INTRODUCING ALGEBRAIC FACTORING USING A GRAPHIC APPROACH

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Introduction

One of the most common topics in beginning algebra is factoring. In fact, factoring is an elementary problem solving technique used in the solution of a variety of problems from elementary to complex. However, most common algebra textbooks treat factoring as a process without giving it a meaning. Factoring is frequently reduced to following a set of rules or procedures.

Many authors begin factoring with a set of rules. One such set might be as follows:

$$x^2 - a^2 = (x - a)(x + a)$$
$$x^2 - 2ax + a^2 = (x - a)(x - a) = (x - a)^2$$
$$x^2 + 2ax + a^2 = (x + a)(x + a) = (x + a)^2$$

This type of presentation usually progresses to a set of exercises that have been rigged to fit the factoring rules. The typical presentation then progresses to a guessing technique for factoring more general polynomials of the form $ax^2 + bx + c$. This common method involves looking at factors of "a" and "c" and trying to write appropriate factors. For example, in factoring $x^2 - x - 6$ a student might guess $(x + 6)(x - 1)$, $(x - 6)(x + 1)$, $(x + 2)(x - 3)$, or $(x + 3)(x - 2)$ and check for accuracy using the FOIL distribution process.

In other words, factoring is frequently reduced to a set of seemingly mindless rules that are soon forgotten by many students. Certainly we want students to be able to factor quickly and with little thought. However, the best learning occurs when the student can relate a process to an experience. In this case, relating factoring to a graphic interpretation.

The Graphic Approach

A better approach in introducing factoring involves graphing. In graphing, students can see a picture of their equation. The rudiments of graphing are introduced early in the algebra curriculum. Students should be able to plot points to see a picture of a function. Factoring can be taught with more meaning using this knowledge. Further, the early integration of computer graphing software or a graphing calculator like the HP-28S or the Casio fx-7000G is strongly recommended. This approach to factoring makes excellent use of available technology.

The graphic approach is a great introduction to factoring. Begin by having the students plot a graph of an equation, like $y = x^2 - x - 6$, by hand using an appropriate domain like from -5 to +5 (see Figure 1). After the students have their graphs, the teacher could also graph the function at the chalkboard followed by having the computer or calculator draw the graph. If students are already familiar with the computer or calculator, graphing by hand might be eliminated.

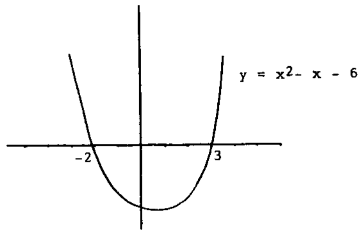

Figure 1

Ask the students to state the x-intercepts. In this case, they are -2 and +3. Write the factors $(x - (-2))(x - (+3)) = (x + 2)(x - 3)$ on the chalkboard. Point out the form of the factors:

$$(x - (\text{intercept}))(x - (\text{intercept})).$$

Now distribute the example to obtain

$$(x + 2)(x - 3) = x^2 - x - 6,$$

the original equation! Thus we have shown that $y = x^2 - x - 6 = (x + 2)(x - 3)$ and that $x^2 + x - 6$ factors to $(x + 2)(x - 3)$. This process clearly shows the relationship between the x-intercepts, or roots, and the factors of the polynomial part of the original equation. Try repeating the graphic approach with several other simple polynomial equations like $y = x^2 + 2x - 8$ and $y = x^2 + 3x - 4$.

After working with such problems for a day or so, try factoring the expression $x^2 - 4x + 4$ by graphing. The sketch of $y = x^2 - 4x + 4$ is shown in Figure 2. In this case, the vertex of the parabola is the x-intercept, x=2. This yields a double root. That is, the factor $(x - 2)$ occurs twice and $x^2 - 4x + 4 = (x - 2)(x - 2)$. A vertex on the x-axis yields a repeated factor. In the initial discussion with a class, it is not necessary to go into greater depth regarding repeated roots.

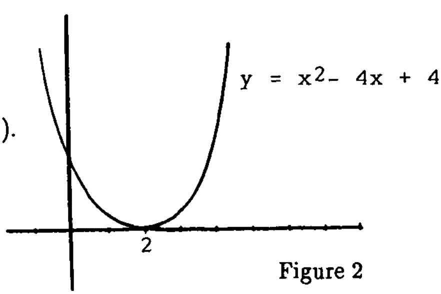

Figure 2

Having students see that the x-intercepts of the graph of their equation yield the factors of the equation gives added meaning to the factoring process. It is for this reason that the graphic relationship should be taught first – prior to factoring rules or trial-and-error factoring. For most students, factoring will have much more meaning because a graph will come to mind when equations are factored.

By using this approach, students can be taught to graphically factor more difficult expressions like $x^3 - x^2 - 6x$, $x^3 - x^2 - 9x + 9$, and $x^4 - 10x^2 + 9$. When the graphic approach is used, these expressions are as easy to factor as the quadratic expressions.

<u>Discussion</u>

It is important to note that although a graph can be used to determine the factors of an equation, the converse is not as easily achieved. For example, given an equation with roots of -3 and +2 says little about the actual graph or equation. Figure 3 shows three graphs with roots -3 and +2. In fact, there are infinitely many functions with roots -3 and +2.

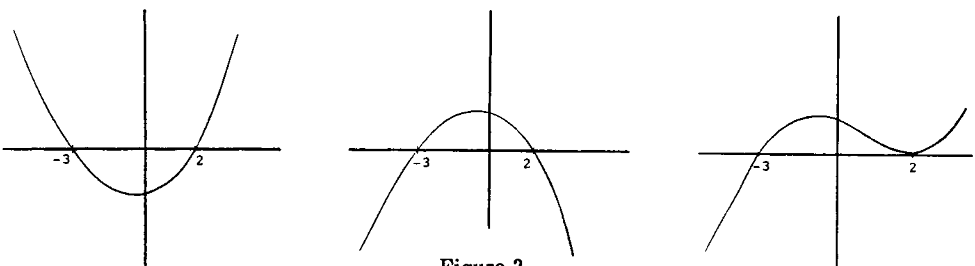

Figure 3

It is not necessary to explore all aspects of graphing with a class while introducing factoring. Graphing is a simple tool to help give meaning to factoring. It is not necessary initially to explore all of the unusual cases – non-integer real roots or irrational roots. You do not have to tell the students everything you know about graphing to use it effectively in the introduction of factoring. The unusual cases can be introduced later as needed.

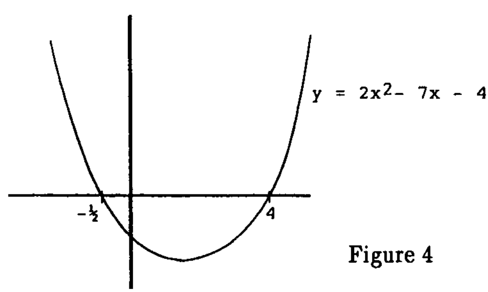

Figure 4

For example, factoring $2x^2 - 7x - 4$. The graph of $y = 2x^2 - 7x - 4$ is shown in Figure 4. The roots occur at -1/2 and +4. The factors are $(x - (-1/2))(x - (+4)) = (x + 1/2)(x - 4) \neq 2x^2 - 7x - 4$. However, if we clear $(x + 1/2)$ of the fraction by multiplying by 2, we get $2(x + 1/2) = (2x + 1)$. Substituting this for $(x + 1/2)$ gives $(2x + 1)(x - 4) = 2x^2 - 7x - 4$. It is appropriate to multiply through by 2 (or any number) since the roots occur at $y = 0$. In fact, we are multiplying both sides of the equation $(x + 1/2)(x - 4) = 0$ by 2.

Graphing also helps students understand why some expressions do not factor – because they do not have x-intercepts. For example, $x^2 + 2x + 2$. Examining the graph of $y = x^2 + 2x + 2$ (Figure 5) shows that the function is located entirely above the x-axis, has no x-intercepts, and does not factor.

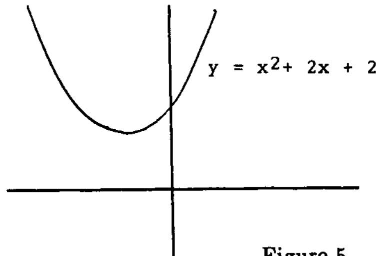

Figure 5

The graphic approach to factoring is not meant to replace all of the old techniques. The factoring rules, guess methods, and quadratic formula still have their place. The graphic approach is taught to add clarity to the process.

Conclusion

The introduction of low cost graphing software and hand-held graphing calculators invites many new approaches to old mathematics. The key is to allow the technology to help us find ways to reinforce mathematical concepts. Factoring is the one area that is greatly aided by the use of graphing. Students learn in depth concepts relating the abstract equation or expression to a concrete picture – the graph.

MORE PENTOMINOES

College Corner 5th graders reported 60 solutions to their pentomino puzzle in the Summer, 1989 issue of this Journal. That class, this year's 5th graders, and puzzling readers have boosted the total to 215! One of Christy's is shown. If you have solutions send them to Bethel Hooven, Union School, College Corner, OH 45003.