frac{\ell_1}{\ell} = \frac{r_1}{R} = \frac{R}{r_o} \quad ,$$

according to a well-known theorem of elementary geometry (circle of Apollonius), to that

$$\frac{r_1}{R\ell_1} = \frac{1}{\ell} \quad \text{if } Q \text{ on } S.$$

On taking this relation into account and putting $r = R$ and $r_1 = R^2/r_o$, eq. (48) becomes

$$4\pi G_S = \frac{2}{\ell} - (4 - A'_1)\frac{R}{r_o^{\,2}}\cos\psi - \frac{3\ell}{r_o^{\,2}} - \frac{3R}{r_o^2}\cos\psi \, \ln \frac{r_o - R\cos\psi + \ell}{2r_o} . \quad (49)$$

The corresponding development of G_S in spherical harmonics is found by inserting (14) in (39) and putting $r = R$ and $r_1/r = R/r_o$:

$$4\pi G_S = \sum_{n=0}^{\infty}{}' \frac{R^n}{r_o^{n+1}} P_n + \frac{R}{r_o^2} \cos\psi + \sum_{n=0}^{\infty}{}' \frac{n+2}{n-1} \frac{R^n}{r_o^{n+1}} P_n + A'_1 \frac{R}{r_o^2} \cos\psi$$

or

$$G_S = \frac{1}{4\pi} \left[\sum_{n=0}^{\infty}{}' \frac{2n+1}{n-1} \frac{R^n}{r_o^{n+1}} P_n + (1+A'_1) \frac{R}{r_o^2} \cos\psi \right] \tag{50}$$

Since the constant A'_1 is arbitrary, we may put

$$A'_1 = -1 \tag{51}$$

This causes the second term on the right-hand side to vanish, and we obtain

$$G_S = \frac{1}{4\pi} \sum_{n=0}^{\infty}{}' \frac{2n+1}{n-1} \frac{R^n}{r_o^{n+1}} P_n(\cos\psi). \tag{50'}$$

By putting $A'_1 = -1$ we have achieved that $G_S(50')$ <u>contains no first degree harmonic</u>.

The sum Σ' contains no term $n = 1$; furthermore we may split off the term $n = 0$ with $P_o = 1$. Then we obtain

$$G_S = \frac{1}{4\pi} \left[S(r_o, \psi) - \frac{1}{r_o} \right], \tag{52}$$

where

$$S(r_o, \psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \frac{R^n}{r_o^{n+1}} P_n(\cos\psi) \tag{53}$$

is Stokes' function as generalized to apply to a point outside the earth ($r_o > R$). For $r_o = R$ this becomes $1/R$ times the ordinary Stokes' function.

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n (\cos \psi) = \operatorname{cosec} \frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi \, ln \left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right). \tag{53'}$$

If we put $A_1 = -1$ in (49), we have

$$4\pi G_S = \frac{2}{l} - \frac{5R}{r_o^2} \cos \psi - \frac{3l}{r_o^2} - \frac{3R}{r_o^2} \cos \psi \, ln \frac{r_o - R\cos\psi + l}{2R_o} \quad ; \tag{54}$$

and adding $1/r_o$, according to (52), gives the usual closed expression for Stokes' function (53).

Stokes' formula is usually given a somewhat simpler derivation, which directly aims at obtaining the values G_S for $r = R$ (or Q on S) only, disregarding the function $G(P,Q)$ for Q not on S. This is sufficient for solving the spherical problem. However, once we want to go beyond the sphere, as in solving the geodetic boundary value problem for the physical surface of the earth, we must know Green's function G for points Q above (or below) S as well; see the two concluding sections of this report.

5. Planar Approximation

For certain purposes it is convenient and sufficiently accurate to replace the sphere by its tangential plane, that is, letting formally $R \to \infty$.

The second term in the boundary condition (11) then goes to zero, and there remains

$$\frac{\partial G}{\partial z} = 0, \tag{55}$$

if we take the plane S as xy-plane and its normal as the z- direction of a spatial rectangular coordinate system. This holds for an arbitrary finite k, so that the third boundary value problem becomes identical with the second boundary value problem of potential theory, or Neumann's problem.

Letting $R \to \infty$, $r_1/R \to 1$ in (48), we obtain

$$G(P,Q) = \frac{1}{4\pi}\left(\frac{1}{\ell} + \frac{1}{\ell_1}\right). \tag{56}$$

This is Green's function for Stokes' or Neumann's problem for the plane. The geometrical relationship is exhibited by Fig. 3; P_1 is the mirror image of P with respect to the plane S.

Since

$$\ell = \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}$$

$$\ell_1 = \sqrt{(x - x_o) + (y - y_o) + (z + z_o)^2} \; ,$$

the partial derivative of G with respect to z becomes

$$\frac{\partial G}{\partial z} = \frac{1}{4\pi} \left(- \frac{z-z_o}{\ell^3} - \frac{z+z_o}{\ell_1{}^3} \right) .$$

On S itself we have $z = 0$, $\ell_1 = \ell$, so that

$$\left.\frac{\partial G}{\partial z}\right)_S = 0 ,$$

in agreement with the boundary condition (55).

For later reference we shall need the second derivative. It is

$$\frac{\partial^2 G}{\partial z^2} = \frac{1}{4\pi} \left[- \frac{1}{\ell^3} - \frac{1}{\ell_1^3} + 3 \frac{(z-z_o)^2}{\ell^5} + 3 \frac{(z+z_o)^2}{\ell_1{}^5} \right] , \tag{57}$$

and on S

$$\left.\frac{\partial^2 G}{\partial z^2}\right)_S = - \frac{1}{2\pi D^3} + \frac{3z_o^2}{2\pi D^5} , \tag{57'}$$

where

$$D = \sqrt{(x - x_o)^2 + (y - y_o)^2 + z_o^2} \tag{58}$$

is equal to ℓ and ℓ_1 for $z = 0$.

6. Variation of Green's Function with the Boundary

We shall now derive a formula expressing the change of Green's function if the boundary surface is slightly deformed. This problem is of immediate geodetic significance because the earth's physical surface may be considered as an ellipsoid deformed by the irregularities of topography.

Let us for the moment return to the introductory section and consider the general third boundary value problem for an arbitrary surface S, the boundary condition being given by (3), and Green's function being defined by (5) and (6).

Let now the surface S being deformed into a neighboring surface $\bar{S}$, so that the distance between S and $\bar{S}$, measured along the normal n to S, is δn (Fig. 4). We set up a new boundary value problem for the surface $\bar{S}$, so that the boundary condition (3),

$$\frac{\partial T}{\partial n} + \alpha T = - f, \tag{59}$$

originally referring to S, now holds on $\bar{S}$, with n still being the normal to S, not to $\bar{S}$. Denote Green's function for this new boundary value problem by $\bar{G}$.

We introduce an abbreviation for the left-hand side of (59):

$$[T] \quad \frac{\partial T}{\partial n} + \alpha T. \tag{60}$$

Using this symbol we may say that the original Green's function G satisfies the boundary condition

$$[G] = 0 \text{ on } S, \text{ or } \quad [G]_S = 0, \tag{61 a}$$

whereas $\bar{G}$ satisfies

$$[\bar{G}] = 0 \text{ on } \bar{S}, \text{ or } \quad [\bar{G}]_{\bar{S}} = 0. \tag{61 b}$$

For small δn we have

$$[\bar{G}]_S = [\bar{G}]_{\bar{S}} - \frac{\partial[\bar{G}]}{\partial n}\,\delta n + \text{higher order terms}$$

or, up to linear terms,

$$[\bar{G}]_S = -\frac{\partial[\bar{G}]}{\partial n}\,\delta n$$

according to (61 b). We put

$$\delta G = \bar{G} - G\,, \tag{62}$$

so that

$$[\delta G]_S = [\bar{G}]_S - [G]_S = -\frac{\partial[\bar{G}]}{\partial n}\,\delta n - 0 = -\frac{\partial[\bar{G}]}{\partial n}\,\delta n\,.$$

As δG is of the order of δn, this equation differs from

$$[\delta G]_S = -\frac{\partial[G]}{\partial n}\,\delta n \tag{63}$$

only by higher order terms.

Since G and $\bar{G}$ both have the same singularity $1/4\pi\ell$ at P, their difference δG is everywhere regular outside S (or outside $\bar{S}$, which is the same for our present objective). Hence $\delta G(P, Q)$, for P fixed, is a harmonic function of Q. Its boundary values on S satisfy the relation (63).

The important point now is that therefore the values of δG outside (or on) S may be found by solving the third boundary value problem for the boundary condition (63). We write (4) in the form

$$T(Q') = - \iint_S G(Q',Q) \; [T(Q)]_S \, dS_Q \quad , \tag{64}$$

because $f = - [T]$, according to (3) and (60). For the sake of clearness, we have written dS_Q instead of dS. Since T is an arbitrary harmonic function, we are permitted to put

$$T(Q) = \delta G(P, Q) \; ,$$

where P is fixed. Hence we obtain

$$\delta G(P, Q') = - \iint_S G(Q', Q) \; [\delta G(P,Q)]_S \, dS_Q$$

and finally, by (63),

$$\delta G(P, Q') = \iint_S G(Q', Q) \frac{\partial[G(P, Q)]}{\partial n_Q} \; \delta n(Q) \, dS_Q . \tag{65 a}$$

Since G, and therefore δG, is symmetric in its arguments, $\delta G(P, Q') = \delta G(Q', P)$, we may interchange P and Q' in the right-hand side of (65 a), obtaining

$$\delta G(P,Q') = \iint_S G(P,Q) \frac{\partial[G(Q',Q)]}{\partial n_Q} \delta n(Q) \, dS_Q \ . \tag{65 b}$$

The equation (65a), or its alternative (65 b), expresses the variation of Green's function for the third boundary value problem if the surface S is subjected to a small variation δn along its normal. As usual in such variational equations, we are dealing with small δn and δG, so that linear expressions are sufficient. These formulas, which might be new, bear some analogy to Hadamard's variational equation for Green's function of the first kind, which solves Dirichlet's problem (Schiffer, 1956, p. 136).

In geodetic applications (Molodensky's problem), S is the geoid, approximated by an ellipsoid or even by a shpere, and $\bar{S}$ is the physical surface of the earth, so that

$$\delta n = h \tag{66}$$

is the elevation above sea level. It is appropriate to use the boundary condition (2) or (3) on the earth's surface $\bar{S}$ as well as at sea level (see next section), so that n is the plumb line (normal to S) rather than the normal to the topographic surface $\bar{S}$. Hence we have

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial h} \ . \tag{67}$$

Using the normal to $\bar{S}$ would result in a more difficult boundary condition, which in addition depends on the deflection of the vertical (Moritz, 1964, p. 21).

The equations (65 a,b) hold for arbitrary α in (59), even in the eigenvalue case. If α corresponds to an eigenvalue, as in Stokes' problem, then (63) still holds, although the right-hand sides of (61 a,b) are different from zero. The reason is that then the right-hand side of (61 b) is practically equal to the right-hand side of (61 a), as represented by (28). This term therefore cancels out in (63).

It might be noted that the boundary condition (59), if applied to S, expressed a third boundary value problem in the strict sense of potential theory, but not so if applied to $\bar{S}$, because n is still normal to S, which is different from the normal to $\bar{S}$. We must bear this in mind if we call (65 a) or (65 b) "the variation of Green's function for the third boundary value problem." But as we have just seen, these variational formulas are precisely those which are relevant for geodetic applications.

7. Application to Molodensky's Boundary Value Problem.

Molodensky's boundary value problem is the determination of the physical surface of the earth $\bar{S}$ from gravity anomalies referred to this surface (Molodenskii et. al., 1962, chapter V; Moritz, 1964, section 5). The boundary condition at $\bar{S}$ is

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T = - \Delta g \quad , \tag{68}$$

where γ is normal gravity. As a shperical approximation (the reference ellipsoid being approximated by a sphere) this can be written

$$\frac{\partial T}{\partial h} + \frac{2}{r} T = - \Delta g \ . \tag{68'}$$

Since the radius vector r may be replaced by a mean radius R of the earth of, say, 6371 km, this becomes

$$\frac{\partial T}{\partial h} + \frac{2}{R} T = - \Delta g \ , \text{ or } [T] = - \Delta g, \tag{69}$$

which is identical with (2).

We are now going to apply the developments of the preceding section to Molodensky's problem. As we have mentioned, S is a shpere representing the reference ellipsoid; Green's function G corresponding to S is expressed by (48) or (54). Green's function $\overline{G}$ for $\overline{S}$, that is, for Molodensky's problem, is given by

$$\overline{G} = G + \delta G \ ,$$

where δG satisfies (65 a) or (65 b), with $\delta n = h$.

Forming

$$\frac{\partial [G]}{\partial n} = \frac{\partial [G]}{\partial h} = \frac{\partial [G]}{\partial r}$$

by differentiation of (48) offers no principal difficulties, but leads to inconvenient formulas. Fortunately, closer investigation show that the planar approximation of section 5 may be applied for the small term δG.

Then

$$[G] = \lim_{R\to\infty} \left(\frac{\partial G}{\partial h} + \frac{2}{R} G\right) = \frac{\partial G}{\partial h} = \frac{\partial G}{\partial z} \quad ,$$

$$\frac{\partial [G]}{\partial h} = \frac{\partial}{\partial z} \left(\frac{\partial G}{\partial z}\right) = \frac{\partial^2 G}{\partial z^2} \quad ,$$

which is given by (57). Hence we have

$$\frac{\partial [G]}{\partial h} = \frac{1}{4\pi} \left[-\frac{1}{\ell^3} - \frac{1}{\ell_1^3} + 3\frac{(H-h)^2}{\ell^5} + 3\frac{(H+h)^2}{\ell_1^5} \right. \quad . \tag{70}$$

Here we may take H and h as heights of P and Q above sea level (Fig. 5), put

$$r = R + h, \; r_o = R + H \tag{71}$$

and compute ℓ and ℓ_1 by means of the usual spherical formulas

$$\ell = \sqrt{r^2 + r_o^2 - 2r_o \, r\cos\psi,} \tag{72a}$$

$$\ell_1 = \sqrt{r^2 + r_1^2 - 2r_1 \, r\cos\psi,} \tag{72b}$$

(see Fig. 1).

Since (70) is multiplied by h in (65a, b) and since we are satisfied with terms linear in the topographic elevation h, we may simplify (70) as

$$\frac{\partial [G]}{\partial h} = \frac{1}{4\pi} \left[-\frac{1}{\ell^3} - \frac{1}{\ell_1^3} + 3H^2 \left(\frac{1}{\ell^5} + \frac{1}{\ell_1^5}\right) \right] \quad . \tag{73}$$

The height H of P, which may be arbitrarily large, cannot be neglected unless P lies on the topographical surface $\bar{S}$ itself. Then we have

$$\left.\frac{\partial[G]}{\partial h}\right)_{P \text{ on } \bar{S}} = -\frac{1}{2\pi \ell_o^3} \quad , \tag{73'}$$

because it is permitted in this case to put $\ell = \ell_1 = \ell_o$ where

$$\ell_o = 2R \sin \frac{\psi}{2} . \tag{74}$$

Hence

$$\frac{\partial[G(P,Q)]}{\partial n_Q} = \frac{\partial[G]}{\partial h}$$

is given by relatively simply formulas. Then δG may be obtained from (65 a) or (65 b). However, if P lies on $\bar{S}$ in (65 a), or if Q' lies on $\bar{S}$ in (65 b), then (73') is used, which becomes strongly infinite as $Q \to P$ or $\ell_o \to 0$. Hence (65 a) cannot be used directly but must be modified .

Again we shall use the planar approximation. It is evidently arbitrary from which level the topographic elevation h is reckoned. Instead of sea level we may also take the level of P_o (Fig. 5). Then h(Q) is replaced by h(Q) - h(P), and (65 a) becomes

$$\delta G(P,Q') = \iint_S G(Q',Q) \frac{\partial[G(P,Q)]}{\partial h(Q)} \{h(Q) - h(P)\} \, dS. \tag{75a}$$

Similarly (65b), on replacing h(Q) by h(Q) - h(Q'), goes over into

$$\delta G(P,Q') = \iint_S G(P,Q) \frac{\partial[G(Q',Q)]}{\partial h(Q)} \{h(Q) - h(Q')\} \, dS. \tag{75b}$$

Now the singularity of (73') for $\ell_o = 0$ has no effect. Hence (65 a) may be applied if P lies above $\bar{S}$, that is, if

$$H > h(P) ;$$

(65 b) is applicable if Q' lies above $\bar{S}$. On the other hand, (75a,b) may be applied in any case; note that h(P) is the height of the topography below P, which is not in general equal to the height H of P above sea level (Fig. 5).

Hence the procedure is as follows. After the elevation h has been taken from a topographical map, the correction δG is computed from one of the formulas (65 a,b) or (75 a,b). Then Green's function for determining the physical surface of the earth is given by

$$\bar{G} = G + \delta G \quad ,$$

where $G = S(\psi)/4\pi R$, Stokes' function being defined by (53'). All this is completely independent of gravity, so that it can be done beforehand, prior to any use of gravity data. Then, if the gravity anomaly Δg is also given around the whole earth, the disturbing potential on or outside the earth can be obtained from

$$T(P) = \iint_{\bar{S}} \bar{G}\ (P,Q) \quad \Delta g(Q)\ d\bar{S}\ . \tag{76}$$

To the approximation applied so far, we have $d\bar{S} = dS$. Hence we may in this formula replace the integration over the earth's surface $\bar{S}$ by an integration over the reference sphere S, obtaining

$$T(P) = \iint_{S} \bar{G}(P,Q)\ \Delta g(Q)\ dS, \tag{77}$$

which, of course, is much easier to use.

Finally we note that Green's function corresponds to the inverted matrix of a system of linear equations. The determination of $\bar{G}$ prior to using the gravity data is the analogue of the <u>indeterminate solution</u> of the system of linear equations. The general concept comprising both inverted matrix and Green's function is denoted by the term <u>resolvent</u>. Thus Stokes' function is the resolvent for Stokes' problem and, to a certain approximation (zero degree in h) also for Molodensky's problem. The function $\bar{G}$ considered in this section is an approximation of first degree in h; so is the function F_c of (Pellinen, 1964, p. 328), which is a particular case obtained in a different way. The resolvents of (Bjerhammar, 1959) and (Hirvonen, 1960) are lying between: theoretically, they are zero degree approximations (Molodensky et. al., 1963); practically, they may give more accurate results than Stokes' formula (Bjerhammar, 1964, p. 11).

8. Two Consequences

Now we shall develop two consequences of (75 a,b), considering P, Q, and Q' as points of the earth's surface $\bar{S}$. We may write (77) in the form

$$T(P) = T_o(P) + \delta T(P), \tag{78}$$

where

$$T_o(P) = \iint_S G(P,Q)\ \Delta g(Q)\ dS \quad , \tag{79}$$

with $G(P,Q) = S(\psi)/4\pi R$, and

$$\delta\, T(P) = \iint_S \delta G(P,Q)\, \Delta g(Q)\, dS. \tag{80}$$

The last equation can be transcribed into the form

$$\delta T(P) = \iint_{S'} \delta G(P,Q')\, \Delta g(Q')\, dS' \quad , \tag{80'}$$

where dS' is dS at Q' and S' denotes the sphere S as the locus of points Q'.

First we use (75 b). Insertion in (80') results in

$$\delta T(P) = \iint_{S'} \iint_S G(P,Q) \frac{\partial[G(Q',Q)]}{\partial h(Q)} \{h(Q) - h(Q')\}\, \Delta g(Q')\, dSdS'.$$

By performing the integration with respect to Q' first we obtain

$$G_1(Q) = \iint_{S'} \frac{\partial[G(Q',Q)]}{\partial h(Q)} \{h(Q) - h(Q')\}\, \Delta g(Q')\, dS' \tag{81}$$

and

$$\delta T(P) = \iint_S G(P,Q)\, G_1(Q)\, dS, \tag{82}$$

or, adding this to (79),

$$T(P) = \iint_S G(P,Q)\, \{\Delta g(Q) + G_1(Q)\}\, dS. \tag{83}$$

We may interpret this in the sense that a correction term G_1 (81) must be added to Δg, in order that T can be computed by means of the simple Stokes' formula. On inserting (73') into (81) we obtain the well-known formula

$$G_1(Q) = \frac{1}{2\pi} \iint_{S'} \frac{h(Q')-h(Q)}{\ell_o^3} \Delta g(Q')\, dS', \qquad (84)$$

where ℓ_o is defined by (74), ψ being the spherical distance between Q and Q'. In more familiar notation, putting $dS' = R^2 d\sigma$, we have

$$G_1(Q) = \frac{1}{16\pi R} \iint_\sigma \frac{h-h_Q}{\sin^3 \frac{\psi}{2}} \Delta g\, d\sigma . \qquad (84')$$

This formula is due to Molodensky (Molodenskii et. al., 1962, p. 122; it is permissible to put $\chi_o = \Delta g/2\pi$), but the present derivation is different. The notation G_1 has been taken over from Molodensky; it has of course nothing to do with Green's function G.

Now we shall develop a second consequence, using (75 a). Insertion in (80') results in

$$\delta T(P) = \iint_{S'} \iint_{S} G(Q',Q) \frac{\partial[G(P,Q)]}{\partial h(Q)} \{h(Q) - h(P)\} \Delta g(Q')\, dS dS' \quad .$$

By performing the integration with respect to Q' first and taking (79) into account we obtain

$$T_o(Q) = \iint_{S'} G(Q',Q) \Delta g(Q')\, dS'$$

and

$$\delta T(P) = \iint_S T_o(Q) \frac{\partial[G(P,Q)]}{\partial h(Q)} \{h(Q) - h(P)\} \, dS.$$

On inserting (73') this becomes

$$\delta T(P) = -\frac{1}{2\pi} \iint_S \frac{h(Q)-h(P)}{\ell_o^{\,3}} T_o(Q) \, dS \tag{85}$$

or, in a more familiar notation,

$$T(P) = -\frac{1}{16\pi R} \iint_\sigma \frac{h-h_P}{\sin^3\frac{\psi}{2}} T_o \, d\sigma, \tag{85'}$$

where now ψ is the spherical distance between P and Q. These equations, which are analogous to (84) and (84'), express the correction δT in terms of the disturbing potential T itself. T_o is the approximation for T that results from applying the simple Stokes' formula. Equation (85) might be new, as well as (65 a,b).

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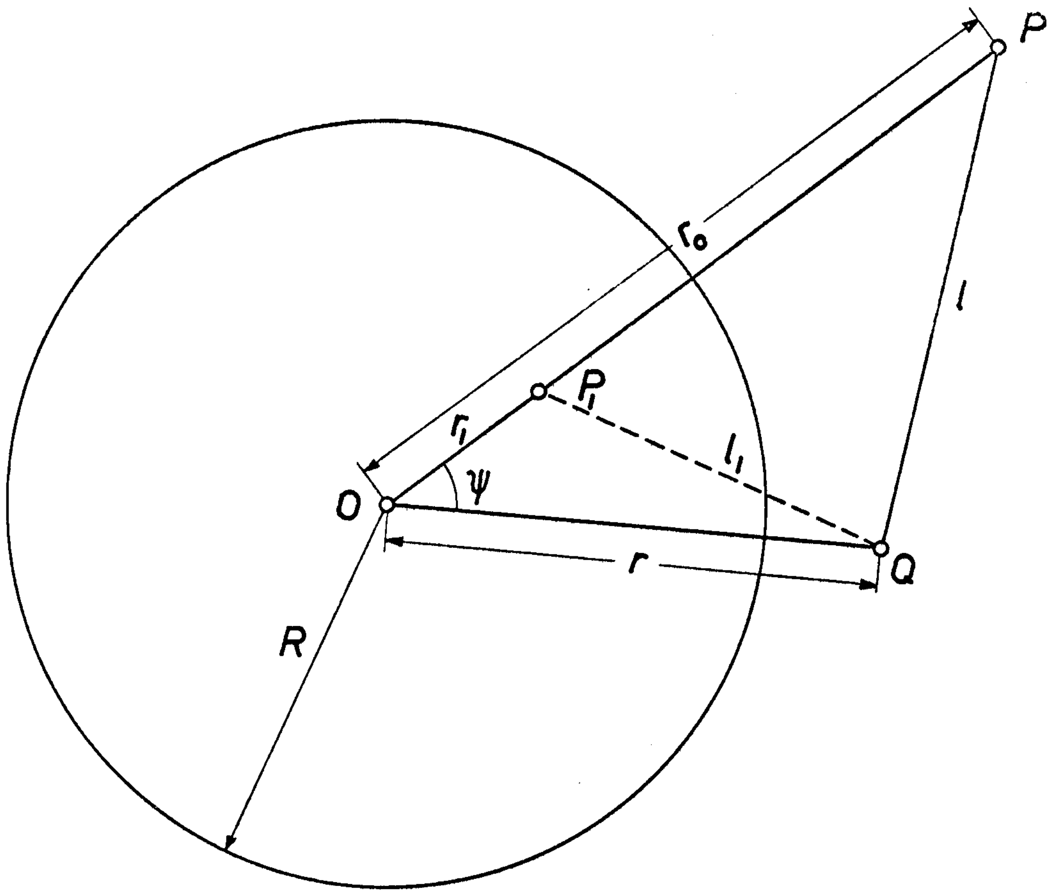

Figure 1

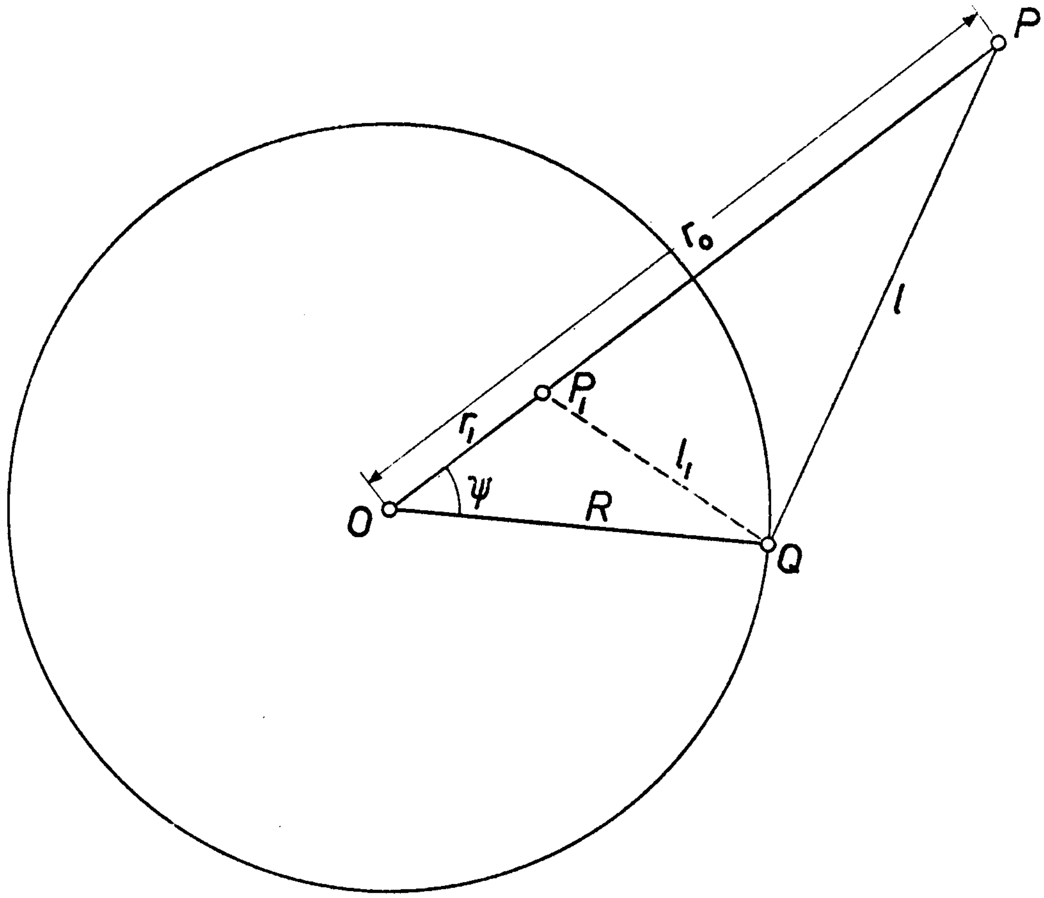

Figure 2

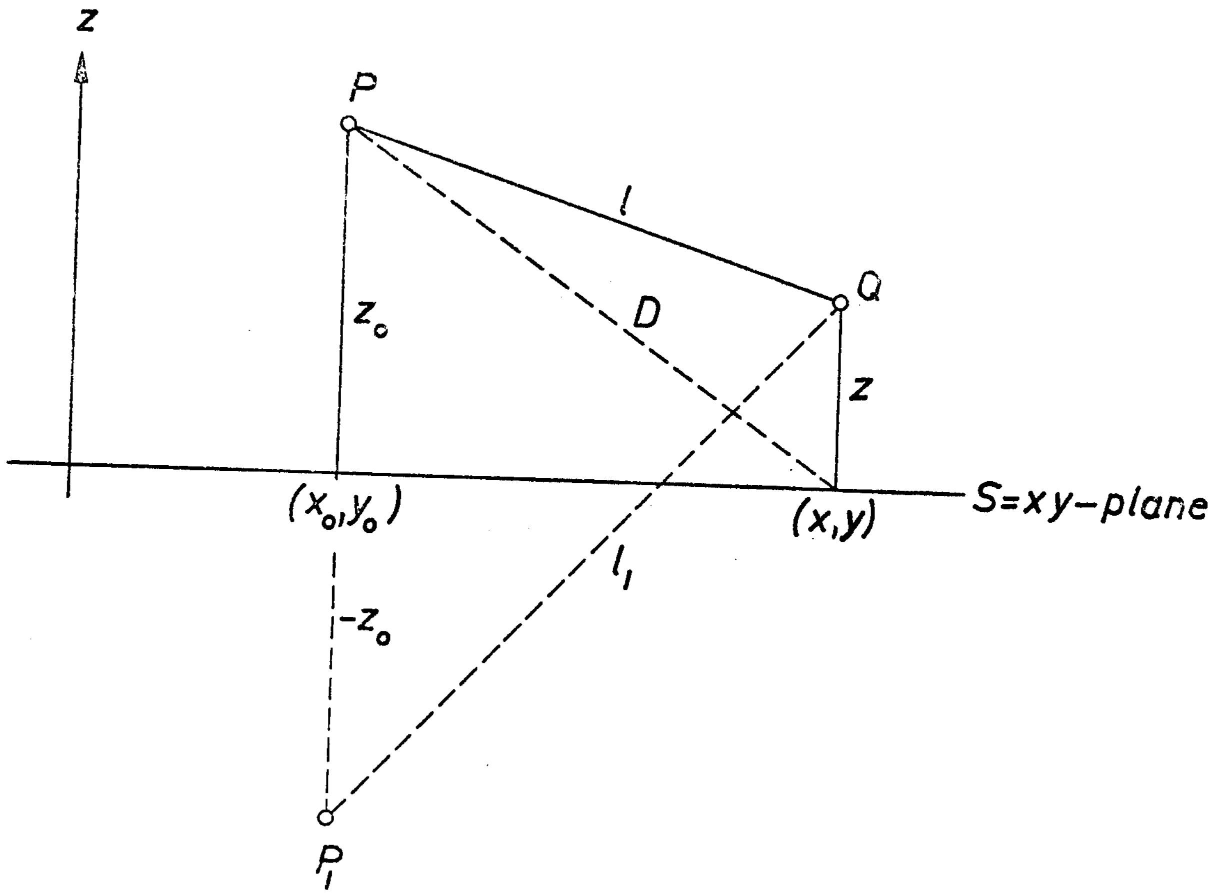

Figure 3

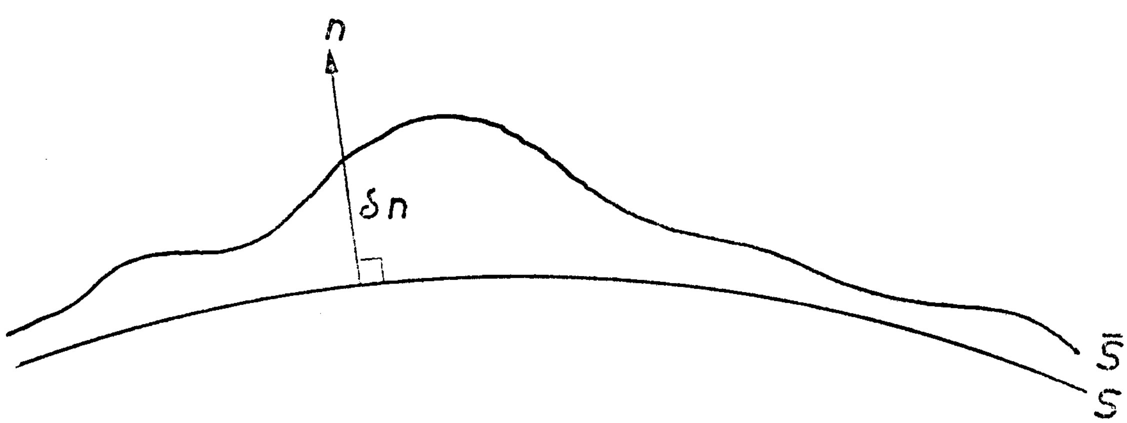

Figure 4

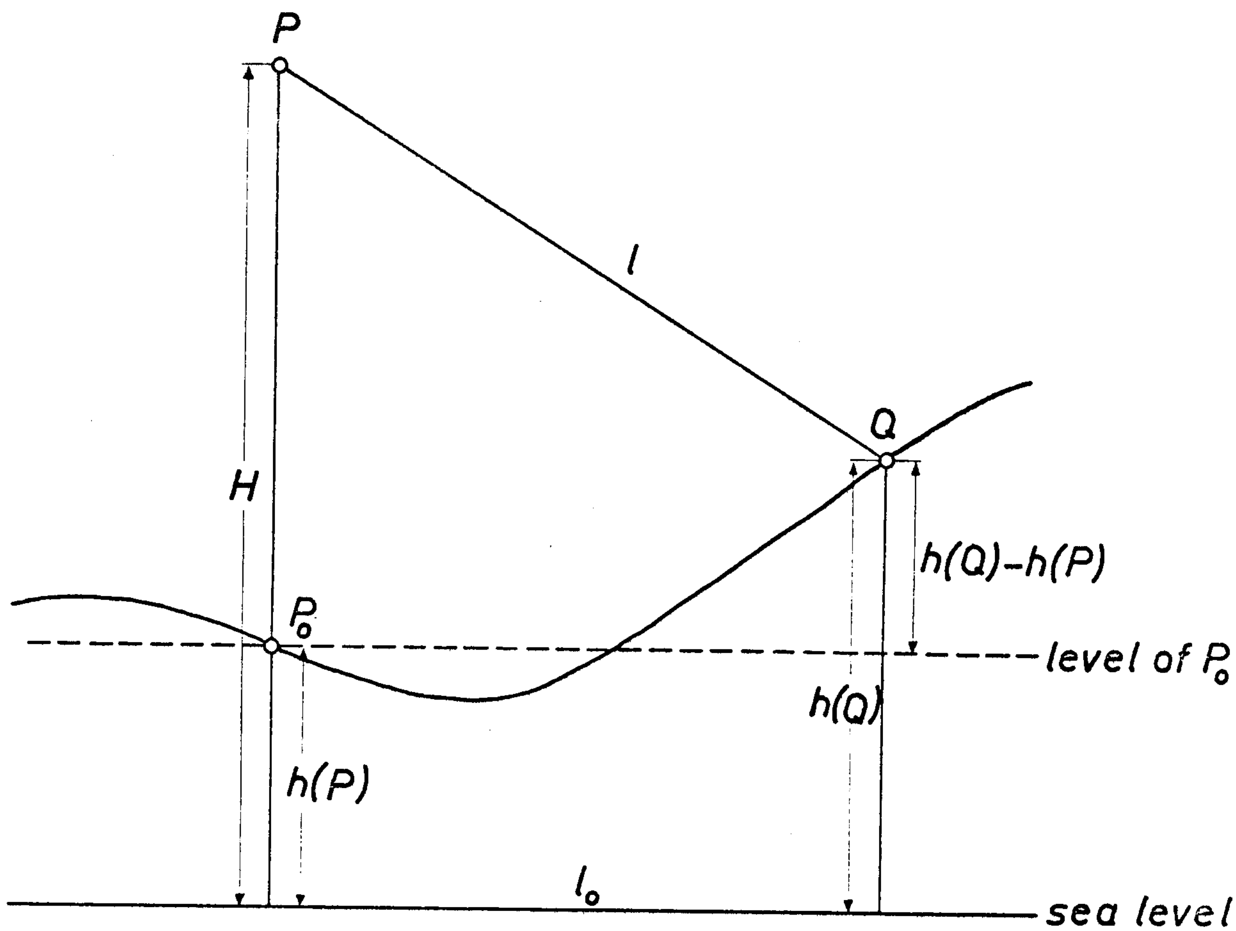

Figure 5

THE COMPUTATION OF THE EXTERNAL GRAVITY FIELD

AND THE GEODETIC BOUNDARY VALUE PROBLEM

By

Helmut Moritz

1. Introduction

The geodetic boundary value problem may be defined as the determination of the earth's physical surface S if the gravity vector $\vec{g}$ and the gravity potential W are given on it. It is none of the three well-known boundary value problems of potential theory, where the surface S is always considered given.

Once the physical surface S has been determined, however, the computation of the gravity field outside the earth from surface data is indeed closely related to the usual boundary value problems of potential theory. Let, for instance, the normal component

$$g_n = \frac{\partial W}{\partial n} \tag{1}$$

of the gravity vector $\vec{g}$ be known at every point of S. It is the negative derivative of the gravity potential W along the surface normal n of S. If g_n is given in addition to S, then the determination of the external gravity field is identical with the <u>second boundary value problem</u> of potential theory or <u>Neumann's problem</u>. It is unessential in this connection that W is not harmonic, owing to the effect of centrifugal force.

According to the theory of Neumann's problem, the solution is formally given by

$$W_P = \iint_S g_n(Q)\ G(P,Q)\ dS. \tag{2}$$

The integration is extended over the physical surface S. Q denotes the variable point on S where the surface element dS is situated; P is the fixed point outside S at which W is to be determined. $G(P,Q)$ is Green's function for our problem. For simplicity we have assumed W to be harmonic in (2); in reality this formula should also take the centrifugal force into account.

Green's function $G(P,Q)$ depends on the surface S. Since the earth's physical surface is extremely irregular and difficult to handle mathematically, it is impossible to determine the function $G(P,Q)$ for it at the present time. Hence a slightly modified approach is taken which, to a certain extent, abandons the rigid formulation in terms of a second boundary value problem.

First, the gravity potential W on S is used as an additional information. Secondly, a normal potential U is introduced, and the disturbing potential $T = W - U$ and the gravity anomaly Δg are used rather than the potential W and the gravity g themselves.

Let this additional information be that S is an equipotential surface $W = W_o =$ const. Then our problem can indeed be solved. We have the well-known formulas

$$T_P = \frac{R}{4\pi} \iint_\sigma S(r_P,\psi)\, \Delta g \, d\sigma \,, \tag{3}$$

$$\Delta g_P = \frac{r_P^2 - R^2}{4\pi r_P} \iint_\sigma \frac{\Delta g}{D^3}\, d\sigma . \tag{4}$$

Fig. 1 illustrates the notations used. R is the mean radius of the reference ellipsoid $U = W_o$, which for the present purpose may be considered as a sphere of radius R. Let H be the elevation of the exterior point P above sea level; then

$$r_P = R + H \tag{5}$$

D is the spatial distance of P from the variable point Q on the sphere representing the reference ellipsoid:

$$D = \sqrt{r_P^2 + R^2 - 2r_P R \cos\psi} \,. \tag{6}$$

ψ is the angular distance of Q from P; $d\sigma$ is the element of solid angle situated at Q. Finally, the function

$$S(r_P,\psi) = \frac{2R}{D} + \frac{R}{r_P} - 3\,\frac{RD}{r_P^2} - \frac{R^2}{r_P^2} \cos\psi \left(5 + 3\, \ln \frac{r_P - R\cos\psi + D}{2r_P}\right) \tag{7}$$

is the generalized Stokes' function.

Since the gravity disturbance vector $\vec{\delta}$ is the gradient of T, that is,

$$\vec{\delta} = \text{grad } T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right), \tag{8}$$

we find its components by forming the partial derivatives of (3).

An alternative way of computing T_P is using a surface layer, a coating, on the reference ellipsoid. Then

$$T_P = \frac{R^2}{2\pi} \iint_\sigma \frac{\mu}{D} \, d\sigma \quad , \tag{9}$$

where the density μ of the coating is given by

$$\mu = \Delta g + \frac{3T}{2R} , \tag{10}$$

Δg and T being taken at sea level (see, e.g., Hirvonen and Moritz, 1963).

These preceding formulas are not rigorously valid for the physical surface S, because S is not equipotential. Compared with the earth's radius R, the elevation h of S above sea level, i.e., above an equipotential surface, are rather small. Thus a development in a power series with respect to h is feasible. The approximation of zero order, neglecting h completely, is then given by equation (3) through (10). The purpose of present paper is to find first order approximations, which are linear in h.

2. A Solution for T

The point of departure is Green's identity

$$4\pi T_P = \iint_S \{T \frac{\partial}{\partial n} (\frac{1}{l}) - \frac{1}{l} \frac{\partial T}{\partial n}\} \, dS, \tag{11}$$

extended over the physical surface S. Fig. 2 shows the notations used.

From (Moritz, 1964, p. 21) we take the formula

$$\frac{\partial T}{\partial n} = (\frac{1}{\gamma} \frac{\partial \gamma}{\partial h} T - \Delta g'') \cos\beta, \tag{12}$$

where

$$\Delta g'' = \Delta g + g_1 \tag{13}$$

and

$$g_1 + - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2). \tag{14}$$

γ is normal gravity, ξ and η are the components of the deflection of the vertical, β is the angle of (maximum) inclination of the terrain, and β_1 and β_2 are the angles of inclination of the north-south and the east-west terrain profile, respectively. To a sufficient approximation,

$$\frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \doteq - \frac{2}{R} \tag{15}$$

and

$$\cos \beta \doteq 1, \tag{16}$$

so that (12) becomes

$$-\frac{\partial T}{\partial n} = \frac{2T}{R} + \Delta g'' \ . \tag{17}$$

This is introduced in (11), giving

$$4\pi T_P = \iint_S \{\frac{\partial}{\partial n} (\frac{1}{\ell}) + \frac{2}{R\ell}\} \ T \ dS + \iint_S \frac{\Delta g''}{\ell} dS \ . \tag{18}$$

We have

$$\frac{\partial}{\partial n} (\frac{1}{\ell}) = - \frac{1}{2\ell^3} \frac{\partial \ell^2}{\partial n} \tag{19}$$

and

$$\frac{\partial \ell^2}{\partial n} = \vec{n} \ . \ \mathrm{grad} \ \ell^2 \ , \tag{20}$$

according to a basic formula of vector analysis. $\vec{n}$ is the unit vector along the normal. From (Moritz, 1964, p. 20) we take the formula

$$\vec{n} = \cos \beta \ (- \frac{\partial h}{\partial x}, - \frac{\partial h}{\partial y} \ , 1) \quad , \tag{21}$$

so that

$$\frac{\partial \ell^2}{\partial n} = \cos\beta \left(- \frac{\partial \ell^2}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial \ell^2}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial \ell^2}{\partial z} \right). \qquad (22)$$

Let the z-axis be the line OQ, and let the x-axis lie in the plane containing the points O, P, and Q, which is the plane of drawing of Fig. 2. Then

$$\frac{\partial \ell^2}{\partial x} = \frac{\partial \ell^2}{r d\psi}, \quad \frac{\partial \ell^2}{\partial y} = 0, \quad \frac{\partial \ell^2}{\partial z} = \frac{\partial \ell^2}{\partial r}, \qquad (23)$$

where

$$r = R + h. \qquad (24)$$

Fig. 2 shows that

$$\frac{\partial h}{\partial x} = \tan\tau \qquad (25)$$

and

$$\ell^2 = r^2_P + r^2 - 2r_P r \cos\psi . \qquad (26)$$

Differentiating (26) with respect to r and ψ, inserting in (23), and taking (25) into account, we find that (22) becomes

$$\frac{\partial \ell^2}{\partial n} = 2 \cos\beta \; (A + h - r_P \sin\psi \tan\tau) , \qquad (27)$$

where

$$A = R - r_P \cos \psi \ . \tag{28}$$

Thus we find from (19)

$$\frac{\partial}{\partial n}\left(\frac{1}{\ell}\right) = - \frac{\cos\beta}{\ell^3}\left(A + h - r_P \sin \psi \tan \tau\right). \tag{29}$$

This formula is rigorous. We now wish to develop quantities involved in (18) as power series with respect to h, retaining only terms linear in the elevation h and the inclination β. Then (26) becomes

$$\ell^2 = r^2_P + R^2 - 2r_P R \cos \psi + 2\left(R - r_P \cos \psi\right) h$$

or, by (6) and (28),

$$\ell^2 = D^2 + 2Ah = D^2\left(1 + \frac{2A}{D^2} h\right) ,$$

so that

$$\frac{1}{\ell} = \frac{1}{D}\left(1 - \frac{A}{D^2} h\right) = \frac{1}{D} - \frac{A}{D^3} h \quad , \tag{30}$$

$$\frac{1}{\ell^3} = \frac{1}{D^3}\left(1 - \frac{3A}{D^2} h\right) = \frac{1}{D^3} - \frac{3A}{D^5} h. \tag{31}$$

Equation (29) assumes the form

$$\frac{\partial}{\partial n}\left(\frac{1}{\ell}\right) = -\frac{1}{D^3}\left\{A + \left(1 - \frac{3A^2}{D^2}\right) h - r_P \sin\psi \tan \tau\right\} \quad . \tag{32}$$

Setting $\cos\beta \doteq 1$, as we have done, is in agreement with our linear development since

$$\cos\beta = 1 - \frac{1}{2}\beta^2 + \ldots \tag{33}$$

differs from 1 only by quadratic terms.

The disturbing potential T is split up into a part T_o independent of h and a part T_1 depending linearly on h:

$$T = T_o + T_1 \quad . \tag{34}$$

Higher order terms T_2, T_3, etc., are neglected. In the same way Δg is split up according to (13), since g_1 depends linearly on the inclination β.

The expressions (32), (30), (34), and (13) are inserted in (18), linear terms are retained only, and zero order and first order terms are separated. Furthermore we may put

$$dS \doteq R^2\left(1 + \frac{2h}{R}\right) d\sigma \doteq R^2 \, d\sigma \ , \tag{35}$$

neglecting h/R in agreement with Molodensky (Molodenskii et. al., 1962, p. 122). Thus (18) becomes

$$\begin{aligned}
4\pi T_{o,P} + 4\pi T_{1,P} = {} & R^2 \iint_\sigma \left(-\frac{A}{D^3} + \frac{2}{RD}\right) T_o\, d\sigma + R^2 \iint_\sigma \frac{\Delta g}{D}\, d\sigma \\
& + R^2 \iint_\sigma \left(-\frac{A}{D^3} + \frac{2}{RD}\right) T_1\, d\sigma + R^2 \iint_\sigma \frac{g}{D^1}\, d\sigma \\
& + R^2 \iint_\sigma \left(-\frac{1}{D^3} + \frac{3A^2}{D^5} - \frac{2A}{RD^3}\right) T_o h\, d\sigma \\
& + R^2 \iint_\sigma \frac{r_P \sin\psi}{D^3} T_o \tan\tau\, d\sigma - R^2 \iint_\sigma \frac{A}{D^3} \Delta g\, h\, d\sigma \; .
\end{aligned} \tag{36}$$

Since the zero order terms are independent of h, the sum of the zero order terms on the left side of this equation must be equal to the sum of the zero order terms on the right side. The same holds for the first order terms.

The zero order terms are of no further interest. The quantity T_o, corresponding to $h = 0$, is naturally obtained from Δg by means of the spherical formula (3):

$$T_{o,P} = \frac{R}{4\pi} \iint_\sigma S(r_P, \psi)\, \Delta g\, d\sigma \; . \tag{37}$$

For T_1 we find from (36)

$$T_{1,P} = \frac{R^2}{4\pi} \iint_\sigma \left(-\frac{A}{D^3} + \frac{2}{RD}\right) T_1 \, d\sigma + \frac{R^2}{4\pi} \iint_\sigma \frac{g_1}{D} \, d\sigma +$$

$$+ \frac{R^2}{4\pi} \iint_\sigma \left(-\frac{1}{D^3} + \frac{3A^2}{D^5} - \frac{2A}{RD^3}\right) T_o h \, d\sigma \tag{38}$$

$$+ \frac{R^2}{4\pi} \iint_\sigma \frac{r_P \sin\psi}{D^3} T_o \tan\tau \, d\sigma - \frac{R^2}{4\pi} \iint_\sigma \frac{A}{D^3} \Delta g \, h \, d\sigma \; .$$

This equation clearly shows that $T_{1,P}$ depends on the products $T_o h$, $T_o \tan\tau$ and $\Delta g\, h$, and on the first order terms g_1 and T_1.

The term T_o on S is given by Stokes' formula

$$T_o = \frac{R}{4\pi} \iint_\sigma \Delta g \, S(\psi) \, d\sigma \quad . \tag{39}$$

The quantity g_1 depends on the deflections of the vertical according to (14). The term T_1 on S may be obtained, for instance, by Molodensky's formula (Molodenskii et. al., 1962, pp. 122-123)

$$T_1 = \frac{R}{4\pi} \iint_\sigma G_1 \, S(\psi) \, d\sigma \quad , \tag{40}$$

where $S(\psi)$ is Stokes' function and G_1 is given by

$$G_1 = \frac{1}{16\pi R} \iint_\sigma \frac{h-hp}{\sin^3\frac{\psi}{2}} \mu d\sigma \quad , \tag{41}$$

μ being defined by (10) .

3. A Connection with Molodensky's Problem

Note that (38) gives T_1 at a point outside S, whereas (40) gives it at a point on S. Equation (40) thus constitutes a solution of the boundary value problem for the earth's physical surface S. For if we know the disturbing potential T along S, then we can find the height anomaly ζ by

$$\zeta = \frac{T}{\gamma} \tag{42}$$

and so determine S itself (Molodenskii et. al., 1962; Moritz, 1964).

Now it is interesting that a solution for T, equivalent to (40) can be found from (38) itself. For this purpose we have to make the transition $P \to P_o$ to a point on the surface S itself. We know that Green's identity (11) changes discontiuously during this transition, 4π being replaced by 2π if P lies on S.

It is convenient first to shift the level from which h is reckoned, so that it contains the point P_o at the foot of the plumb line through P (Fig. 2). We thus have to replace h in (38) by $h - h_P$ and to reckon

H from P_o. Then the transition $P \to P_o$ is effected; it means putting $H = 0$. We then have

$$D = 2R \sin \frac{\psi}{2} = \ell_o , \qquad (43)$$

$$A = R - R \cos \psi = 2R \sin^2 \frac{\psi}{2} = \frac{\ell_o^2}{2R} . \qquad (44)$$

Inserting this in (38), replacing h by $h - h_P$ and 4π by 2π, we obtain after a transformation the requires some care,

$$T_1 - \frac{3R}{4\pi} \iint_\sigma \frac{T_1}{\ell_o} d\sigma = \frac{R^2}{2\pi} \iint_\sigma \frac{g_1}{\ell_o} d\sigma + \varkappa , \qquad (45)$$

where

$$\varkappa = \frac{R^3}{2\pi} \iint_\sigma \frac{- \delta \cos \psi + \tau \sin \psi}{\ell_o^3} T_o \, d\sigma . \qquad (46)$$

Here we have replaced $\tan \tau$ by τ and introduced the quantity

$$\delta = \frac{h - h_P}{R} . \qquad (47)$$

Equation (45) is the approximation linear in h of an equation originally given by Molodensky (Molodenskii et. al., 1962, p. 90; Moritz, 1964, p. 21). Molodensky calls this an integro-differential equation, because he considers ξ and η, which are the horizontal derivatives of T, as unknown. We shall assume ξ and η to be given, however; approximate values are sufficient. Then (45) is a linear integral equation for T_1.

Its solution can be found to be

$$T_1 = \frac{R}{4\pi} \iint_\sigma g_1 \, S(\psi) \, d\sigma + \varkappa + \frac{3}{8\pi} \iint_\sigma \varkappa \, S(\psi) \, d\sigma \, . \tag{48}$$

This equation serves the same purpose as Molodensky's solution (40).

The integral equation (45) was also obtained by de Graaff-Hunter (1960), but his way of solving it is not entirely correct.

4. A Solution for Δg

The first order term grad T_1 of the gravity disturbance vector

$$\vec{\delta} = \text{grad } T = \text{grad } T_0 + \text{grad } T_1 \tag{49}$$

may be found by forming the partial derivatives of (38).

An expression for the corresponding first order term of the gravity anomaly Δg outside the earth was derived in (Moritz, 1962 a). In the following an improved derivation, based on this paper, will be given. The notation will be slightly changed. The quantities R, α, r, $\text{tg}\alpha \cos(\varphi-\psi)$, Q of the 1962 paper are now denoted by ℓ, β, s, $-\tan\tau$, P, respectively, in order to conform with the notation used in the present paper.

As we did in the previous paper, we shall replace the sphere by its tangential plane; this corresponds to letting $R \to \infty$. This is possible here because only a limited area surrounding P_o is relevant for the computation of Δg_P . As a plane approximation we obtain from (17) by setting $R \to \infty$

$$- \frac{\partial T}{\partial n} = \Delta g'' = \Delta g + g_1 , \tag{50}$$

$$- \frac{\partial T}{\partial H} = \Delta g_P . \tag{51}$$

Inserting this in equation (97) of (Moritz, 1962 a,) we find

$$\Delta g_P = - \frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H^2}{D^5}\right) T \; dxdy + \frac{H}{4\pi} \iint \frac{\Delta g + g_1}{D^3} \; dxdy$$

$$- \frac{3H}{4\pi} \iint \left(\frac{3}{D^5} - \frac{5H^2}{D^7}\right) T \; h \; dxdy + \frac{3H}{4\pi} \iint \frac{s}{D^5} \; T \tan \tau \; dxdy$$

$$- \frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H^2}{D^5}\right) \Delta g \; h \; dxdy.$$

Now we set again

$$T = T_o + T_1 ,$$

$$\Delta g_P = \Delta g_{o,P} + \Delta g_{1,P} .$$

This is inserted in the preceding equation, and the zero and the first order terms are separated.

The zero order terms, corresponding to $h = 0$, yield

$$\Delta g_{o,P} = - \frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H^2}{D^5}\right) T_o \, dxdy + \frac{H}{4\pi} \iint \frac{\Delta g}{D^3} \, dxdy \quad .$$

The solution of this equation is given by

$$\Delta g_{o,P} = \frac{H}{2\pi} \iint \frac{\Delta g}{D^3} \, dxdy \ , \tag{52}$$

because this is the well-known plane approximation of (4). The first order terms give

$$\begin{aligned} \Delta g_{1,P} = & - \frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H^2}{D^5}\right) T_1 \, dxdy + \frac{H}{4\pi} \iint \frac{g_1}{D^3} \, dxdy \\ & - \frac{3H}{4\pi} \iint \left(\frac{3}{D^5} - \frac{5H^2}{D^7}\right) T_o h \, dxdy + \frac{3H}{4\pi} \iint \frac{s}{D^5} T_o \tan \tau \, dxdy \\ & - \frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H^2}{D^5}\right) \Delta g \, h \, dxdy \ . \end{aligned} \tag{53}$$

This is the desired solution. It differs from eq. (100) of (Moritz, 1962 a) by the terms containing T_1 and g_1, which were omitted there by mistake.

5. Solution by Means of a Surface Layer

The coating method for an equipotential surface, equations (9) and (10), can be generalized for the physical surface S of the earth. We represent the disturbing potential T outside S as the potential of a surface layer on S of density χ :

$$T_P = \iint_S \frac{\chi}{\ell}\, dS \quad . \tag{54}$$

We again apply our usual approximation, linear in h. By (30) and (35) we have

$$T_P = R^2 \iint_\sigma \left(\frac{1}{D} - \frac{A}{D^3}\, h\right) \chi \, d\sigma \quad . \tag{55}$$

The surface layer χ is split up into a part χ_o independent of h and a part χ_1 proportional to h:

$$\chi = \chi_o + \chi_1 \quad . \tag{56}$$

This corresponds to (34). Then (55) becomes

$$T_P = T_{o,P} + T_{1,P} = R^2 \iint_\sigma \frac{\chi_o}{D}\, d\sigma + R^2 \iint_\sigma \frac{\chi_1}{D}\, d\sigma - R^2 \iint_\sigma \frac{A}{D^3}\, \chi_c \, d\sigma \ ;$$

that is

$$T_{o,P} = R^2 \iint_\sigma \frac{\chi_o}{D}\, d\sigma \ , \tag{58}$$

$$T_{1,P} = R^2 \iint_\sigma \frac{\chi_1}{D}\, d\sigma - R^2 \iint_\sigma \frac{A}{D^3}\, \chi_o\, h\, d\sigma \quad . \tag{59}$$

The quantities χ_o and χ_1 on S can be determined by making the transition $P \to P_o$ to a point on the surface S itself. This is quite analogous to the transformation of (38) described in section 3, by which a solvable linear integral equation for T_1 was found. In our present case a linear integral equation for χ is obtained, which was found and solved by Molodensky. According to (Molodenskii et. al, 1962, p. 122) we have

$$2\pi\chi_o = \Delta g + \frac{3}{8\pi} \iint_\sigma \Delta g\, S(\psi)\, d\sigma = \Delta g + \frac{3T_o}{2R} = \mu \ , \tag{60}$$

$$2\pi\chi_1 = G_1 + \frac{3}{8\pi} \iint_\sigma G_1\, S(\psi)\, d\sigma = G_1 + \frac{3T_1}{2R} \quad , \tag{61}$$

where G_1 is given by (41). Equation (60) also follows directly by comparing (55), for $h = 0$, and (9).

If we compare (59) with (38), we see that the coating method yields a considerably simpler formula for computing T_1. It will be remembered that T_1 is the effect of the topographic height h, T_o corresponding to a level earth, $h = 0$.

The gravity disturbance vector $\vec{\delta}$ may be found in a relatively simple way as the gradient of (57). The coating method is, however, less suited for computing external gravity anomalies Δg_P .

6. Solution by Reduction to Sea Level

It can be shown (Moritz, forthcoming paper) that Molodensky's solution, equations (39) through (41), may be transformed in a purely formal manner into

$$T = \frac{R}{4\pi} \iint_{\sigma} \left(\Delta g - \frac{\partial \Delta g}{\partial h} h\right) S(R + h_P, \psi) \, d\sigma \quad . \tag{62}$$

The geometrical interpretation of this new equation is obvious. The gravity anomalies Δg at the physical surface of the earth are reduced "in free air" to sea level by adding the term $- h \, \partial\Delta g/\partial h$. Then T at the point P_o of the physical surface, which has the elevation h_P above sea level, is computed by using eq. (3), which is Stokes' formula generalized for points above sea level.

This is the "free-air reduction to sea level" described in (Moritz, 1964 sec. 6.4.). The indicated formal derivation of (62) shows that the difficulties of analytical continuation have no effect on an approximation linear in h. These difficulties, inherent in free-air reduction to sea level, were the reason why Molodensky, who used this method in 1949 for solving his integral equation, gave it up later on. He then developed another solution, which is based on a coating on the physical surface and corresponds to equations (39) through (41).

However, free-air reduction to sea level has enormous practical advantages, as emphasized by Bjerhammar (1963). It reduces the problem to the simple case of a bounding level surface, to which spherical formulas such as equations (3) through (10) may be applied. The computation of the external gravity field, of spherical harmonics, etc., can thus be handled rigorously by means of the conventional formulas.

All we have to do is to reduce Δg to sea level, obtaining

$$\tilde{\Delta g} = \Delta g - \frac{\partial \Delta g}{\partial h} h , \qquad (63)$$

as an approximation linear in h, and use $\Delta\tilde{g}$ instead of Δg in the spherical formulas (3) through (10). The effect of topographical elevation h, which otherwise involves complicated formulas, is thus automatically taken into account.

The vertical gradient of the gravity anomalies may be computed by well-known formulas such as the integral expression

$$\left.\frac{\partial \Delta g}{\partial h}\right)_P = -\frac{2}{R}\Delta g_P + \frac{R^2}{2\pi}\iint_\sigma \frac{\Delta g - \Delta g_P}{\ell_o^3}\, d\sigma, \qquad (64)$$

where again

$$\ell_o = 2R \sin\frac{\psi}{2}$$

As we have seen, free-air reduction to sea level is valid at least to an approximation linear in h. It may be expected, however, to be accurate to a considerably higher approximation. For this reason, Bjerhammar has suggested to compute the reduced anomalies $\Delta\tilde{g}$ by solving equation

$$\Delta g_P = \frac{r_P^2 - R^2}{4\pi r_P}\iint_\sigma \frac{\Delta\tilde{g}}{D^3}\, d\sigma \qquad (65)$$

for $\Delta\tilde{g}$. It is eq. (4) applied to a point P at the physical surface of the earth which has the elevation h_P above sea level; there is

$$r_P = R + h_P \quad . \tag{66}$$

Formula (65) may be considered as a linear integral equation for $\Delta\tilde{g}$ if the surface anomalies Δg are given.

Bjerhammar (1963) has also described how (65) may be solved. A different way of solving the integral equation (65) is an iterative method that is well suited for pratical computation.

For this purpose we must transform (65). Consider a fictitious anomalous point mass at the center of the spherical earth. Its attraction is proportional to $1/r^2$, so that the gravity anomaly caused by this point mass may be written, omitting a constant factor

$$\Delta g = (\frac{R}{r})^2 \quad .$$

At sea level, $r = R$, we have

$$\Delta\tilde{g} = 1 \ .$$

Apply now eq. (65) to this special function, finding

$$(\frac{R}{r_P})^2 = \frac{r_P^2 - R^2}{4\pi r_P} \iint_\sigma \frac{d\sigma}{D^3} \quad . \tag{67}$$

Let $\Delta\tilde{g}_P$ denote the value of $\Delta\tilde{g}$ corresponding to Δg_P, i.e., lying on the same radius vector OP. Multiply (67) by $\Delta\tilde{g}_P$ which is a constant with respect to the integration, and subtract from (65). This gives

$$\Delta g_P - \left(\frac{R}{r_P}\right)^2 \Delta\tilde{g}_P = \frac{r_P^2-R^2}{4\pi r_P} \iint_\sigma \frac{\Delta\tilde{g}-\Delta\tilde{g}_P}{D^3}\, d\sigma \,.$$

The factor $(R/r_P)^2$ of $\Delta\tilde{g}_P$ can be set equal to 1 without loss of accuracy. Thus we obtain, as an alternative to (65), the integral equation

$$\Delta\tilde{g}_P = \Delta g_P - \frac{r_P^2-R^2}{4\pi r_P} \iint_\sigma \frac{\Delta\tilde{g}-\Delta\tilde{g}_P}{D^3}\, d\sigma \,, \tag{68}$$

which recommends itself for iterative solution by means of an automatic computer.

As a first approximation set

$$\Delta\tilde{g}^{(1)} = \Delta g \,,$$

then compute a second approximation

$$\Delta\tilde{g}_P^{(2)} = \Delta g_P - \frac{r_P^2-R^2}{4\pi r_P} \iint_\sigma \frac{\Delta\tilde{g}^{(1)}-\Delta\tilde{g}_P^{(1)}}{D^3}\, d\sigma \,,$$

a third approximation

$$\Delta\tilde{g}_P^{(3)} = \Delta g_P - \frac{r_P^2-R^2}{4\pi r_P} \iint_\sigma \frac{\Delta\tilde{g}^{(2)}-\Delta\tilde{g}_P^{(2)}}{D^3}\, d\sigma \,,$$

and so forth. The convergence of this process should be very rapid.

In this way the "downward continuation" of the gravity anomalies from the physical surface (Δg) to sea level ($\Delta\tilde{g}$) may be effected. It is characteristic for all problems of downward continuation that they cannot be solved directly, but only by iterative inversion of the formula for upward continuation. The iterative process just explained may also be performed as a plane approximation, bu replacing the sphere by its tangential plane. This is described in (Moritz, 1962 b).

7. Conclusion

In the present paper we have described and compared three methods for computing the gravity field outside the earth, taking account of the fact that the free-air anomalies are given on the physical surface of the earth, which is not a level surface. The direct formulas (38) for T and (53) for Δg are very complicated. The soltuion (59) for T according to the coating method is simpler, but this method does not provide a suitable solution for Δg. Practically the best method is the free-air reduction to sea level.

According to this method, the solution is effected in two steps:

(1) Downward continuation of Δg from the physical surface to sea level, using (63) or an iterative solution of (68).

(2) Computation of the external gravity field using the simple spherical formulas (3) through (10). This is a process of upward continuation.

In this way the irregularities of the earth's physical surface are fully are fully taken into account.

Appendix

Estimates for the Effect of Topographic Elevation on the External Gravity Anomalies.

We shall now give some estimates for the magnitude of $\Delta g_{1,P}$ according to eq. (53). First we neglect the terms containing T_1 and g_1 which are unknown. The geoidal undulation N and hence $T_o = \gamma N$ may often be considered constant over limited areas. Then the sum of the third and fourth term of (53) becomes, omitting the constant factor $(3H/4\pi)T_o$:

$$-\iint \left(\frac{3}{D^5} - \frac{5H^2}{D^7}\right) h \, dxdy + \iint \frac{s}{D^5} \tan \tau \, dxdy$$

$$= \int_{\alpha=o}^{2\pi} \int_{s=o}^{\infty} \left\{ \frac{\partial}{\partial s} \left(\frac{s^2}{D^5}\right) h + \frac{s^2}{D^5} \frac{\partial h}{\partial s} \right\} dsd\alpha$$

$$= \int_{\alpha=o}^{2\pi} \int_{s=o}^{\infty} \frac{\partial}{\partial s} \left(\frac{s^2}{D^5} h\right) dsd\alpha = \int_{\alpha=o}^{2\pi} \left\{ \frac{s^2}{(s^2 + H^2)^{5/2}} h \right\}_{s=o}^{s=\infty} d\alpha = 0 \ .$$

Here we have transformed to polar coordinates s, α by

$$x = s \cos \alpha \ , \ y = s \sin \alpha, \ dxdy = s \, dsd\alpha \tag{69}$$

and used the fact that

$$\tan \tau = \frac{\partial h}{\partial s}, \quad \left(-\frac{3}{D^5} + \frac{5H^2}{D^7}\right) s = \frac{\partial}{\partial s} \frac{s^2}{D^5} \quad .$$

Thus we have found the important result that the third and the fourth term of (53) cancel if T_o can be considered constant over the area of integration. Then there remains only the last term of (53),

$$\delta = -\frac{1}{4\pi} \iint \left(\frac{1}{D^3} - \frac{3H}{D^5}\right) \Delta g \ h \ dxdy \ . \tag{70}$$

According to the mean value theorem of integral calculus, this is equal to

$$\delta = \frac{\overline{\Delta g}}{4\pi} \iint \left(-\frac{1}{D^3} + \frac{3H^2}{D^5}\right) h \ dxdy = \overline{\Delta g} \cdot J \quad , \tag{71}$$

where $\overline{\Delta g}$ is a certain mean value of Δg taken over the area of integration

We shall now compute the factor

$$J = \frac{1}{4\pi} \iint \left(-\frac{1}{D^3} + \frac{3H^2}{D^5}\right) h \ dxdy \tag{72}$$

for two different models.

First, let the terrain consist of a conical mountain of height h_1 and slope β, superposed on a plane (Fig. 3). The point P is situated on the axis of the cone, H being its elevation above the plane. Then

$$h = \begin{cases} (a-s) \tan \beta & \text{if } s < a \\ 0 & \text{otherwise} \end{cases} \quad . \tag{73}$$

Thus we find on transforming to polar coordinates according to (69)

$$J = J_1 = \frac{1}{4\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{a} \left(- \frac{1}{D^3} + \frac{3H^2}{D^5} \right) (a-s) \tan \beta \; s \, ds d\alpha \quad . \tag{74}$$

Since a and β are constants, and

$$D = \sqrt{s^2 + H^2} \tag{75}$$

this integral can be evaluated in closed form by the usual methods, giving

$$J_1 = \frac{1}{2} \tan \beta \left(\ln \frac{a+d}{H} - \frac{a}{d} \right) . \tag{76}$$

On introducing the angle ω according to Fig. 3, this may be written

$$J_1 = \frac{1}{2} \tan \beta \left(\ln \cot \frac{\omega}{2} - \cos \omega \right). \tag{77}$$

A second model will now be considered. It represents a so-called "two-dimensional" body, extending from infinity to infinity in one horizontal direction, the cross-section normal to this direction being the same everywhere. This direction is taken normal to the plane of Fig. 3, so that this figure now represents a cross-section of the new model. The same notations will be used as in the first model. Fig. 4 shows the difference between Model 1 and Model 2.

For the second model, assume the corrdinate axes as shown in Fig. 4. Then we have corresponding to (73),

$$h = \begin{cases} (a-x) \tan \beta & \text{if } x < a \\ 0 & \text{otherwise} \end{cases} , \tag{78}$$

so that, corresponding to (74),

$$J = J_2 = \frac{1}{4\pi} \int_{x=-a}^{a} \int_{y=-\infty}^{\infty} \left(- \frac{1}{D^3} + \frac{3H^2}{D^5}\right) (a-x) \tan \beta \, dxdy \quad . \tag{79}$$

Since

$$D = \sqrt{x^2 + y^2 + H^2} \quad ,$$

this integral can again be evaluated in closed form. After somewhat laborious calculations we obtain

$$J_2 = - \frac{2}{\pi} \tan \beta \, \ln \sin \omega \quad .$$

The effect may be expected to be largest if P lies close to the top of the mountain, that is, if

$$H \doteq h_1, \quad \omega \doteq \beta \quad .$$

The following table gives values of J_1 and J_2 for this limiting case, for several inclinations $\beta = \omega$:

$\beta = \omega$	J_1	J_2
2°	0.053	0.074
10°	0.128	0.196
20°	0.145	0.249
40°	0.102	0.236
60°	0.043	0.159

J_1 or J_2 times $\overline{\Delta g}$ gives the effect δ of topographic elevation on the computed Δg at P. It is seen to be much larger for the second model, which represents a large-scale deviation from the plane, whereas the first model represents a local irregularity only. Both effects are seen to be largest for inclinations of about 20°.

As a matter of fact, these values imply quite extreme conditions. In practice, the correction δ should hardly exceed 10% of $\overline{\Delta g}$ even for low altitudes H in rugged areas.

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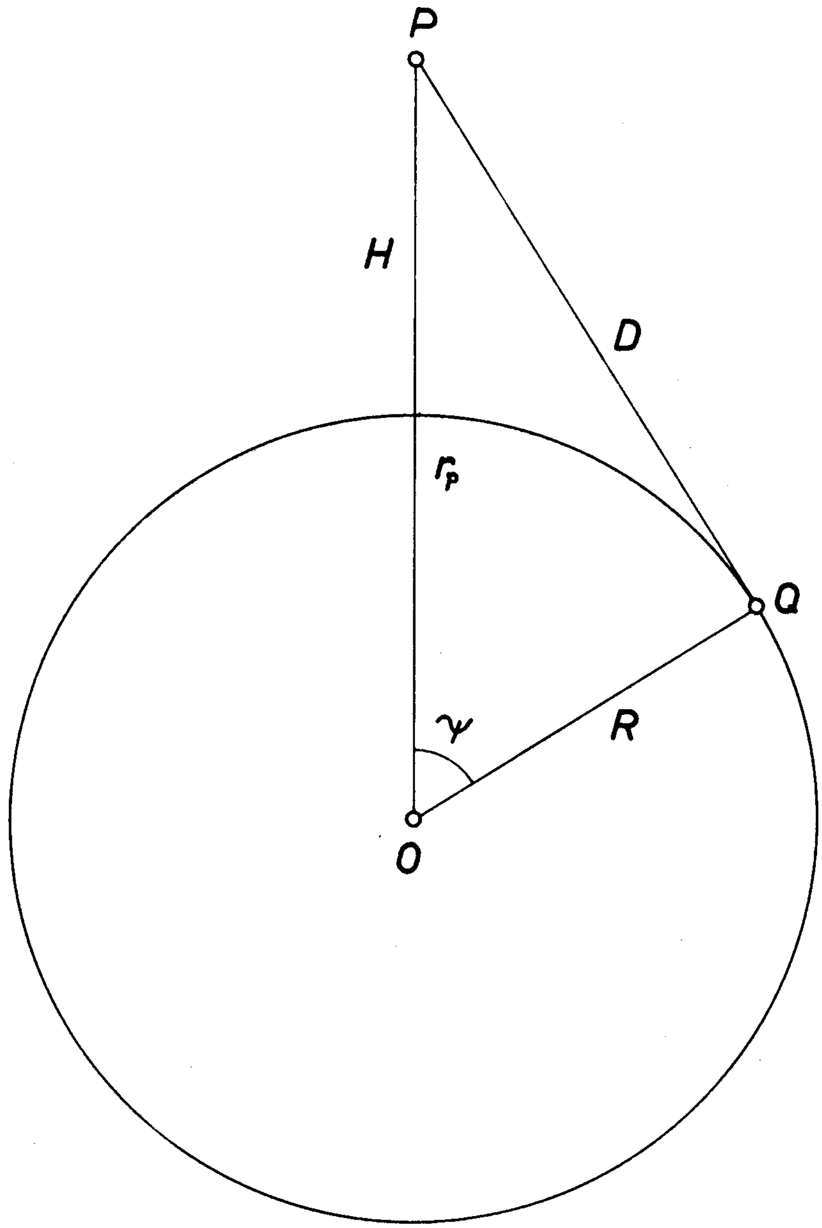
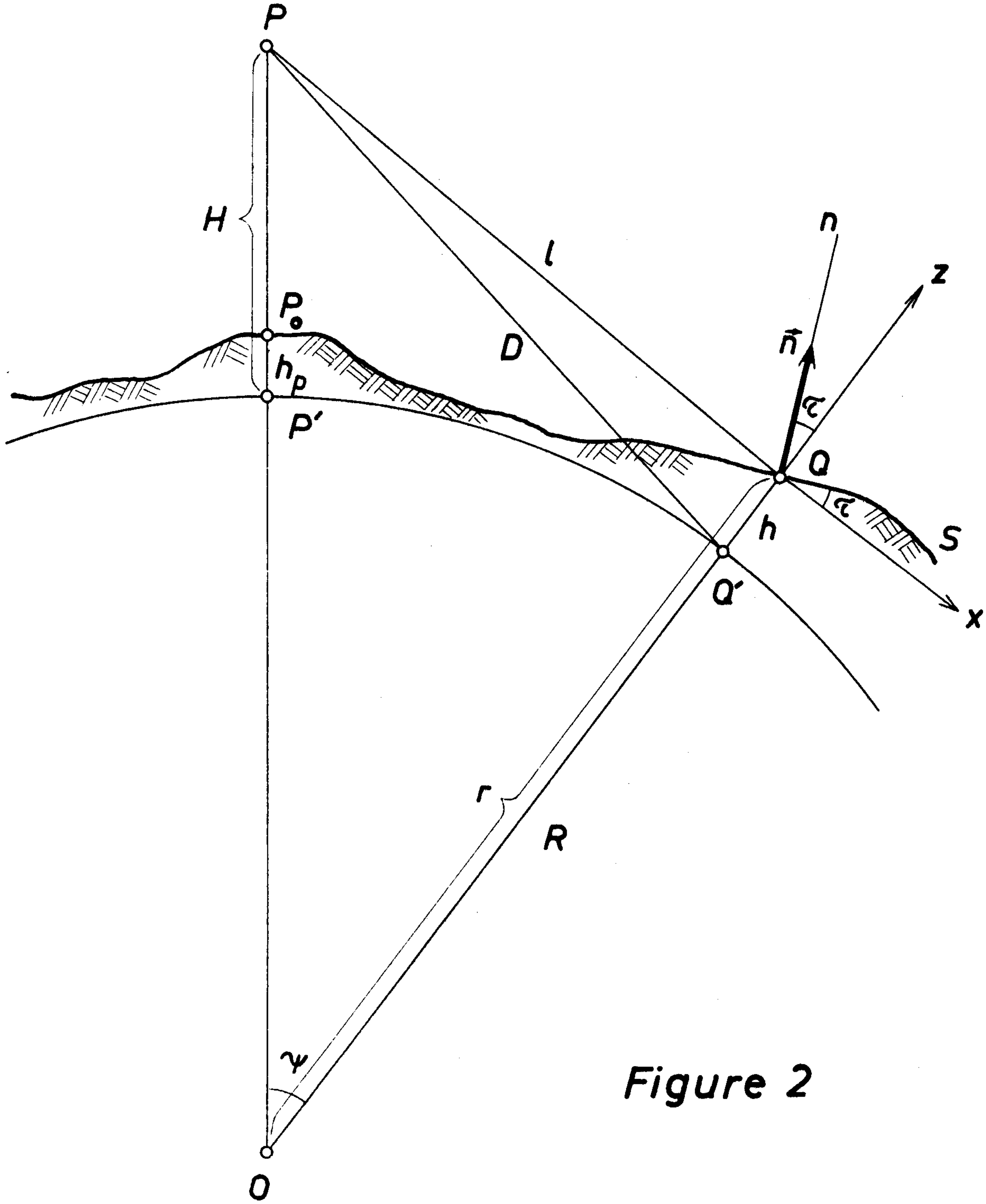


Figure 1



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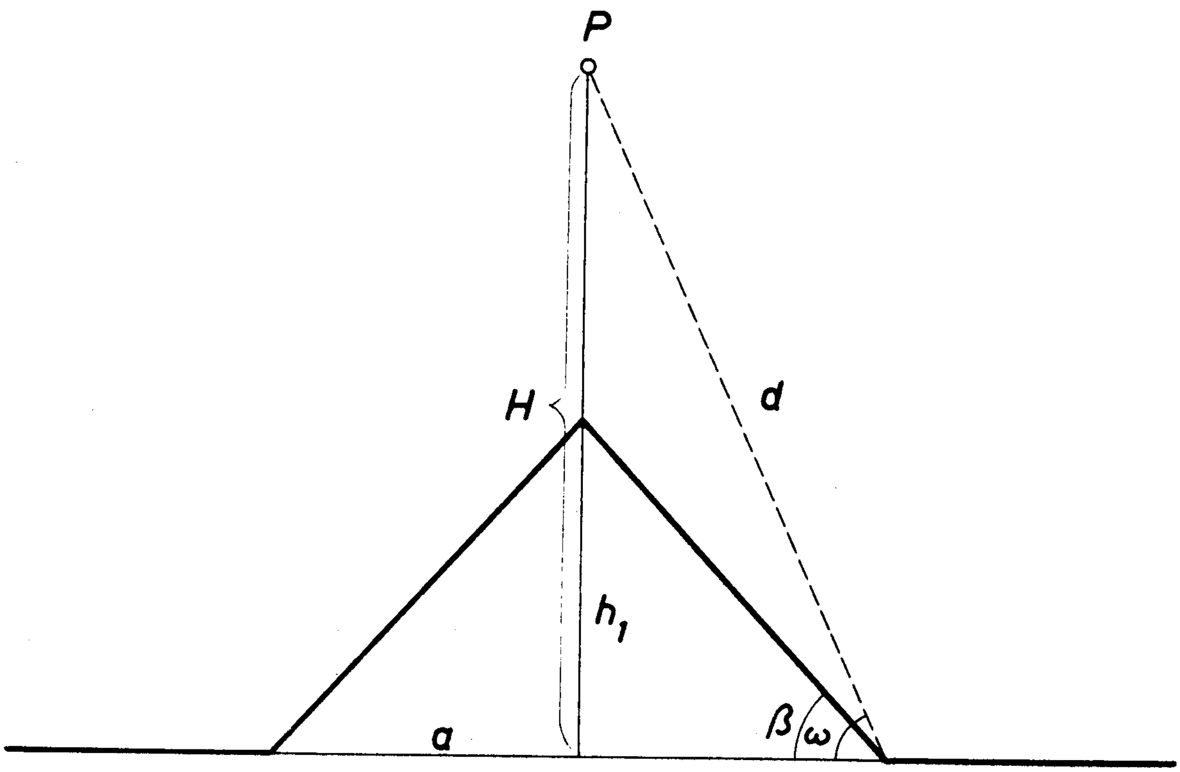

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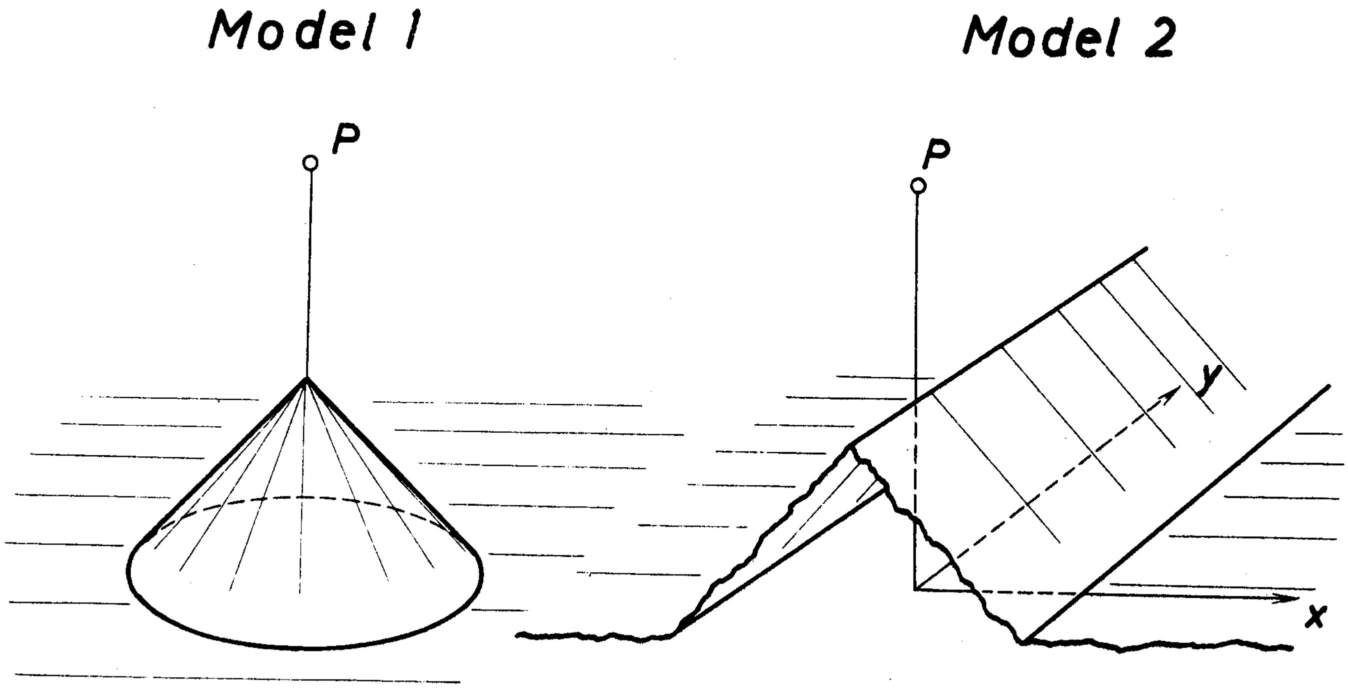

Figure 4

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13. ABSTRACTS

External Gravity Field and the Boundary Value Problems. In the usual way of computing the external gravity field, the earth is considered as a level surface although, strictly speaking, the free-air gravity anomalies refer to the non-level physical surface of the earth.

The main purpose of the present paper is to give formulas and, as an appendix, some estimates for the effect of topographic height on these computations. As application of Green's identities yields direct, but complicated, formulas for the effect of the disturbing potential T and the gravity anomaly Δg outside the earth.

Green's Functions in Physical Geodesy. Of the two main approaches to boundary value problems -- the use of Green's functions and of integral equations -- the latter approach has been chosen almost exclusively for the geodetic boundary value problem. The present report shows how Green's functions may be used for re-deriving known formulas and also for obtaining new results.

Formulas for the third boundary value problem for the sphere are developed and then specialized to Stokes' problem. As a limiting case, the boundary value problem for the plane is briefly considered. Then a formula for the variation of Green's function with the boundary surface is developed and applied to the problem of Molodensky.

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