OCTM

Contest Corner

Problem Solving is About Seeing Relationships

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Introduction

"The *Standards for Mathematical Practice* describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important 'processes and proficiencies' with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy)." - *Common Core State Standards Initiative: Preparing Students for College & Careers*

The Ohio Department of Education adopted the *Common Core State Standard*, (CCSS), on June 7, 2010. The gradual transition to the CCSS must be completed by 2014. It is recommended that districts begin this process now, since the State Board adopted the model curriculum in March 2011.

What does this mean for mathematics teachers? It means that mathematics education must consist of rich problems that will promote students' efficiency in problem solving, reasoning and proof, and the ability to communicate one's mathematical thinking through proper terminology and representations. Constructivism, the learning theory described in the practice standards, demands that students not only develop procedural fluency, but a conceptual understanding of mathematical concepts and operations. It is important that teachers understand the practice standards to ensure that the content standards are introduced, understood, and communicated by both the instructor and the students in the manner called for by these standards. So let's take a look at the CCSS Practice Standards for Mathematics, as written by the Common Core State Standards Initiative, 2011.

Make sense of problems and persevere in solving them

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary.

Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Reason abstractly and quantitatively

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: (a) the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents, and (b) the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Construct viable arguments and critique the reasoning of others

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Model with mathematics

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.

Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Use appropriate tools strategically

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.

Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Attend to precision

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Look for and make use of structure

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspectives. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.

Look for and express regularity in repeated reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $\frac{y-2}{x-1} = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

The *Standards for Mathematical Practice* describe ways in which students ought to engage with mathematics content as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development must connect the mathematical practices to mathematical content in mathematics instruction, (CCSS Initiative, 2011). We illustrate this idea in the following section through careful consideration of the *Film Developing Problem* (Gross, 1999).

The Film Developing Problem (Gross, 1999)

Consider the following task taken from *Math as a Second Language, Vermont Math Initiative* (Gross, 1999).

> There are two photography stores in town that do custom film developing, *Perfect Picture* (abbreviated PP) and *Dynamic Developers* (abbreviated DD). At PP the cost to develop one roll of specialty film is $12 but any additional rolls of film cost only $10. At DD the cost of developing one roll of film is $24 but each additional roll is developed at a cost of only $8. For what number of rolls of film is the cost of developing the same at PP and DD?
>
> A) Organize the data in a table to determine at what number of rolls of film the cost is the same.
>
> B) Graph the two photography stores cost on the same graph. Give the coordinates for the point of intersection and explain its significance in this problem.
>
> C) Develop an equation for the total cost of *n* number or rolls of film for each photography store.

The problem, as stated, removes the problem solving component from the problem. The reader is given the three ways to solve the problem and each student is expected to solve the problem in all three ways. By simply removing parts A), B) and C) the reader is left with a more engaging problem, that requires the student to read the problem, decipher the information and devise a plan to find a solution.

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This problem now has multiple entry points, allowing students to approach this problem from a basic mathematics, geometry, or algebraic perspective. Having the student construct viable arguments and present his/her finding to a small group or the whole class, allows all students to evaluate the thought processes of others and decide if the presented solution makes sense. Through students sharing of solutions, all perspectives may be presented so that students can evaluate the various ways to solve the problem. In this problem, students will be encouraged to make sense of the mathematics content by constructing their own solution and sharing the solution for others to validate or dispute. Students become the source of truth and the instructor becomes the facilitator.

The Travel Problem is another example of a problem that is stated in such a way that the student is not required to produce a means to solve the problem, but must use the stated method to find the solution.

The Travel Problem (Long, DeTemple, Millman, 2009)

Consider the following task taken from *Mathematical Reasoning for Elementary Teachers* (Long, DeTemple, Millman, 2009).

Following a weekend in Cincinnati, Emily, LaToya, and Krystal end their travel together at the intersection of I-71 and I-75 (the Brent Spence Bridge). From that intersection, Emily travels north toward Cleveland at a speed of 55 miles per hour, LaToya travels south toward Lexington at a speed of 70 miles per hour, and Krystal drives west towards Indiana at a speed of 65 miles per hour.

A) Graph the location of each person on a coordinate grid after one hour, with the intersection of I-71and I-75 being the origin. Calculate how far apart Emily and LaToya are after one hour. How far apart and Emily and Krystal after one hour? Show your work.

B) How far apart are Emily and Krystal after two hours? How far apart are LaToya and Krystal after two hours? Show your work.

C) Write an equation representing how far apart Emily and Krystal are after *t* hours. Write an equation for how far apart LaToya and Krystal are after *t* hours. Which of the two friends are farthest apart after driving consistently for three hours at the given speeds?

This problem would more closely align to the CCSS for Mathematics Practice Standards, if worded in the following way.

Following a weekend in Cincinnati, Emily, LaToya, and Krystal end their travel together at the intersection of I-71 and I-75 (the Brent Spence Bridge). From that intersection, Emily travels north toward Cleveland at a speed of 55 miles per hour, LaToya travels south toward Lexington at a speed of 70 miles per hour, and Krystal drives west towards Indiana at a speed of 65 miles per hour. Which of the two friends are farthest apart after driving consistently for three hours at the given speeds?

As educators seek to design instruction that produces mathematically proficient students, in both the *Mathematical Practice Standards* and *Mathematical Content Standards*, teachers and contest writers must assure that they are not taking the problem solving out of problems. Rich problems must promote students efficiency in problem solving, reasoning and proof, and the ability to communicate one's mathematical thinking through proper terminology and representations. The *Contest Corner* will continue to seek such problems that will promote the CCSS of *Mathematics Practice Standards*.

References

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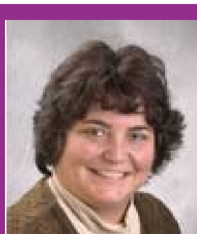

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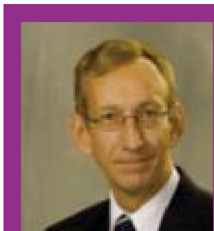

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