FROM THE MIDPOINT FORMULA TO GENERALIZED DIVISION OF LINE SEGMENTS

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To most students of Algebra 2 and above, the division of a line segment in the coordinate plane is limited to what is classically known as "The Midpoint Formula." Many texts barely touch on this interesting relationship and often dismiss its derivation as beyond the scope of the text or leave it to the reader. Often only a few sentences of superficial explanation accompany the statement of the formula. Indeed, this may be all some students can handle, but an advanced or honors group deserves much more.

Most students of Algebra 2, and others who might study the midpoint formula, have completed a course in Plane Geometry. This means that they already have begun to develop an understanding of a ratio and its applications. The ratio and a little algebra are all that is needed not only to derive the midpoint formula, but to generalize and extend it into what the author calls the division of the line segment. The procedure for dividing the line segment into equal parts will lead to greater and easier use of this technique by the student and will therefore expand problem solving ability while enhancing a full understanding of underlying concepts.

A pattern arises in the generalization that greatly simplifies the division of the line segment, just as Pascal's Triangle (or the combination approach) vastly simplifies the expansion of binomials raised to high powers.

Figure 1 shows line segment AB which is to be bisected at midpoint (a,b). From the geometry of ratios, it is known that

$$\frac{1}{a - x_1} = \frac{1}{x_2 - a} .$$

Since "a" is the x-coordinate of the midpoint, simply solve this ratio to get the standard result

$$x_2 - a = a - x_1$$
$$2a = x_1 + x_2$$
$$a = \frac{x_1 + x_2}{2}.$$

A similar ratio relates to the y-axis the corresponding result

$$b = \frac{y_1 + y_2}{2}.$$

Therefore, the point that bisects segment AB is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

This is a derivation that is easily understood by better students. The derivation easily generalizes to trisecting a line segment.

Figure 2 shows a setup similar to Figure 1. This time (a,b) is located so that 1/3 of the length of $\overline{AB}$ is to the left and 2/3 is to the right (A ratio of 1:2). The dividing ratio now becomes

$$\frac{1}{a - x_1} = \frac{2}{x_2 - a}$$

or

$$x_2 - a = 2a - x_1$$
$$3a = 2x_1 + x_2$$
$$a = \frac{2x_1 + x_2}{3}.$$

Similarly, $b = \dfrac{2y_1 + y_2}{3}$ which gives one of the two trisecting points as $\left(\dfrac{2x_1 + x_2}{3}, \dfrac{2y_1 + y_2}{3}\right)$.

By letting the length to the left of (a,b) equal 2/3 of the total length with the length to the right equalling 1/3 of the length (ratio of 2:1), the dividing ratio $\dfrac{2}{a - x_1} = \dfrac{1}{x_2 - a}$ yields the other trisection point as $\left(\dfrac{x_1 + 2x_2}{3}, \dfrac{y_1 + 2y_2}{3}\right)$.

The process easily generalizes. By applying the same ratios to the general setup in Figures 3 and 4 we get

$$\frac{p}{a - x_1} = \frac{q}{x_2 - a} \qquad \frac{p}{b - y_1} = \frac{q}{y_2 - b}$$

$$px_2 - pa = qa - qx_1 \qquad py_2 - pb = bq - qy_1$$

$$a = \frac{qx_1 + px_2}{p + q} \qquad b = \frac{qy_1 + py_2}{p + q}$$

Now it is possible to simply write down the points that divide segment AB by substituting p and q into the formula

$$\left(\frac{qx_1 + px_2}{p + q}, \frac{qy_1 + py_2}{p + q}\right). \qquad \text{(EQUATION 1)}$$

If we maintain a left-to-right orientation from (x_1,y_1) to (x_2,y_2), it is possible to establish an easily remembered pattern for dividing a line segment into any desired number of pieces. To divide segment AB of figures 3 and 4 into four equal pieces will require working with the three substitutions:

1) let $p = 1$ and $q = 3$
2) let $p = 2$ and $q = 2$
3) let $p = 3$ and $q = 1$.

This maintains a left-to-right orientation for finding the points that divide AB into four equal parts. It is also important to keep $A(x_1,y_1)$ left of $B(x_2,y_2)$ for a left to right orientation. The results of substituting 1, 2, and 3 above into EQUATION 1 are shown in Table 1 under $N = 4$. Note that in Table 1 the values of p and q are shown to the right of each point along the vertical line. The results of the above procedure are also included for $N = 5$ and $N = 6$.

Additional entries for greater values of N and close inspection of the results show that an interesting pattern has emerged in the triangular shape of Table 1. Note that left-to-right orientation has been maintained throughout Table 1.

The pattern is shown mathematically for a line segment that is divided into "N" equal parts. What this shows is that if we maintain left-to-right orientation of (x_1,y_1) and (x_2,y_2), the coefficients of x_1 and y_1 count down from left-to-right beginning with N-1 and ending with 1, while the coefficients of x_2 and y_2 count up from left-to-right beginning with 1 and ending with N-1. The denominator is N in all cases.

The verbal description of this pattern sounds much more complicated than is the case. An inspection of Table 1 gives a much clearer picture of the ease with which this pattern applies.

The following specific examples illustrate the process for concrete points:

EXAMPLE 1: Find the points that divide the line segment connecting (2,1) and (8,4) into 3 equal parts.

SOLUTION: To divide into three equal parts we apply the pattern established and shown in Table 1. Let the coefficients of x_1 and y_1 count down from 2 to 1 and the coefficients of x_2 and y_2 count up from 1 to 2 while dividing by 3. The required points are

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right) \text{and} \left(\frac{x_1 + 2x_2}{3}, \frac{y_1 + 2y_2}{3}\right).$$

Let $(2,1) = (x_1,y_1)$, since this point is left of $(8,4)$, and $(8,4) = (x_2,y_2)$.

$$\left(\frac{2(2) + 8}{3}, \frac{2(1) + 4}{3}\right) \text{and} \left(\frac{2 + 2(8)}{3}, \frac{1 + 2(4)}{3}\right)$$

the required points are (4,2), (6,3).

EXAMPLE 2: Find the points that divide the line segment connecting (2,-8) and (-3,2) into 5 equal parts.

SOLUTION: Again the established pattern is applied. The four required points are shown in Table 1 with N = 5. Now since (-3,2) is left of (2,-8), we set $(-3,2) = (x_1,y_1)$ and $(2,-8) = (x_2,y_2)$. Substitution yields the solution (-2,0), (-1,-2), (0,-4), (1,-6), (2,-8).

In conclusion, the process of deriving the midpoint formula gives an excellent opportunity to generalize the important procedure of dividing a line segment. In fact, the results are so nice that dividing a line segment into "N" equal parts can be as easy (and mechanical) as simply bisecting a line segment. The generalization is far more useful than the one specific case and greatly expands the student's understanding and problem solving ability.

FIGURE 1

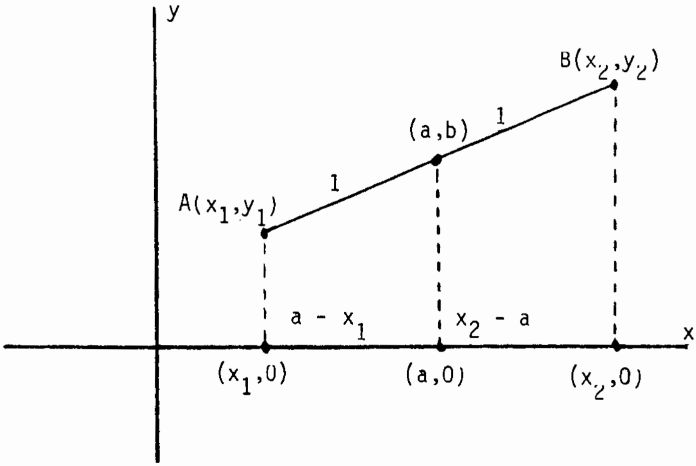

FIGURE 2

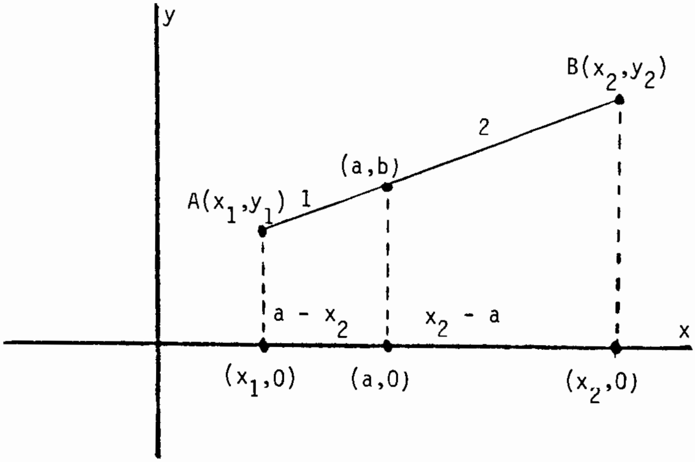

FIGURE 3

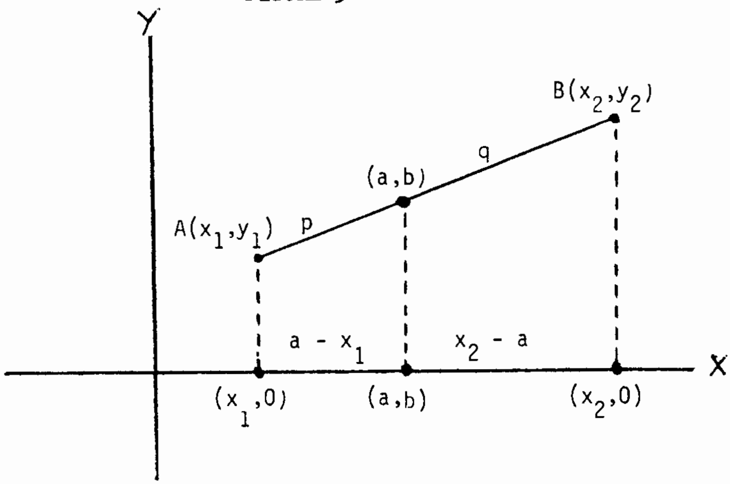

FIGURE 4

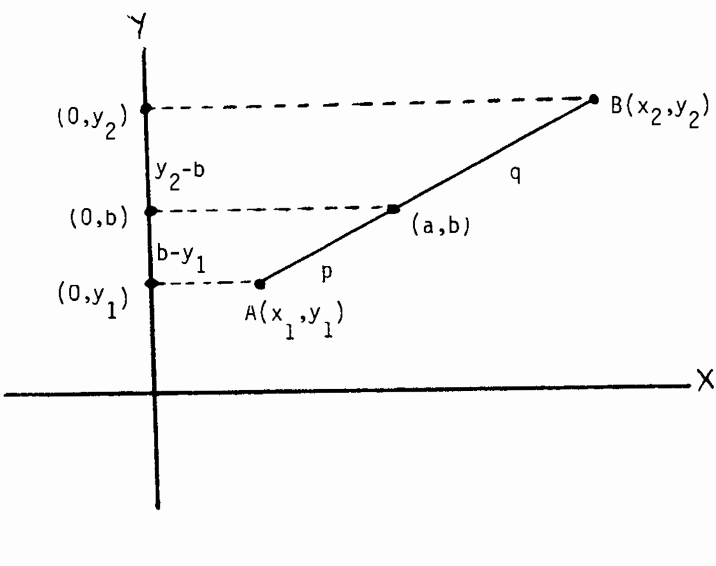

TABLE 1*

N (number of parts after division)	$(a,b)\Big\vert_q^p$ or $\left(\frac{qx_1 + px_2}{p+q}, \frac{qy_1 + py_2}{p+q}\right)\Big\vert_q^p$
2	$\left(\frac{x1 + x2}{2}, \frac{y1 + y2}{2}\right)\Big\vert_1^1$
3	$\left(\frac{2x1 + x2}{3}, \frac{2y1 + y2}{3}\right)\Big\vert_2^1$ $\left(\frac{x1 + 2x2}{3}, \frac{y1 + 2y2}{3}\right)\Big\vert_1^2$
4	$\left(\frac{3x1 + x2}{4}, \frac{3y1 + y2}{4}\right)\Big\vert_3^1$ $\left(\frac{2x1 + 2x2}{4}, \frac{2y1 + 2y2}{4}\right)\Big\vert_2^2$ $\left(\frac{x1 + 3x2}{4}, \frac{y1 + 3y2}{4}\right)\Big\vert_1^3$
5	$\left(\frac{4x1 + x2}{5}, \frac{4y1 + y2}{5}\right)\Big\vert_4^1$ $\left(\frac{3x1 + 2x2}{5}, \frac{3y1 + 2y2}{5}\right)\Big\vert_3^2$ $\left(\frac{2x1 + 3x2}{5}, \frac{2y1 + 3y2}{5}\right)\Big\vert_2^3$ $\left(\frac{x1 + 4x2}{5}, \frac{y1 + 4y2}{5}\right)\Big\vert_1^4$
6	$\left(\frac{5x1 + x2}{6}, \frac{5y1 + y2}{6}\right)\Big\vert_5^1$ $\left(\frac{4x1 + 2x2}{6}, \frac{4y1 + 2y2}{6}\right)\Big\vert_4^2$ $\left(\frac{3x1 + 3x2}{6}, \frac{3y1 + 3y2}{6}\right)\Big\vert_3^3$ $\left(\frac{2x1 + 4x2}{6}, \frac{2y1 + 4y2}{6}\right)\Big\vert_2^4$ $\left(\frac{x1 + 5x2}{6}, \frac{y1 + 5y2}{6}\right)\Big\vert_1^5$
N	$\left(\frac{(N-1)x1 + x2}{N}, \frac{(N-1)y1 + y2}{N}\right)$ $\left(\frac{(N-2)x1 + 2x2}{N}, \frac{(N-2)y1 + 2y2}{N}\right)$ . . . $\left(\frac{x1 + (N-1)x2}{N}, \frac{y1 + (N-1)y2}{N}\right)$

*Note: the numbers <u>following</u> the x's and y's in this table are subscripts.