THE OHIO JOURNAL OF SCIENCE

VOL. LII | OCTOBER, 1952 | NO. 6

CRITERIA FOR THE CONTINUITY OF LINEAR FUNCTIONALS IN REAL LINEAR SPACES[1]

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1. INTRODUCTION

Many criteria for the continuity of linear functionals in various linear topological spaces are known, for example, in Banach spaces the Hilbert—boundedness, expressed by the formula $|U(x)| \leqslant M||x||$, $M > O$, is equivalent to the continuity of $U(x)$.

V. L. Klee, Jr., and the author have found, independently from one another, that these criteria have a common source in some very general topological properties of the space and Klee has even disclosed that the intersection property of open sets is many times irrelevant or can be replaced by a weaker condition (see condition (S) below). The main purpose of this paper is to throw more light on this topic. We shall deal with spaces endowed with an operation invariant paratopology (i.e., Klee's (α)—space; Klee, Jr., 1951) and with a "symmetry"—condition (S') which is slightly weaker than Klee's condition (S), (see below).

It is striking how little is required to assure the continuity, and that only a small drop of a condition is sufficient for it (see Th. 8.2). This fact can be explained by the circumstance that a linear space is an abelian group, and being a group is a very strong property.

We shall give necessary and sufficient conditions for the continuity of linear functionals in spaces much more general than the linear topological spaces. These cover all known criteria in the last ones. The proofs are very simple.

As a side result we give a generalization of the known theorem by La Salle (1941) on the existence of a non-trivial linear continuous functional.

Some properties of convex sets obtained by the author in collaboration with W. D. Berg (in infinite dimensional spaces where no topology was supposed, in 1949–1951, Berg and Nikodým, 1952) have shown themselves useful in the underlying work. As the tool we use ordering limits which are more general than Moore-Smith directed set limits (Moore and Smith, 1922; Birkhoff, 1937).

2. TERMINOLOGY AND NOTATIONS

To avoid misunderstanding we give some definitions and explanation of symbols to be used.

[1]Some results of this paper were presented to the 56th Summer Meeting of the American Mathematical Society in Minneapolis, Minnesota, September 7, 1951.

By a *linear space* we understand an abelian group L with real multipliers, (see e.g., Banach, 1932). The elements of the linear space are termed ***points*** or *vectors*. The null-vector is denoted by θ.

By a *linear variety in* L we understand a non empty set E of vectors such that, if $x, y \in E$ and λ, μ are real numbers, then $\lambda x+\mu y \in E$. Any translation of a linear variety will be termed *flat*.

By a ***hyperplane*** we understand a linear variety F in L such that (1) $F \neq L$, (2) there exists a vector x_o such that the smallest linear variety $\{F, x_o\}$ spanned by F and x_o coincides with L. A translation of a hyperplane is termed *hyperflat*.

If G is a non empty set of vectors, λ a number, and x a vector, then $G + x$, $[\lambda G]$ denotes the set of all vectors $y + x$ [resp. λy] where $y \in G$. If G_1, G_2 are not empty sets of vectors, then $G_1 + G_2$ denotes the set of all vectors $x_1 + x_2$, where $x_1 \in G_1$ and $x_2 \in G_2$.

The empty set (or class) is denoted by Λ, the union of sets E, F is denoted by $E \cup F$, and their intersection by $E \cap F$. The symbol $E \subseteq F$ means that E is a subset of F. If $E \subseteq F$, $E \neq F$, we write $E \subset F$. The union of all sets E belonging to a not empty class K of sets denoted by $\bigcup E$, $(E \in K)$. The symbol $x \bar{\in} E$ means that x does not belong to E.

By a *convex set* we understand a set C such that if $x, y \in C$, $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda + \mu = 1$, then $\lambda x + \mu y \in C$. A convex set may be empty. By a *convex hull* of a set G of points we understand the smallest convex set containing G.

The *linear functional in* L is a real valued function $U(x)$ defined for all $x \in L$ and satisfying the condition $U(\lambda x + \mu y) = \lambda U(x) + \mu U(y)$ for any real λ, μ and any vectors $x, y \in L$.

A linear functional is called *trivial*, if $U(x) = 0$ for all $x \in L$. By a *topology* in L we understand a class T of subsets of L satisfying the following conditions (1) $\Lambda \in T$; (2) $L \in T$; (3) if $E, F \in T$, then $E \cap F \in T$; (4) if K is not any empty class of sets of T, then

$$\bigcup E, \ (E \in K)$$

also belongs to T. If we drop (3), we obtain a ***paratopology***, (*p top.*), (Klee, 1951). The sets of T are termed (T)–***open*** *sets*, and their complements with respect to L, (T)–*closed sets*. No separation axiom is required, neither the continuity of the vector-operations.

If **R** is a relation (Russell and Whitehead, 1910), by its *domain* we understand the set $\mathfrak{D}$**R** of all elements a such that there exists b with a **R** b; by its range $\mathfrak{R}$ we understand the set of all elements b such as there exists a with a **R** b.

A relation **R** is termed *ordering* if (1) $\mathfrak{D}$**R** = $\mathfrak{R}$**R** and is not empty, (2) a **R** b, b **R** c imply a **R** c, (3) if $a \in \mathfrak{D}$**R**, then a **R** a, (4) a **R** b, b **R** a is equivalent to $a = b$ for any $a, b \in \mathfrak{D}$**R**.

A function $f(a)$ or $\{f_a\}$, defined for all $a \in \mathfrak{D}$**S** (where **S** is an ordering) is termed **S** – *sequence*.

If the range of an **S**-sequence belongs to a set provided with a p. top., we say that $\{f_a\}$ *tends to* g if for every open set α, containing g, there exists $a_o \in \mathfrak{D}$ **S** such that for every a with a_o **S** a, we have $f_a \in \alpha$. We write $f_a \rightarrow g$. We call g **S**-*limit*. Notice that g may be not unique.

If L is a linear space provided with a p. top. and G is a set of points in L, than by the *closure* $\bar{G}$ of G, we understand the set of all limits of all **S**-sequences $\{x_a\}$, for all orderings **S**, $x_a \in G$. A real valued function $\Phi(x)$ defined on L is said to be *continuous at* x_o with respect to (T) if for every $\epsilon > O$ there exists a (T)–open set U with $x_o \in U$ such that for every $x \in U$ we have

$$|\Phi(x) - \Phi(x_o)| < \epsilon.$$

A necessary and sufficient condition for the continuity of Φ at x_o is that for every ordering sequence $\{x_a\}$ we have $\Phi(x_a) \rightarrow \Phi(x_o)$.

For a function $\Phi(x)$ defined on L with a p. top. (T) the following are equivalent: (1) $\Phi(x)$ is continuous at each $x_o \in L$, (2) for every real λ and μ the set $x \mid \lambda < f(x) < \mu$ is (T)-open.

3. SOME PROPERTIES OF LINEAR FUNCTIONALS

Let $f(x)$ be a linear functional in L, and α a real number. We define

$$E_\alpha = \{x | f(x) = \alpha\}.$$

We easily see that $E_\alpha = E_o + x_\alpha$, where x_α is any point of E_α.

If $\alpha \neq \beta$, then $E_\alpha \cap E_\beta = \Lambda$. We have

$$L = \bigcup E_\alpha, (-\infty < \alpha < +\infty).$$

If $f(x)$ is not trivial, then E_α is a hyperflat. If L_1 is a hyperplane, $x_o \bar{\in} L - L_1$, $s \neq O$, then there exists a unique linear functional $f(x)$ such that

$$f(x_o) = s, E_o = L_1.$$

Proof. Since x_o is independent of L_1, any vector $x \in L$ possesses a unique representation

$$x = y + \lambda x_o \text{ where } y \in L_1.$$

Put

$$f(x) = \lambda s.$$

One proves readily that $f(x)$ is a linear functional having the required properties. The uniqueness is obvious.

4. SPACES WITH THE TRANSLATION, MULTIPLICATION AND (S') PROPERTY

Let L be a linear space endowed with a p. top. (T). We say that (T) has the *translation property*, if for every (T)–open set E and for every vector x_o the translation $E + x_o$ is also (T)–open. We say that (T) has the *multiplication property*, if for every $\lambda \neq O$ and every (T)–open set, the set $\lambda . E$ is also (T)–open. If (T) has these two properties, and **R** is an ordering, we have:

(1) if $x_\alpha \rightarrow x$, then $x_\alpha + y \rightarrow x + y$;

(2) if $x_\alpha \rightarrow x$, then $\lambda x_\alpha \rightarrow \lambda x$ for every real λ.

Remark. It may happen that $x_\alpha \rightarrow x_o$, $y_\alpha \rightarrow y_o$, but $x_\alpha + y_\alpha$ may not tend to $x_o + y_o$. Similarly it may happen that $x_\alpha \rightarrow x_o$, $\lambda_\alpha \rightarrow \lambda_o$, but $\lambda_\alpha x_\alpha$ may not tend to $\lambda_o x_o$.

4. Theorem. If (1) L is a linear space,

(2) (T) a p. top. on it possessing the translation and multiplication property,

(3) $f(x)$ a linear functional on L,

(4) $x_o \in L$,

(5) $f(x)$ is (T)–continuous at x_o,

then $f(x)$ is (T)–continuous everywhere in L.

Proof. Let $x_a \to y$. We have $x_a - y + x_o \to x_o$, hence

$$f(x_a - y + x_o) \to f(x_o), \text{ and then}$$

$$f(x_a) - f(y) \to 0, \text{ which completes the proof.}$$

Thus the linear functional is either everywhere continuous or everywhere discontinuous.

Let us remark that the space with a p. top. satisfying these conditions of operation-invariance, coincides with the Klee's (α)-space characterized by the condition that $f(x) = y + rx$, where x is a variable vector, is continuous for every vector y and every real number r.

We say that a p. top. (T) on L has the *property* (S'), if $G \in T$, $\theta \in G$ imply the existence of G' such that $\theta \in G' \in T$, $G' \subseteq G \cap (-1)\, G$. This condition is slightly more general that the Klee's condition (S) stating that if $\theta \in G \in T$, then $G \cap (-1)\, G \in T$.

5. A CRITERION FOR THE CONTINUITY OF LINEAR FUNCTIONALS

We shall prove the theorem:

If (1) L is a linear space,

(2) (T) a p. top. on L possessing the translation and multiplication property,

(3) $f(x)$ a linear non-trivial functional on L, $M_o = \{x \mid f(x) = o\}$,

then the following are equivalent:

I. $f(x)$ is (T)–continuous,

II. There exists an open set $G \neq \Lambda$ contained in a strip parallel to M_o.

Proof. Suppose I. The set $\{x \mid -1 < f(x) < 1\}$ is open and is a strip parallel to M_o, so II follows.

Suppose II. Let $G \neq \Lambda$ be an open set in the strip $\langle M_1, M_2 \rangle$ in which the boundary-hyperflats M_1, M_2 are different, parallel to M_o, and included in $\langle M_1, M_2 \rangle$. The set $G_1 = M_o + G - x_o$, where $x_o \in G$, is open, for it is the union of translations of G. It is also the union of some hyperflats parallel to M_o. We have $\theta \in G_1$. It follows that $G_2 = \bigcup \lambda G_1$, $(0 < \lambda \leqslant 1)$, is a strip parallel to M_o, and then

$$G_3 = \bigcup \left(1 - \tfrac{1}{n}\right) G_2, \ (n = 1, 2, \ldots),$$

is so. G_3 is also parallel to M_o. By a suitable translation G_3 can be transformed into another strip G_4 with $\theta \in G_4 = (-1)G_4$. This strip does not contain its boundary which is composed of two hyperflats, M_1', M_2' parallel to M_o, and is a (T)–open set. Hence the linear functional $g(x)$, defined by putting $g(x) = 1$ for $x \in M_1$, is (T)–continuous. Indeed $k . G_4 + x_1$ where $k > 0$, and x_1 is any vector, is (T)–open so the inverse image $g^{-1}(\alpha, \beta)$ is an open set for any α, β where $\alpha < \beta$.

We have $M_o = \{x \mid g(x) = 0\}$ and then $f(x) = \lambda . g(x)$ for a suitable constant, and consequently $f(x)$ is (T)–continuous.

The above theorem implies the following one:

5.1. Theorem. If L is a linear space provided with a p. top (T) having the translation, multiplication and (S') property, then a necessary and sufficient condition that a linear functional on L be (T)–continuous is that there exists at least one (T)–open set $G \neq \Lambda$ on which $f(x)$ is bounded from above (from below).

Since every linear topological space (see Kolmogoroff 1934, and v. Neumann 1935) has the translation, multiplication and (S') property, the criterion worded in theorem 5.1, applies to all linear topological spaces.

6. GENERALIZATION OF A THEOREM BY LA SALLE

The obtained very general theorem allows us to generalize the following by La Salle (1941); (concerning examples of lin. pop. spaces where every continuous linear functional is trivial, see Nikodým, 1931, and Day, 1940).

If L is a linear topological space then the following are equivalent:

I. There exists in L at least one linear continuous functional not identically equal O;

II. There exists in L at least one not empty open convex set which does not coincide with L.

We first prove the following Klee's lemma:

6.1. If (1) L is a linear space with a p. top (T) satisfying the translation and multiplication condition,

(2) G is a (T)–open set of points,

then the convex hull of G is also a (T)–open set.

Proof. Let F be the convex hull of G. We easily prove (Berg and Nikodým, 1952), that

$$F = \bigcup \langle x_1, x_2, \ldots, x_n \rangle \ (n = 1, 2, \ldots; x_1, x_2, \ldots, x_n \in G),$$

where $\langle x_1 \ldots, x_n \rangle$ is the set of all points

$$y = \Sigma \ \alpha_{i=1}^{n} x_i \text{ where } \alpha \geqslant 0, \ \Sigma_{i=1}^{n} \alpha_i = 1.$$

Let $x \in F$. We have

$$x \in \alpha_1 G + \alpha_2 G + \ldots + \alpha_m G \subseteq F \text{ for some } m \text{ and } \alpha_i, \text{ for which } \alpha_i > 0, \ \alpha_1 + \ldots + \alpha_m = 1.$$

By the multiplication property, $\alpha_i G$ is open. Now we notice that, if E, F are open, so is $E + F$. Indeed, $E + F = \bigcup (E + y)$, $(y \in F)$, and by the translation property $E + y$ is also open. Having this, we prove, by induction, that $\alpha_i G + \ldots + \alpha_m G$ is open, which completes the proof.

6.2. Suppose there exists a (T)–open set $G \neq \Lambda$ contained in a halfspace N whose boundary, which is a hyperflat, be denoted by ρ.

Let $\rho_0 = \rho + (-z_0)$, where $z_0 \in$ p.

ρ_0 is a hyperplane. Let $y_0 \in N$ but $y_0 \in \rho$.

The vector $y_0 - z_0$ is independent of ρ_0. Let P be the set of all vectors

$$(y_0 - z_0) \ . \ \lambda + y_1 \text{ where } \lambda \geqslant 0, \ y_1 \in \rho_0.$$

P is a halfspace whose boundary is ρ_0. The set

$$N_0 = \bigcup (G + x) \, , (x \in F)$$

is a halfspace containing G and a (T)–open set too.

Let $u_0 \in N_0$, $u_0 \in \rho_0$. There exists a linear functional $f(x)$ on L such that $f(x) = 0$ for $x \in \rho_0$ and $f(u_0) = 1$. This functional is bounded from above or from below on the open set N_0, hence, by theorem 5.1., it is a (T)–continuous functional on L.

Now suppose that there exists a linear continuous functional $f(x)$ with $f(x) \neq O$. The set $\{x | f(x) > O\}$ is a (T)–open set and a halfspace.

If we take account of the lemma, we can state the following:

6.3. Theorem. If L is a linear space provided with a p. top. (T) having the translation, multiplication and (S') property, then the following are equivalent.

I. There exists on L a (T)–continuous linear functional not identically equal O;

II. There exists a not empty (T)–open set contained in a half-space;

III. There exists a non empty (T)–open convex set contained in a halfspace.

6.4. The theorem of La Salle is covered in this theorem. Indeed his theorem deals with linear topological spaces, and they all possess the translation, multiplication and (S') property. A not empty open set contains necessarily a linearly inner point, i.e., a point x_0 such that every straight line passing through it contains an open segment which contains x_0. Consequently (Berg and Nikodým, 1952) every convex not empty open set is a convex body. If a convex body does not coincide with the whole space, it possesses at least one boundary point y_0.

Now Dieudonné (1941) has proved that there exists a hyperflat ρ passing through y_0 and such that the convex body is contained in one of the two halfspaces determined by ρ. Hence, if there exists in L at least one open convex set $\neq \Lambda$ not identical with whole space, then there exists a linear continuous not trivial functional. The converse theorem is obvious.

7. SPACES WITH THE TRANSLATION, MULTIPLICATION AND STAR-PROPERTIES

We start with an auxiliary theorem.

Theorem. If (1) L is a linear space provided with a p. top. (T) having the translation , multiplication and (S')-property,

(2) $f(x)$ is a (T)–discontinuous linear functional on L,

(3) $\theta \in G \in T$, then there exists an stream-sequence $x_a \rightarrow \theta$, such that $f(x_a) \rightarrow +\infty$, $x_\alpha \in G$.

Proof. Define the relation **R** (Whitehead and Russell, 1910), whose field is the class of all (T)–open sets, containing θ and $\in G$, in the following way:

$$U_1 \mathbf{R} U_2 . = . U_2 \subseteq U_1.$$

R is an ordering.

Let **S** be a relation whose field is composed of all ordered couples (U, p) where $U \in \mathbf{\Phi} \mathbf{R}$ and $p > 1$. We define

$$(U_1, p_1) \mathbf{S} (U_2, p_2) \text{ by } U_1 \mathbf{R} U_2, p_1 \leqslant p_2.$$

S is also an ordering. We have supposed that $f(x)$ is discontinuous. Hence, by theorem 5.1., on each $U \in \mathbf{\Phi} \mathbf{R}$, the functional $f(x)$ is not bounded from above. Hence for every U and $p > 1$ there exists $x_{U, \mathrm{p}}$ with $f(x_{U, \mathrm{p}}) \geqslant p$, $x_{U, \mathrm{p}} \in U$.

We see that the **S**–sequence $\{x_{U, \mathrm{p}}\}$ tends to θ. But we also see that $f(x_{U, \mathrm{p}}) \rightarrow +\infty$. This completes the proof. The above theorem yields another condition for continuity of a linear functional, easy to word.

7.1. Let E be a set of points in L and x_0 a point $\in E$. We say that E is *starred at* x_0, if for every $x \in E$, $x \neq x_0$, the closed segment $\langle x_0, x \rangle$ belongs to E.

We say that a p. top. (T) *has the star-property*, if for every (T)–open set G containing x_0 there exists a (T)–open set G_1 starred at x_0 such that $G_1 \subseteq G$.

It is not in general true that a space L with a Hansdorff topology, having the translation and multiplication property, has also the star property. Later on we shall have an example showing this. Following the lines of the construction of this example, we can even have an example on the two-dimensional plane.

8. ANOTHER CRITERION FOR THE CONTINUITY OF LINEAR FUNCTIONALS

The above auxiliary theorem yields the

Theorem. If (1) L is a linear space provided with a p. top. (T) possessing the translation, multiplication, (S^1) and star property,

(2) $f(x)$ is a linear discontinuous functional on L,

(3) $G \neq \Lambda$ is a (T)–open set,

(4) a a number,

then there exists in G a point x_0 such that $f(x_0) = a$.

Proof. Suppose that

$$\theta \in G. \qquad (1)$$

and take $a > 0$.

Since (T) has the star property, we can find an open set G_1 starred at θ with $G_1 \subseteq G$.

Since $f(x)$ is discontinuous, there exists an ordering **R** and an **R**-sequence $\{x_a\}$ with $x_\alpha \rightarrow \theta$, $x_\alpha \in G_1$ and $f(x_a) \rightarrow +\infty$, (therom 7).

There exists γ such that, if γ **R** α, we have

$$f(x_\alpha) \geqslant a.$$

We have

$$x\,\gamma \in G_1, f(x\,\gamma) \geqslant a > 0.$$

Put

$$y = a \cdot \frac{x\,\gamma}{f(x\,\gamma)}.$$

We have $y \in G_1$, because $0 < \dfrac{a}{f(x\,\gamma)} \leqslant 1$,

and we have

$$f(y) = \frac{a}{f(x\gamma)} \cdot f(x\,\gamma) = a,\ y \in G.$$

Thus, if $a > 0$, then there exists, in G, a point at which $f(x)$ admits the value a.

Let $a < 0$. The functional $g(x) = -f(x)$ is also discontinuous; hence there is, in G, a point z at which $g(z) = -a$. Hence $f(z) = a$.

Since $f(\theta) = 0$, $\theta \in G$, we conclude that for every a there exists $y \in G$ with $f(y) = a$.

Now let G be an arbitrary not empty (T)–open set. Let $x_0 \in G$. Put

$$G_1 = G + (-x_0).$$

Since G_1 is an open set containing θ, there exists y with $y \in G_1$, $f(y) = a - f(x_0)$.

Putting $z = y + x_0$,

we have

$$f(z) = f(y) + f(x_0) = a, \text{ and } z \in G_1 + x_0 = G.$$

The theorem is proved.

8.2. As a consequence we have the

Theorem. If (1) L is a linear space provided with a p. top. (T) possessing the translation, multiplication, (S') and star properties,

(2) $f(x)$ is a linear functional on L,

the following are equivalent:

I. $f(x)$ is (T)—continuous;

II. There exists a number a and a (T)–open not empty set G such that $f(x)$ does not admit the value of a on G.

8.3. Let us remark that if we drop the star property, the thesis may be not true.

E.g. Let L be a real infinite-dimensional Hilbert space with usual metric topology (T) given by the norm. The open unit sphere S:

$$||x|| < 1$$

is a convex body in L.

Now the following theorem is true (Berg and Nikodým, 1952): If E is a convex body in a linear space, then there exists a hyperplane ρ such that every hyperflat obtained by a translation of ρ has at least one common point with E.

Let ρ_o be such a hyperplane in L for the convex body S. Take a vector ξ independent of ρ_o, and put

$$\rho_1 = \rho_o + \xi.$$

Let $f(x)$ be the linear functional defined on L by the conditions

$$f(x) = 0 \text{ on } \rho_o, f(\xi) = 1.$$

Consider the set

$$A = \bigcup_{k=1}^{\infty} \frac{1}{k} \rho_1,$$

and take the class K of all sets

(1) $\ldots . G - \bigcup_{p=1}^{n} (\alpha_p A + x_p)$, $(n = 1, 2, \ldots .)$, where G is a (T)–open set, x_p a vector and α_p a real number $\neq 0$. Denote by (T') the smallest topology containing K. It possesses the translation and multiplication property and is a Hausdorff topology.

The functional $f(x)$ is (T')–discontinuous.

Indeed let $G' \neq \Lambda$ be a (T')–open set. There exists $G_1 \in K$ with $G_1 \neq \Lambda$ and $G_1 \subseteq G'$. G_1 has the form (1) where $G \in T$, $G \neq \Lambda$. Since G is a (T)–open set, $\neq \Lambda$, there exists $\beta > 0$ and y_o such that $S_1 = y_o + \beta S \subseteq G$. Since every hyperflat parallel to δ_o cuts S, every hyperflat parallal to δ_o also cuts S_1. As there is an at most denumerable number of hyperflats, $\delta_1, \delta_2, \ldots .$ parallel to δ_o and such that

$$G_1 = G - \bigcup_m \delta_m,$$

it follows that all hyperflats parallel to δ_o cut G', excepting perhaps a denumerable number of them. Hence $f(x)$ is neither bounded from above nor from below on G_1, and then, by theorem 5.1, $f(x)$ is (T')–discontinuous.

Nevertheless $f(x)$ does not admit the value $\frac{1}{2}$ on the (T')–open set $S - A$.

8.4. As a consequence we have the following theorem:

Theorem. If (1) L is a linear space provided with a p. top. (T) having the translation, multiplication, (S') and the star property,

(2) $f(x)$ is a linear functional on L,
then the following are equivalent:

I. $f(x)$ is (T)—continuous,

II. The set $\{x|f(x) = O\}$ is (T)—closed.

Proof. II follows easily from I. Put

$$E_o = \{x|f(x) = O\}.$$

The set $L - E_o$ is (T)–open, and on it $f(x)$ does not admit the value O. Hence by theorem 8.2, $f(x)$ is (T)–continuous.

8.5. Every linear topological space (Wehausen, 1938, v. Neumann, 1935) possesses not only the translation and the multiplication property, but it also has the star property (Hyers, 1939). Consequently the theorems 8.2 and 8.4 hold for them, and then they hold for all Fr. Riesz-Banach spaces (Banach, 1932) and for the F—spaces (Banach, 1932).

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The Ohio Journal of Science award for excellence in research will be given this year to a member of the Zoology Section of The Ohio Academy of Science. Information about eligibility and rules may be found on page 216 of the July, 1952, number of the Journal.