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Report No. 21

Studies on the Accuracy of the Computation of Gravity in High Elevations

by

Helmut Moritz

Prepared by

Geophysics Research Directorate
Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

Contract No. AF 19(604)-6201
Project No. 7600
Task No. 76002
(OSURF Project 1058)
Technical Paper No. 1058-7

The Ohio State University
Research Foundation
Columbus 12, Ohio

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W. A. Heiskanen, Director

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FOREWORD

This report was prepared by Dr. Helmut Moritz, Research Associate, of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF 19(604)-6201, OSURF Project No. 1058, under the supervision of Dr. Weikko A. Heiskanen, Director of the Institute. The contract covering this research is administered by the Geophysics Research Directorate, Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Engineers.

STUDIES ON THE ACCURACY OF THE COMPUTATION

OF GRAVITY IN HIGH ELEVATIONS

By Helmut Moritz

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STUDIES ON THE ACCURACY OF THE COMPUTATION OF GRAVITY IN HIGH ELEVATIONS

1. Introduction and Summary

The study of gravity in high elevations has gained practical importance during the last years. It is necessary, especially in connection with checking airborne gravimetric surveys, not only to compute these gravity values from gravity measured on the earth's surface, but also to know the accuracy obtainable.

In the following, a general theory of accuracy of these computations is sketched and numerical values given. Further investigations are to follow.

The central conception is the covariance function $C(s)$ which was introduced into gravimetry by R.A. Hirvonen [5]. The theory of errors and of their propagation is more complicated and superficially completely dissimilar to ordinary theory of errors--ordinary propagation of errors involves functions of one or several variables, but here we have integrations. But in reality there are intimate connections between these two cases.

After the theoretical foundations are summarized, some detailed investigations concerning different computational methods are given.

The errors arising from representation, interpolation and from replacing the integrals by sums are studied, and finally a possibility of computing the influence of topographical irregularities and of the neglected outer zones is outlined.

The results may provide an idea of the accuracy obtainable by different methods.

2. Statistical Foundations

2.1. The Error Function

The gravity anomaly Δg is a function of the geographical coordinates φ, λ:

$$\Delta g = \Delta g(\varphi, \lambda) .$$

We now assume that, for some reason, each gravity anomaly has an error ϵ, this error being also a function of φ, λ:

$$\varepsilon = \varepsilon(\varphi, \lambda) . \tag{1}$$

If this error ϵ is assumed to be a continuous function, then the errors in two neighboring points are necessarily correlated. The standard (or mean square) error m is defined by

$$m^2 = M(\varepsilon^2) , \tag{2}$$

M being the mean taken over all possible realizations of the error ϵ of one point. Because of the correlation the error covariance

$$(3) \qquad \sigma(P, P') = M(\varepsilon\varepsilon') = \sigma(\varphi, \lambda, \varphi', \lambda')$$

will not vanish everywhere, $\sigma(P,P')$ denoting the mutual influence of the errors $\epsilon=\epsilon(\varphi,\lambda)$ in point P and $\epsilon'=\epsilon(\varphi',\lambda')$ in P'. By (2) and (3),

$$\sigma(P,P) = \sigma(\varphi,\lambda,\varphi,\lambda) = m^2.$$

Function (3) is fundamental for the entire theory of errors concerning gravity anomalies. We call it "<u>error covariance function</u>" or shorter, "<u>error function</u>."

If we assume the same conditions everywhere on the sphere and, furthermore, in every direction then $\sigma(P,P')$ will depend only on the spherical distance s of points P and P':

$$(4) \qquad \sigma(P,P') = \sigma(s)$$

where

$$(5) \qquad s = \arccos\left[\sin\varphi \sin\varphi' + \cos\varphi \cos\varphi' \cos(\lambda-\lambda')\right].$$

The function $\sigma(s)$ will, of course, be symmetric with maximum at s=0 (being, in fact, defined only for $s \geqq 0$) and will be practically zero for greater distances, thus having a form similar to Fig. 1. In many cases it can be represented by functions

$$(6) \qquad \sigma(s) = \frac{\sigma_0}{1+k^2s^2}$$

or

$$(7) \qquad \sigma(s) = \sigma_0 e^{-c^2s^2},$$

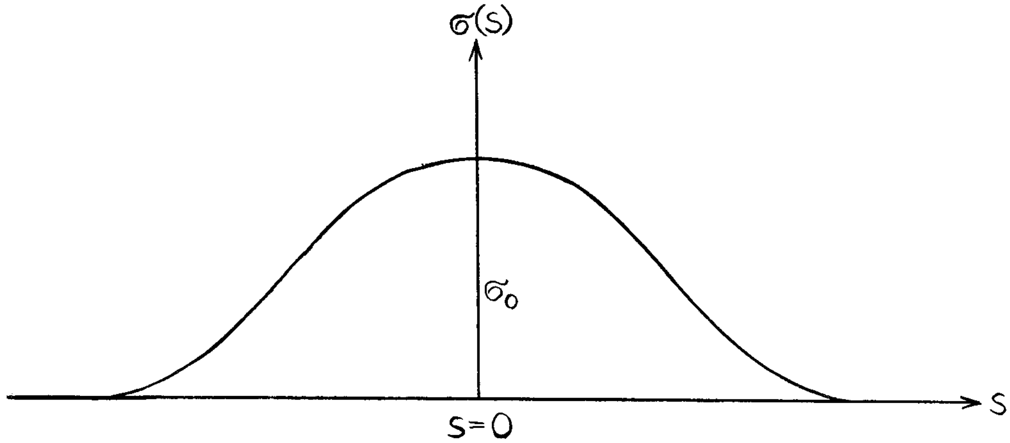

Figure 1

σ_o, k, c being suitable constants.

How this function $\sigma(s)$ can be determined will be discussed later.

This error function $\sigma(s)$ must not be confused with the covariance function $C(s)$, defined by

$$C(s) = M(\Delta g \Delta g') \tag{8}$$

with $s=\overline{PP'}$ as before. This covariance function also may be represented by a function of form (6):

$$C(s) = \frac{C_0}{1+\left(\frac{s}{d}\right)^2}, \tag{9}$$

d being a constant [6]. Its importance is to be seen in the fact that it provides a complete description of the statistical behavior of gravity anomalies. For this reason it can be used for deriving the error function $\sigma(s)$ for representation and interpolation errors.

2.2. Propagation of Errors

Many gravimetric computations are the evaluation of some integral, say Stokes' formula. Therefore, we can apply the theory of propagation of errors in integration developed in detail in [9].

We summarize: if

$$F = \int_{\alpha}^{\beta} l(x) f(x) \, dx \tag{10}$$

and $\sigma(x,x')$ is the error function of $f(x)$ then the standard error m_F of F is given by

$$m_F^2 = \int_{x=\alpha}^{\beta} \int_{x'=\alpha}^{\beta} l(x) \, l(x') \, \sigma(x,x') \, dx \, dx' . \tag{11}$$

There is a striking analogy to the well-known case of a linear function of variables $x_1, \ldots, x_n$ with error matrix (σ_{kl}): if

$$F = \sum_1^n l_k x_k \tag{12}$$

then

$$m_F^2 = \sum_{k=1}^n \sum_{k'=1}^n l_k l_{k'} \sigma_{kk'} . \tag{13}$$

There corresponds: function $f(x)$ to vector (x_k),

variable x to index k ,

integral to sum.

Analogies can be traced even further, for, as we shall see in the next section, a function f(x) is equivalent to a vector with infinitely many components.

The special problem we are concerned with in this paper is the computation of gravity in high elevations. Here, under suitable conditions, we can use a simplified formula, valid for the plane:

$$\Delta g_H = \frac{H}{2\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \Delta g(x,y) \frac{dx\,dy}{D^3} \tag{14}$$

where

$$D = (H^2+x^2+y^2)^{1/2}. \tag{15}$$

H is the elevation of point Q in which Δg_H is to be computed and xyz are rectangular coordinates, the z-axis being vertical and passing through Q so that Q has the coordinates (0,0,H).

Denoting the error of Δg_H by ϵ_H we have by (14)

$$\epsilon_H = \frac{H}{2\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \epsilon(x,y) \frac{dx\,dy}{D^3}$$

and

$$\epsilon_H^2 = \frac{H^2}{4\pi^2} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} \epsilon(x,y)\,\epsilon(x',y') \frac{dx\,dy}{D^3} \frac{dx'\,dy'}{D'^3}.$$

Taking the mean M we find for the mean square error m_H

$$(16) \quad m_H^2 = \frac{H^2}{4\pi^2} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} \sigma(x,y,x',y') \frac{dx\,dy}{D^3} \frac{dx'dy'}{D'^3}.$$

Of course, (14) and (16) correspond immediately to (10) and (11). We see again the fundamental importance of the error function: given $\sigma(s)$, the standard error m_H can be computed by (16) without any theoretical difficulties, for instance using numerical integration.

We now give an approximate evaluation of (16) with the special error function(7):

$$\sigma(s) = \sigma_0 e^{-c^2 s^2}$$

where

$$s^2 = (x-x')^2 + (y-y')^2.$$

$\sigma(s)$ being appreciably different from zero only for small s, i.e., for $(x,y) \doteq (x',y')$, we can put in (16) approximately $D' \doteq D$, getting

$$m_H^2 = \frac{H^2}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_0 e^{-c^2 s^2} \frac{dx\,dy\,dx'dy'}{(H^2+x^2+y^2)^3} =$$

$$= \frac{H^2}{4\pi^2} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{1}{(H^2+x^2+y^2)^3} \left[\int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} \sigma_0 e^{-c^2 s^2} dx'dy' \right] dx\,dy.$$

The double integral in the square brackets is evaluated first by transforming it to polar coordinates s, α:

$$\int_{x'=-\infty}^{\infty}\int_{y'=-\infty}^{\infty}\sigma_0 e^{-c^2s^2}dx'dy' = \int_{\alpha=0}^{2\pi}\int_{s=0}^{\infty}\sigma_0 e^{-c^2s^2} s\,ds\,d\alpha =$$

$$= 2\pi\sigma_0\int_0^{\infty} e^{-c^2s^2} s\,ds = \pi\frac{\sigma_0}{c^2} \quad .$$

Hence we get for m_H

$$m_H^2 = \frac{H^2}{4\pi^2}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\frac{\pi\frac{\sigma_0}{c^2}}{(H^2+x^2+y^2)^3}\,dx\,dy\,.$$

Introducing again polar coordinates

$$r = \sqrt{x^2+y^2}\ ,\quad \varphi = \operatorname{arc\,tg}\frac{y}{x}$$

we find

$$(17) \qquad m_H^2 = \frac{H^2}{4\pi^2}\cdot\pi\frac{\sigma_0}{c^2}\cdot\int_{\varphi=0}^{2\pi}\int_{r=0}^{\infty}\frac{r\,dr\,d\varphi}{(H^2+r^2)^3}$$

and finally

$$(18) \qquad m_H^2 = \frac{1}{8H^2}\,\frac{\sigma_0}{c^2}\ .$$

As this formula shows, the standard error m_H of Δg_H is inversely proportional to the height H of Q. Furthermore, it depends on the quotient $\frac{\sigma_0}{c^2}$. σ_0, according to (7), is the square of the standard error of surface anomaly Δg, c^2 is expressing the correlation: the smaller it is, the larger is the correlation. m_H, therefore, depends not only on the mean square error of the surface anomalies, but also on the correlation of the errors of these anomalies.

In a similar way we could also compute the correlation of the errors of two points in the same elevation, getting a new error function σ_H. The general problem can be formulated this way: Δg_H is related to the surface anomaly Δg by an integral transformation. The problem is to compute σ_H from σ, i.e., to study the propagation of the error function with respect to integral transformations. A general solution is found in [9].

2.3. Representation by Orthogonal Functions

A function which is defined on the earth's surface can be developed in a series of orthogonal functions. If the earth is considered an ellipsoid, these orthogonal functions are Lamé's functions; in the case of a sphere we have the spherical harmonics; if it is sufficient to approximate the earth's surface by a plane, we take a Fourier or a Bessel series (cf. [10]).

If we denote a complete set of such orthogonal functions by ψ_1, ψ_2, ψ_3, . . . we can write

$$\Delta g = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + \ldots = \sum_k a_k \psi_k .$$

The a_k's are generalized Fourier coefficients, they form an infinite vector

$$(a_1, a_2, a_3, \ldots\ldots).$$

The significance of this terminology lies in the fact that we can operate with the Fourier coefficients in exactly the same way as with finite vectors: their accuracy is characterized by an error matrix (σ_{kl}) which is, of course, now infinite. Furthermore, integration of the function corresponds to a linear transformation of the Fourier coefficients which may be simpler: to the Stokes integral there corresponds a very simple transformation of spherical harmonics.

In the same way the integral formulas for the computation of gravity in higher elevations can be translated into linear transformations of the generalized Fourier coefficients which are of utmost simplicity, having always the following form. If the surface anomaly

$$\Delta g = \sum_k a_k \psi_k , \tag{19a}$$

then the gravity anomaly at height H is

$$\Delta g_H = \sum_k A_k \psi_k \tag{19b}$$

where

$$A_k = a_k c_k(H)$$

and $c_k(H)$ is a simple function of elevation H.

For instance, in spherical harmonics

$$\Delta g = \sum_{m,n} a_{nm} P_{nm}(\cos\vartheta) e^{im\lambda},$$

$$\Delta g_H = \sum_{m,n} A_{nm} P_{nm}(\cos\vartheta) e^{im\lambda}$$

where

$$A_{nm} = \frac{a_{nm}}{(1 + H/R)^{n+2}},$$

or in Fourier series

$$\Delta g = \sum_{m,n} a_{mn} e^{i(mx+ny)},$$

$$\Delta g_H = \sum_{m,n} A_{mn} e^{i(mx+ny)}$$

where

$$A_{mn} = a_{mn} e^{-\sqrt{m^2+n^2}\, H}$$

(for the sake of brevity complex notation has been used). Here we have two indices m,n instead of one index k in (19), but this is no essential difference since we can always arrange the orthogonal functions of two indices m,n so that they are numbered by one index k only.

In the same way we can develop the error (1):

$$\varepsilon(\varphi,\lambda) = \sum_k \varepsilon_k \psi_k(\varphi,\lambda), \tag{20a}$$

and the error function (3)

$$(21) \quad \sigma(\varphi,\lambda,\varphi',\lambda') = \sum_k \sum_l \sigma_{kl}\, \psi_k(\varphi,\lambda)\, \psi_l(\varphi',\lambda').$$

The σ_{kl} form an infinite matrix as has been already mentioned; we have the relation

$$(22) \qquad \sigma_{kl} = M(\varepsilon_k \varepsilon_l).$$

The error of Δg_H is, according to (19b), given by

$$(20b) \qquad \varepsilon_H(\varphi,\lambda) = \sum_k \varepsilon_k c_k(H)\, \psi_k(\varphi,\lambda)$$

and the standard error by

$$(23) \quad m_H^2(\varphi,\lambda) = \sum_k \sum_l \sigma_{kl}\, c_k(H)\, c_l(H)\, \psi_k(\varphi,\lambda)\, \psi_l(\varphi,\lambda).$$

All these formulas appear deceptively simple, at least in comparison with the integral formulas (14) and (16). Their practical use, however, is severely handicapped by very slow convergence of the series, at least for elevations that are not very great. In addition, integrations are not avoided for we need them to compute the coefficients a_k and σ_{kl}, as is well known.

These few remarks concerning representation by orthogonal functions must suffice in this connection. The reader who wants to penetrate deeper into the striking analogies between integrations and linear transformations is again referred to [9].

For the sake of completeness, however, it may be added that the error matrix σ_{kl} is diagonal and therefore the errors ϵ_k are independent if the error function σ is a function only of s (5) and if we develop it into a series of spherical harmonics with pole in one end point P of s. That follows by reasoning similar to analogous considerations for the covariance function C(s), given by Kaula [7].

3. Accuracy of Computations

As we have already mentioned, gravity in higher elevations can, in suitable cases, be computed by the integral

$$\Delta g_H = \frac{H}{2\pi} \iint \Delta g \frac{dA}{D^3} . \tag{24}$$

The integration is to be extended over the entire plane, dA=dxdy being the element of area.

In practice we need integrate only over a limited area surrounding point P and the integration is performed by a suitable summation. We consider here two methods, a manual computation by means of templates, and another one which is suited for high-speed computing machines.

3.1. High Speed Machine Computation

This method is treated first because its theory of accuracy is somewhat simpler.

In some areas we have a very dense net of gravity stations, the areas, furthermore, being rather flat. In this case we can avoid the tedious work of interpolation and of drawing a map of contour curves for gravity anomalies, necessary for manual computation.

We lay a rectangular grid over the whole region, A_i being one rectangle or block. Then the integral (24) may be replaced by the sum

$$\Delta g_H = \frac{H}{2\pi} \sum_i \Delta g_i \frac{A}{\overline{D_i^3}} \tag{25}$$

where

$$\frac{1}{\overline{D_i^3}} = \frac{1}{A} \iint_{A_i} \frac{dA}{D^3} \tag{26}$$

is the mean of $1/D^3$ over the block. The size of the blocks should be chosen as small as possible but so that, in general, at least one point falls within each block--exceptions will be treated later.

Δg_i in (25) is the observed free-air gravity anomaly in this point P_i situated in block A_i. A better result would follow if instead we could take the mean anomaly $\overline{\Delta g_i}$ over the whole block A_i. But as it is unknown we let the point value Δg_i in point P_i represent the mean $\overline{\Delta g_i}$. Thus the error of this assumption is called <u>error of representation</u>.

But even if we could take the correct mean $\overline{\Delta g_i}$ formula (25) is still not quite correct because of replacing the integral by a sum. The error resulting from this is the <u>error of integration</u>.

First we consider only the contribution of one block A_i, omitting the index i. The total error is

$$\eta = \frac{H}{2\pi} \iint_A \frac{\Delta g - \Delta g_i}{D^3}\, dA\,, \tag{27}$$

Δg_i being, as before, the gravity in the observed point.

We can split up η into two parts:

$$\eta = \eta_1 + \eta_2 \tag{28}$$

where

$$\eta_1 = \eta_{repr.} = \frac{H}{2\pi} \iint_A \frac{\overline{\Delta g} - \Delta g_i}{D^3} \, dA \tag{29}$$

is the error of representation and

$$\eta_2 = \eta_{integr.} = \frac{H}{2\pi} \iint_A \frac{\Delta g - \overline{\Delta g}}{D^3} \, dA \tag{30}$$

the error of integration. $\overline{\Delta g}$ is defined by

$$\overline{\Delta g} = \overline{M}(\Delta g) \equiv \frac{1}{A} \iint_A \Delta g \, dA \tag{31}$$

where $\overline{M}(\Delta g)$ denotes the mean over the individual block A (not to be confused with $M(\Delta g)$!).

This splitting up of the total error will prove to be very convenient, the more so as $\eta_{repr.}$ and $\eta_{integr.}$, on the average, are uncorrelated. This can be readily seen. The covariance of η_1 and η_2 is

$$M(\eta_1 \eta_2) \, ,$$

on the average, therefore,

$$\overline{M}\left[M(\eta_1 \eta_2)\right] = M\left[\overline{M}(\eta_1 \eta_2)\right] = M\left[\eta_2 \overline{M}(\eta_1)\right] = 0$$

for, according to (29),

$$\overline{M}(\eta_1) = 0 \, .$$

Hence the average correlation between η_1 and η_2 is zero. We can, therefore, treat the errors of representation and of integration separately.

3.1.1. Errors of Representation

We consider the following four cases

a) The gravity anomaly in point P, Δg_P, represents the gravity anomaly in the center of the block, Δg_o, taking the position of P into account (Fig. 2).

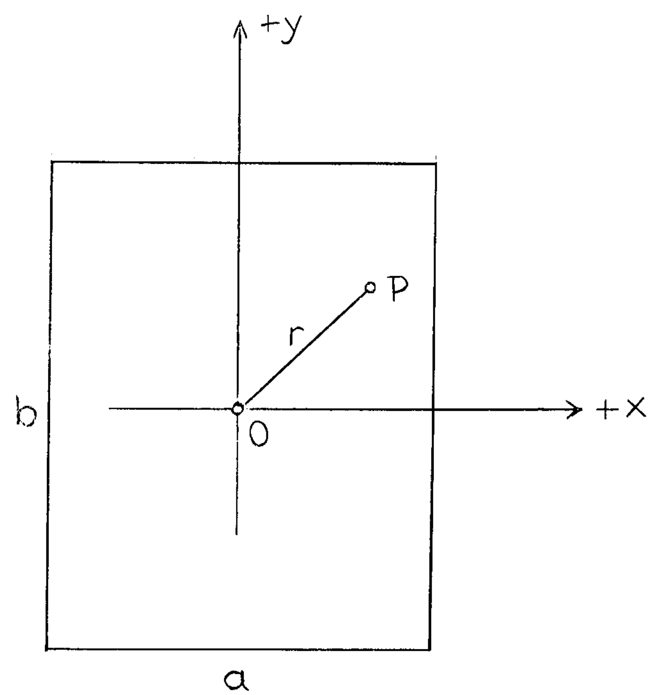

Figure 2

The mean square error of this assumption, m_a, is given by

$$m_a^2 = M(\Delta g_P - \Delta g_0)^2 = M(\Delta g_P^2) - 2M(\Delta g_P \Delta g_0) + M(\Delta g_0^2).$$

Introducing the covariance function (8) we get

$$(32) \qquad m_a^2 = 2C(0) - 2C(r), \quad r = \overline{OP}.$$

More general, the error covariance function corresponding to m_a is

$$(33) \quad \sigma_a(P,P') = M(\Delta g_P - \Delta g_0)(\Delta g_{P'} - \Delta g_0) = M(\Delta g_P \Delta g_{P'} - \Delta g_0 \Delta g_P - \Delta g_0 \Delta g_{P'} + \Delta g_0^2)$$

and therefore

$$(34) \qquad \sigma_a(P,P') = C(\overline{PP'}) - C(\overline{OP}) - C(\overline{OP'}) + C(0).$$

These formulas hold when the position of P (and P') inside the block is known and is taken into account. But more often we are interested only in the average effect, P lying anywhere in the block.

We have, therefore, to form the average over the whole block. We can formulate more precisely:

b) <u>Δg_P represents the gravity anomaly in the center, Δg_0, regardless of the position of P within the block</u>.

Denoting by a and b the sides of the rectangular block and laying the coordinate axes x,y, as in Fig. 2 we get

$$m_b^2 = \overline{M}(m_a^2) \equiv \frac{1}{A}\iint_A m_a^2 \, dA = \frac{1}{ab}\int_{x=-a/2}^{a/2}\int_{y=-b/2}^{b/2} m_a^2(x,y)\,dx\,dy,$$

and by (32)

$$(35) \qquad m_b^2 = 2C(0) - \frac{2}{ab}\int\limits_{x=-a/2}^{a/2}\int\limits_{y=-b/2}^{b/2} C(r)\,dx\,dy,$$

x,y being the coordinates of the variable point P and $r = \sqrt{x^2+y^2}$.

Somewhat more complicated but also more important are the following cases:

c) <u>Δg_P represents the mean $\overline{\Delta g}$ of the block with regard to the position of P.</u>

Now we have by (31)

$$\overline{\Delta g} = \frac{1}{ab}\int\limits_{\xi=-a/2}^{a/2}\int\limits_{\eta=-b/2}^{b/2} \Delta g(\xi,\eta)\,d\xi\,d\eta.$$

Therefore,

$$m_c^2 = M(\Delta g_P - \overline{\Delta g})^2 = M\left[\frac{1}{ab}\int\limits_{\xi=-\frac{a}{2}}^{\frac{a}{2}}\int\limits_{\eta=-\frac{b}{2}}^{\frac{b}{2}}(\Delta g_P - \Delta g(\xi,\eta))\,d\xi\,d\eta\right]^2 =$$

$$= \frac{1}{(ab)^2}\int\limits_{\xi=-\frac{a}{2}}^{\frac{a}{2}}\int\limits_{\eta=-\frac{b}{2}}^{\frac{b}{2}}\cdot\int\limits_{\xi'=-\frac{a}{2}}^{\frac{a}{2}}\int\limits_{\eta'=-\frac{b}{2}}^{\frac{b}{2}} M\left[(\Delta g(\xi,\eta)-\Delta g_P)(\Delta g(\xi',\eta')-\Delta g_P)\right]d\xi\,d\eta\,d\xi'\,d\eta'.$$

The integrand, as can be seen from (33), is just the error function σ_a with the points $(\xi,\eta),(\xi',\eta')$, P replacing P, P', O, respectively.

Hence,

$$m_c^2(x,y) = \frac{1}{(ab)^2}\int\limits_{-a/2}^{a/2}\int\limits_{-b/2}^{b/2}\int\limits_{-a/2}^{a/2}\int\limits_{-b/2}^{b/2}\sigma_a(\xi,\eta,\xi',\eta')\,d\xi\,d\eta\,d\xi'\,d\eta'$$

and by (34)

$$m_c^2(x,y) = \frac{1}{(ab)^2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} C(\xi,\eta,\xi',\eta')\,d\xi\,d\eta\,d\xi'\,d\eta' - \tag{36}$$

$$-\frac{2}{ab}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} C(x,y,\xi,\eta)\,d\xi\,d\eta + C(0),$$

x,y being the coordinates of point P.

There remains now to consider the case

d) <u>Δg_P represents the mean Δg, regardless of the position of point P.</u>

Here we must average (integrate) m_c^2 over the block:

$$m_d^2 = \overline{M}(m_c^2) \equiv \frac{1}{ab}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} m_c^2(x,y)\,dx\,dy. \tag{37}$$

Inserting m_c^2 (x,y) from (36) would only complicate the formula.

This last case is practically the most important one.

3.1.2. <u>Approximations and Numerical Values</u>

For numerical application of these formulas we may use the covariance function (9),

$$C(s) = \frac{C_0}{1+(s/d)^2} \tag{38}$$

with numerical constants

$$C_0 = 337\ (\text{mgal})^2,\qquad d = 40\ \text{km} \tag{39}$$

which were computed by Hirvonen for the State of Ohio [6] (our notation s,d corresponds to Hirvonen's d,D, respectively).

Developing (38) into a power series yields

$$C(s) = C_0\left(1 - \frac{s^2}{d^2} + \frac{s^4}{d^4} - + \dots\right),$$

hence in first approximation

$$C(s) = C_0\left(1 - \frac{s^2}{d^2}\right) \equiv C_0 - \alpha^2 s^2 \tag{40}$$

where

$$\alpha^2 = \frac{C_0}{d^2} = 0.211 \,.$$

In order to test for which distances s this approximation is sufficient, let us compare the correct value $\frac{1}{1+(s/d)^2}$ with its approximation $1-(s/d)^2$.

s_{km}	$\frac{1}{1+(s/d)^2}$	$1-(s/d)^2$
1	1.00	1.00
5	0.98	0.98
10	0.94	0.94
20	0.80	0.75
35	0.57	0.23

This means that we can safely replace (38) by (40) for distances up to 20 km.

This taken for granted, we compute the errors of representation, (32), (35), (36), and (37) by means of the approximate function (40), easily finding

$$m_a^2 \doteq m_c^2 \doteq 2\alpha^2(x^2+y^2), \tag{41}$$

$$m_b^2 \doteq m_d^2 \doteq \frac{\alpha^2}{6}(a^2+b^2)\,. \tag{42}$$

This result shows that in first approximation we get the same results for the point value in the center and the mean value of the block. That means that for small blocks (up to sides of 20 km) we may, in general, without appreciable error, take the point value at the center instead of the mean of the block. This is especially important for the manual template method for it is easier to read the point value at the center than to estimate the mean.

Introducing the numerical values (39) one has for 5' x 5' blocks (7 x 9 km in Ohio):

$$m_b \doteq m_d \doteq \pm 2.1 \text{ mgal} . \tag{43}$$

3.1.3. Correlation between Neighboring Blocks

As we have seen in Section 2.2, not only the mean square error of the surface gravity anomalies, the initial data for computing high-elevation gravity, is important but also their correlation.

Therefore, we must compute the correlation between adjacent blocks. As we have just found, we may replace the mean of the block by the point value at its center, thereby obtaining a considerable simplification. For the covariance between the errors of points 1 and 2 (centers of adjacent blocks) we get easily

$$\sigma_{a,12} = M(\Delta g_P - \Delta g_1)(\Delta g_{P'} - \Delta g_2) =$$

$$= M(\Delta g_P \Delta g_{P'} - \Delta g_1 \Delta g_{P'} - \Delta g_2 \Delta g_P + \Delta g_1 \Delta g_2);$$

(44) $$\sigma_{a,12} = C(\overline{PP'}) - C(\overline{1P'}) - C(\overline{2P}) + C(\overline{12}).$$

Inserting the approximation (40) we find

(45) $$\sigma_{a,12} = \alpha^2\left(2xx' + 2yy' - ax - by + ax' - by' - \frac{a^2}{2} + \frac{b^2}{2}\right),$$

the coordinate system being chosen as in Fig. 3.

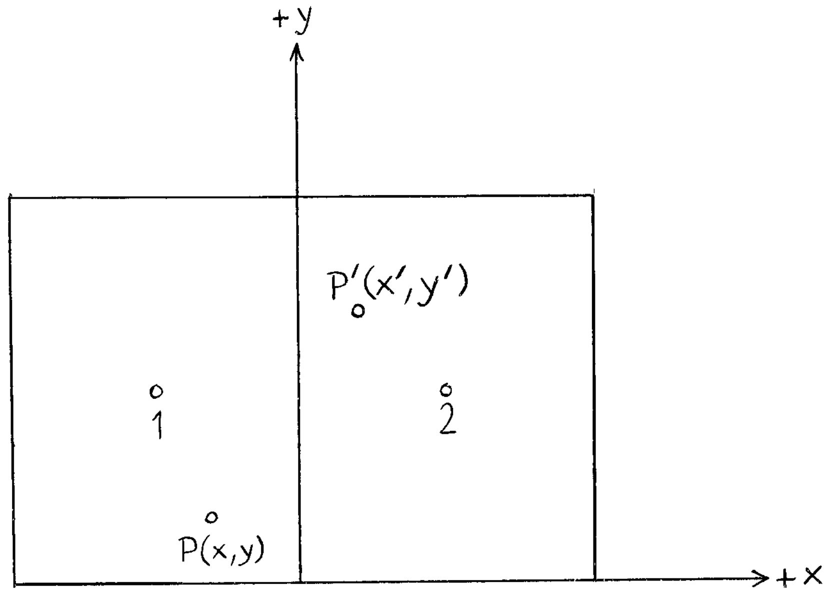

Figure 3

In order to get the average for random points P and P′ we must integrate P over block 1 and P′ over block 2, finding

$$\sigma_{b,12} = \overline{M}\left(\sigma_{a,12}\right) = 0$$

and therefore also

(46) $$\sigma_{d,12} = 0.$$

The average correlation between two adjacent blocks is zero. This important result considerably simplifies further computations.

3.1.4. Total Effect of the Error of Representation

Any two neighboring blocks being not correlated we can employ the usual formula of propagation of errors. From (25) follows

$$m_H^2 = \frac{H^2}{4\pi^2} \sum_i \left(m_i \frac{A}{D_i^3} \right)^2 .$$

All m_i are equal, m_i=m, m being given by the error of representation taken from (42) or (43). A=ab is the area of a block.

Easy manipulations yield

$$m_H^2 = \frac{H^2}{4\pi^2} m^2 A \sum_i \frac{A}{D_i^6} \doteq \frac{H^2}{4\pi^2} m^2 A \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{r\,dr\,d\varphi}{(H^2+r^2)^3} .$$

Comparing this with (17) gives

$$\pi \frac{\sigma_0}{c^2} = m^2 A$$

and

(47)
$$\frac{\sigma_0}{c^2} = \frac{A}{\pi} m^2 .$$

Therefore by (18),

(48)
$$m_H^2 = m^2_{H,\,repr.} = \frac{1}{H^2} \frac{A}{8\pi} m^2 .$$

Eq. (47) shows that the influence of the error of representation on our problem is equivalent to an error function (7) with any values σ_0 and c which are connected by (47).

Numerical evaluation by (39) gives

$$(49) \qquad m_{H,repr.} = \pm \frac{3.4}{H_{(km)}} \text{ mgal.}$$

3.1.5. <u>No Observations in the Block</u>

So far we have always assumed that in each block there is one observed gravity value. This should be the rule, i.e., the size of the blocks should be chosen accordingly. If we have more observed gravity stations in a block A_i, then for Δg_i, we take the arithmetical mean of all of them. The accuracy can only increase by this.

But it may also occur that we have no observations in the block. Then we could take for Δg_i the arithmetical mean of the values of the four immediately adjacent squares. We now compute the error of this assumption. On this assumption we have (Fig. 4):

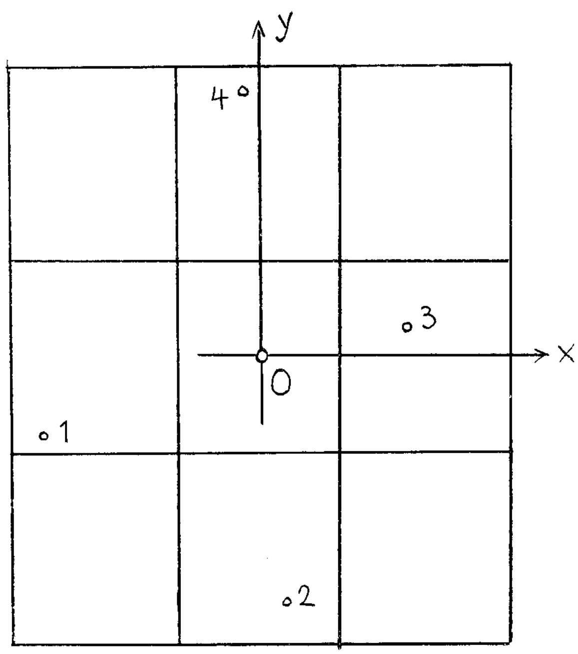

Figure 4

$$\varepsilon = \Delta g_0 - \frac{1}{4}(\Delta g_1 + \Delta g_2 + \Delta g_3 + \Delta g_4).$$

By squaring and averaging we get in the usual way

$$\begin{aligned} m^2 = \frac{1}{16}\Big[&20\,C(0) - 8\,C(\overline{01}) - 8\,C(\overline{02}) - 8\,C(\overline{03}) - 8\,C(\overline{04}) + \\ &+2C(\overline{12}) + 2C(\overline{13}) + 2C(\overline{14}) + 2C(\overline{23}) + 2C(\overline{24}) + 2C(\overline{34})\Big]. \end{aligned} \tag{50}$$

If we again approximate C(s) by (40) we find

$$\begin{aligned} m^2 = \frac{\alpha^2}{8}(&x_1^2 + x_2^2 + x_3^2 + x_4^2 + \\ &+2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4 + \\ &+y_1^2 + y_2^2 + y_3^2 + y_4^2 + \\ &+2y_1y_2 + 2y_1y_3 + 2y_1y_4 + 2y_2y_3 + 2y_2y_4 + 2y_3y_4). \end{aligned} \tag{51}$$

In order to get the average value for points 1,2,3,4 lying arbitrarily within their respective blocks, we must integrate point 1 over block 1, point 2 over block 2, etc. So the average standard error is given by

$$\bar{m}^2 = \frac{1}{(ab)^4} \int_{x_1=-\frac{3a}{2}}^{-\frac{a}{2}} \int_{y_1=-\frac{b}{2}}^{\frac{b}{2}} \int_{x_2=-\frac{a}{2}}^{\frac{a}{2}} \int_{y_2=-\frac{3b}{2}}^{-\frac{b}{2}} \int_{x_3=\frac{a}{2}}^{\frac{3a}{2}} \int_{y_3=-\frac{b}{2}}^{\frac{b}{2}} \int_{x_4=-\frac{a}{2}}^{\frac{a}{2}} \int_{y_4=\frac{b}{2}}^{\frac{3b}{2}} m^2\, dx_1\, dy_1\, dx_2\, dy_2\, dx_3\, dy_3\, dx_4\, dy_4. \tag{52}$$

Evaluation by (51) yields

$$\bar{m}^2 = \frac{\alpha^2}{24}(a^2 + b^2). \tag{53}$$

From this we compute for 5' x 5' blocks

$$\bar{m} = \pm 1.1 \text{ mgal}. \tag{54}$$

Comparison with (43) shows this error to be smaller than the error of representation. This seems to be quite paradoxical but, on the other hand, there is a strong correlation with the neighboring blocks so that the final result is certainly not better. In any case, however, the error caused by a block without observations will not be very great.

3.1.6. Error of Integration

For this error, resulting from one compartment, we have found (30)

$$\eta_2 = \eta_{integr.} = \frac{H}{2\pi}\iint_A \frac{\Delta g - \overline{\Delta g}}{D^3}\, dA .$$

By (31),

$$\iint_A \frac{\overline{\Delta g}}{D^3}\, dA = \overline{\Delta g}\iint_A \frac{dA}{D^3} = \iint_A \Delta g\, dA \cdot \frac{1}{A}\iint_A \frac{dA}{D^3} =$$

$$= \frac{1}{\overline{D}^3}\iint_A \Delta g\, dA = \iint_A \frac{\Delta g}{\overline{D}^3}\, dA$$

where, as previously,

$$\frac{1}{\overline{D}^3} = \frac{1}{A}\iint_A \frac{dA}{D^3},$$

the mean value of $\frac{1}{D^3}$ over the block.

Therefore,

$$(55)\quad \eta_2 = \frac{H}{2\pi}\iint_A \left(\frac{1}{D^3} - \frac{1}{\overline{D}^3}\right)\Delta g\, dA = \frac{H}{2\pi}\int_{x_0-\frac{a}{2}}^{x_0+\frac{a}{2}}\int_{y_0-\frac{b}{2}}^{y_0+\frac{b}{2}} F(x,y)\Delta g\, dx dy$$

if we set

$$(56)\qquad F(x,y) = \frac{1}{D^3} - \frac{1}{\overline{D}^3},$$

the coordinate system being assumed as in (14), x_0, y_0 being the coordinates of the center of the block. Hence we get for the contribution, μ, of one block to the mean square integration error of Δg_H:

$$(57)\qquad \mu^2 = \frac{H^2}{4\pi^2}\int_{x_0-\frac{a}{2}}^{x_0+\frac{a}{2}}\int_{y_0-\frac{b}{2}}^{y_0+\frac{b}{2}}\int_{x_0-\frac{a}{2}}^{x_0+\frac{a}{2}}\int_{y_0-\frac{b}{2}}^{y_0+\frac{b}{2}} F(x,y)\,F(x',y')\,C(s)\,dx\,dy\,dx'\,dy'$$

where, as usual,

$$s^2 = (x-x')^2 + (y-y')^2.$$

From definition (56) follows

$$(58)\qquad \iint_A F(x,y)\,dx\,dy = 0.$$

If we assume

$$C(s) = C_0 - \alpha^2 s^2$$

we have, therefore,

$$\mu^2 = \frac{H^2}{4\pi^2}\iint_A\iint_A (-\alpha^2)\,F(x,y)\,F(x',y')\left[(x-x')^2+(y-y')^2\right]dx\,dy\,dx'\,dy'.$$

Developing (56) in a series of powers of $(x-x_0)$ and $(y-y_0)$ we find

$$(59)\qquad F(x,y) = \frac{1}{D^3} - \frac{1}{\bar{D}^3} = -\frac{3x_0}{D_0^5}(x-x_0) - \frac{3y_0}{D_0^5}(y-y_0) + \ldots$$

Now we can perform the integration and find at last

$$(60)\qquad \mu^2 = \frac{\alpha^2 H^2}{32\pi^2}\,\frac{a^2 b^2}{D_0^{10}}\left(x_0^2 a^4 + y_0^2 b^4\right).$$

If the blocks are squares, $a = b$, we have

$$(61)\qquad \mu = \frac{\alpha H}{4\pi\sqrt{2}}\,\frac{a^4 r_0}{D_0^5}$$

where

$$r_0^2 = x_0^2 + y_0^2,$$
$$D_0^2 = H^2 + r_0^2.$$

Now we want to compute the total effect on Δg_H of the errors of every block. It is not correct to compute the square of the total standard integration error m_H, according to the ordinary formula of error propagation, by simply adding the squares of the contributions, μ_i, of all compartments:

(62) $$m_H^2 = \sum_i \mu_i^2 \quad ,$$

for there is correlation between the blocks. But in order to get some idea of the order of magnitude we may do this, after all, finding for square blocks by (61)

$$m_H^2 = \sum_i \mu_i^2 = \frac{\alpha^2 H^2}{32\pi^2} \sum_i \frac{r_i^2}{D_i^{10}} A^4 =$$
$$= \frac{\alpha^2 H^2}{32\pi^2} A^3 \sum_i \frac{r_i^2 A}{D_i^{10}} \doteq \frac{\alpha^2 H^2}{32\pi^2} A^3 \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{r^3 dr d\varphi}{(H^2+r^2)^5} .$$

The double integral can be easily evaluated, giving

$$\int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \frac{r^3 dr d\varphi}{(H^2+r^2)^5} = \frac{\pi}{12H^6} .$$

Therefore, we get for the total error of integration, approximately,

(63) $$m_H^2 = m_{H,\,integr.}^2 = \frac{\alpha^2}{384\pi} \frac{A^3}{H^4}$$

or

$$(64) \qquad m_{H,\,integr.} = \pm\, 0.0288\, \alpha \frac{\sqrt{a^3 b^3}}{H^2} .$$

With the numerical values (39) we find for 5' x 5' blocks

$$(65) \qquad m_{H,\,integr.} = \pm \frac{6.6}{H^2} \text{ mgal.}$$

For H = 8 km this is

$$\pm\, 0.1 \text{ mgal.}$$

This error, therefore, is much smaller than the error of representation.

Formula (63) is, of course, only approximate. The exact formula is, according to (57),

$$(66) \qquad m^2_{H,\,integr.} = \frac{H^2}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, F(x',y')\, C(s)\, dx\, dy\, dx'\, dy',$$

F(x,y) being given by (56) where $\frac{1}{D^3}$ is variable from block to block.

3.1.7. Some Remarks

The use of formula (25) is restricted to higher elevations by the size of the blocks. Moreover, it would be best if the observed point in the central block were situated in its center so that Δg_H is computed for this point. For practical reasons, however, it is better to regard the grid as fixed, say formed by coordinate lines. In this case we may use an artifice: we do the computation as if the observation point were situated in the center, but afterwards we let Δg_H refer to the original position of this point. By this, as can be readily seen, we obtain a similar result to shifting the grid so that its center coincides with the observed gravity station.

It would be simpler to use instead of (25) the formula

$$\Delta g_H = \frac{H}{2\pi} \sum_i \Delta g_i \frac{A}{D_{o,i}^3} \tag{67}$$

where $D_{o,i}$ is the value of D for the center of block A_i, i.e., to replace $\bar{D}$ by D_o. In first approximation this is the same, for

$$\frac{1}{\bar{D}^3} = \frac{1}{D_o^3} + \left(\frac{5x_o^2}{8D_o^7} - \frac{1}{8D_o^5} \right) a^2 + \left(\frac{5y_o^2}{8D_o^7} - \frac{1}{8D_o^5} \right) b^2 + \ldots \tag{68}$$

But formula (25) is to be preferred: if Δg is constant throughout block A, then (25) yields the correct result whereas (67) does not. (67) is, therefore, subject to a small systematical error.

In any case, this method can be applied only if the gravity net is sufficiently dense and if the topography is flat so that the correlation of the free air anomalies with topographic elevation can be neglected.

If we have mountainous areas this correlation with topography can be accounted for (cf. [11]) but the errors may be larger and their theory more complicated.

3.2. Manual Template Method

This method is described in detail in [6]. Using polar coordinates r, φ we have

$$\Delta g_H = \frac{H}{2\pi} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \Delta g \frac{r\,dr\,d\varphi}{D^3} . \tag{69}$$

This integral can be approximated by the sum

$$\Delta g_H = \sum_i k_i \overline{\Delta g_i} \tag{70}$$

where for each compartment

$$k = \frac{H}{2\pi} \int_{\varphi=\varphi_1}^{\varphi_2} \int_{r=r_1}^{r_2} \frac{r\,dr\,d\varphi}{D^3} = \frac{H}{n}\left(\frac{1}{D_1} - \frac{1}{D_2}\right). \tag{71}$$

D_1 refers to the inner radius r_1, D_2 to the outer radius r_2 of each zone, $n = 2\pi : (\varphi_2 - \varphi_1)$ is the number of compartments in the zone (Fig. 5).

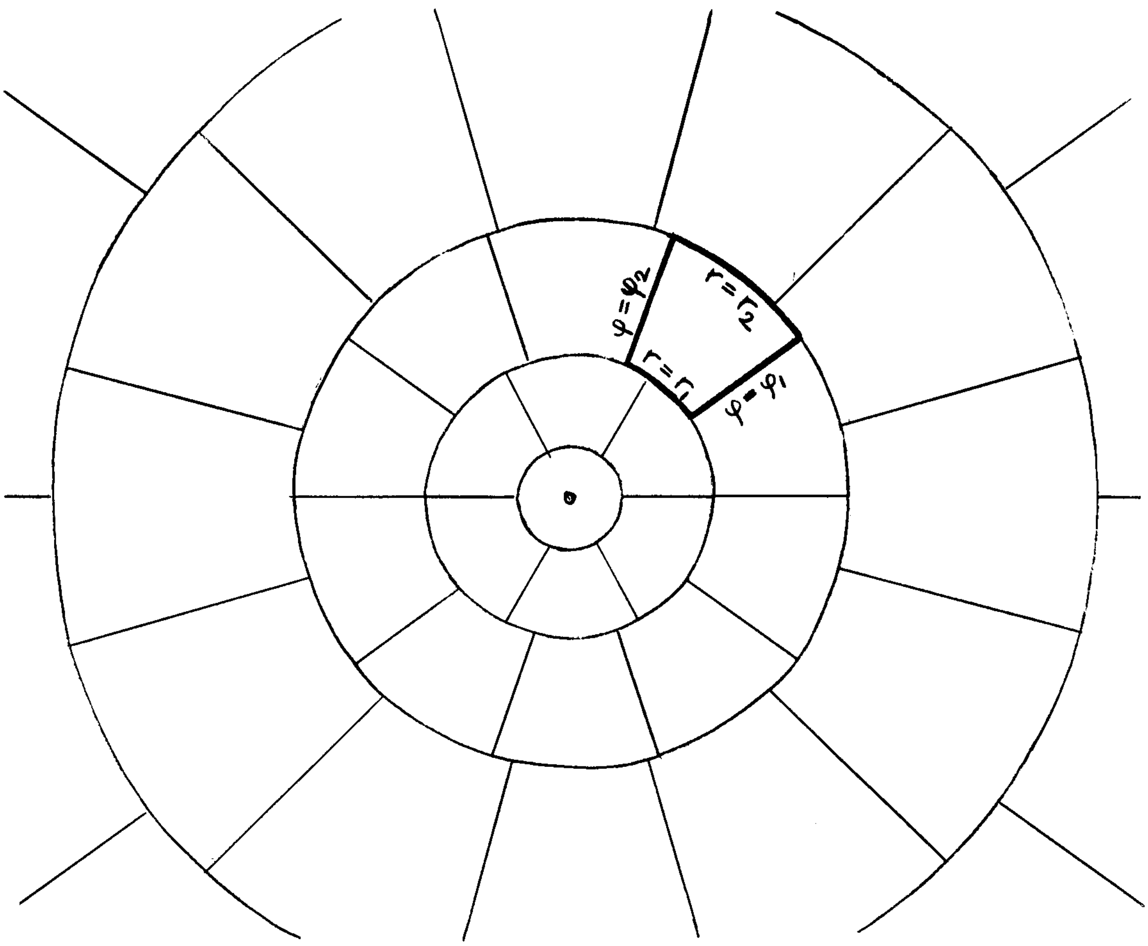

Figure 5

The mean value $\overline{\Delta g}$ of each compartment can be estimated from a map of free air gravity anomalies.

In this method we have also an error of integration, caused by replacing the integral (69) by the sum (70). But the main source of error is the interpolation and drafting the contour curves of gravity anomalies. Thus, we have an error of interpolation which here takes the place of the error of representation in the former method.

3.2.1. Maximum Error of Interpolation

The interpolation of the contour curves can be viewed, geometrically, in this way: between each three neighboring gravity stations the gravity surface is replaced by a plane, thereby approximating the gravity surface by a polyhedron.

For simplicity we assume the gravity stations to be regularly arranged so as to form the points of a lattice of equilateral triangles (Fig. 6). First we consider only the center 0 of one triangle. The

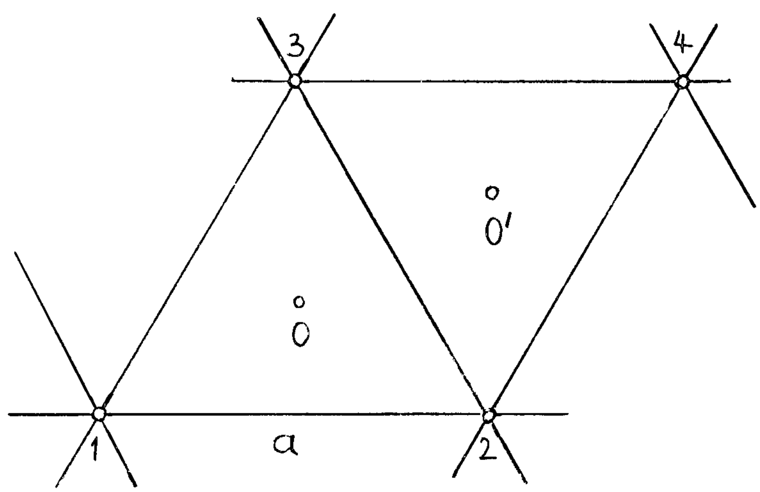

Figure 6

actual gravity is Δg, the interpolated value

$$\frac{1}{3}(\Delta g_1 + \Delta g_2 + \Delta g_3).$$

The error, therefore, is

$$\varepsilon = \varepsilon_{interp.} = \Delta g - \frac{1}{3}(\Delta g_1 + \Delta g_2 + \Delta g_3). \tag{72}$$

The mean square error is

$$m^2 = M(\varepsilon^2) = M\left[\Delta g - \frac{1}{3}(\Delta g_1 + \Delta g_2 + \Delta g_3)\right]^2.$$

By squaring and considering (8) we find easily

$$m^2 = m^2_{interp.} = \frac{4}{3}C(0) - 2C\left(\frac{a}{\sqrt{3}}\right) + \frac{2}{3}C(a), \tag{73}$$

a being the side of the triangles.

This is the mean square error in O, the center of the triangle. Let us now compute the correlation of the errors in O and in O', the center of the adjacent triangle. It is defined by

$$\sigma_{12} = M(\varepsilon\varepsilon'),$$

ϵ being given by (72) and ϵ' by

$$\varepsilon' = \Delta g' - \frac{1}{3}(\Delta g_2 + \Delta g_3 + \Delta g_4).$$

In the same way as before we find

$$\sigma_{12} = \frac{2}{9}C(0) - \frac{1}{3}C\left(\frac{a}{\sqrt{3}}\right) + \frac{2}{3}C(a) - \frac{2}{3}C\left(\frac{2a}{\sqrt{3}}\right) + \frac{1}{9}C(a\sqrt{3}). \tag{74}$$

If we approximate the correlation function by (40),

$$C(s) = C_0 - \alpha^2 s^2,$$

we get

$$m^2 = \sigma_{12} = 0 .$$

In first approximation, therefore, the interpolation error is zero.

Hence we must use the unabbreviated formula (38), finding with numerical values (39) for Ohio (average distance of gravity stations a=11 km)

$$(75) \qquad \begin{aligned} m^2 &= 0.8\,(mgal)^2, \\ \sigma_{12} &= 0.6\,(mgal)^2. \end{aligned}$$

The correlation coefficient

$$(76) \qquad r_{12} \equiv \frac{\sigma_{12}}{m^2} = 0.8 \; ;$$

there is, therefore, a very strong correlation.

3.2.2. Average Error of Interpolation

So far we have considered the center of the triangle where, in general, the deviation of the interpolated from the correct anomalies is greatest. The average values for an arbitrary point of the triangle are, of course, smaller.

A mathematically exact process of averaging leads to very complicated integrals. It seems, therefore, to be more economical to resort to an approximate procedure. In the direction of z we plot the mean deviations m. They are known in points 1, 2, 3--being zero there--and in 0 where m is given by (73). Through these four points we pass a paraboloid of revolution, the equation of which is

$$z^2 = m^2\left[1 - \frac{3}{a^2}(x^2+y^2)\right]. \tag{77}$$

The average value of z^2 inside the triangle is $\bar{m}^2$, the square of the average standard error. We find

$$\bar{m}^2 = \frac{1}{A}\iint_A z^2 dx\, dy = \frac{2}{A}\int_{x=0}^{\frac{a}{2}}\int_{y=-\frac{a}{6}\sqrt{3}}^{\frac{a}{\sqrt{3}}-x\sqrt{3}} m^2\left[1-\frac{3}{a^2}(x^2+y^2)\right]dx\,dy$$

where $A = \frac{1}{4}a^2\sqrt{3}$ is the area of the triangle. After performing the integration we find for the <u>average mean square error of interpolation</u>,

$$\bar{m}^2 = \frac{3}{4}m^2. \tag{78}$$

Multiplying σ_{12} by the same factor $\frac{3}{4}$, we get the <u>average correlation</u> between two arbitrary points which are at distance $\overline{OO'} = \frac{a}{\sqrt{3}}$ apart,

$$\bar{\sigma}_{12} = \frac{3}{4}\sigma_{12}. \tag{79}$$

If we assume the average error function of interpolation to be represented by an expression of form (7),

$$\sigma(s) = \sigma_0 e^{-c^2s^2},$$

this function, therefore, is known for two values s:

$$s=0 \ldots\ldots \sigma(0) = \bar{m}^2,$$

$$s=\frac{a}{\sqrt{3}} \ldots\ldots \sigma\left(\frac{a}{\sqrt{3}}\right) = \bar{\sigma}_{12}.$$

Hence, both constants σ_0 and c can be determined:

$$\sigma_0 = \overline{m}^2 ,$$
$$c^2 = \frac{3}{a^2} \ln\left(\frac{\overline{m}^2}{\overline{\sigma}_{12}}\right) .$$

3.2.3. Numerical Values and Total Effect

Numerical evaluation for Ohio yields

$$\sigma_0 = 0.60 = \overline{m}^2 , \quad c^2 = 0.0071 \tag{80}$$

and by (18)

$$m_H = m_{H,\,interp.} = \pm \frac{3.2}{H}\ \mathrm{mgal} , \tag{81}$$

i.e. almost the same result as (49).

Comparing the corresponding errors of both methods, the point value and the template method, we obtain the same effect m_H on Δg_H though the mean errors of the surface gravity anomalies are

$$m_{repr.} = \pm 2.1\ \mathrm{mgal}$$

by (43) for the error of representation and

$$m_{interp.} = \sqrt{\sigma_0} = \pm 0.8\ \mathrm{mgal}$$

by (80) for the other method. In the point value method the standard error is larger but there is less correlation than in the template method. These numerical results, of course, only apply to our special case.

3.2.4. Error of Integration

Similar to (30) or (55), we find for the template method, for one compartment,

$$\eta = \eta_{integr.} = \frac{H}{2\pi}\iint_A \frac{\Delta g - \overline{\Delta g}}{D^3}\,dA = \frac{H}{2\pi}\iint_A \left(\frac{1}{D^3} - \frac{1}{\overline{D}^3}\right)\Delta g\,dA \tag{82}$$

where in our case, according to (71),

$$\frac{1}{\overline{D}^3} = \frac{1}{A}\int_{\varphi=\varphi_1}^{\varphi_2}\int_{r=r_1}^{r_2} \frac{r\,dr\,d\varphi}{D^3} = \frac{2\pi}{HA}k. \tag{83}$$

Putting

$$\frac{1}{D^3} - \frac{1}{\overline{D}^3} = F(r) \tag{84}$$

we get

$$\eta = \frac{H}{2\pi}\int_{\varphi=\varphi_1}^{\varphi_2}\int_{r=r_1}^{r_2} F(r)\,\Delta g\; r\,dr\,d\varphi \tag{85}$$

and for the standard error of one compartment

$$\mu^2 = \frac{H^2}{4\pi^2}\int_{\varphi=\varphi_1}^{\varphi_2}\int_{r=r_1}^{r_2}\int_{\varphi'=\varphi_1}^{\varphi_2}\int_{r'=r_1}^{r_2} F(r)\,F(r')\,C(s)\,r\,r'\,dr\,d\varphi\,dr'\,d\varphi' \tag{86}$$

where s is the plane distance of points (r,φ) and (r',φ'):

$$s^2 = r^2 + r'^2 - 2rr'\cos(\varphi-\varphi'). \tag{87}$$

In good approximation the compartment bounded by the lines $\varphi=\varphi_1$, $\varphi=\varphi_2$, $r=r_1$ and $r=r_2$ can be considered a rectangle. If n, the number of compartments in one ring, is chosen appropriately, they can be made almost squares with side $\Delta r = r_2 - r_1$. We can, therefore, apply (61) (with $a = \Delta r$), finding

$$\mu = \frac{\alpha H}{4\pi\sqrt{2}} \cdot \frac{r_0}{D_0^5} (\Delta r)^4. \tag{88}$$

The mean effect of one compartment on Δg_H is, therefore, proportional to

$$\frac{r_0}{D_0^5} (\Delta r)^4.$$

This result has, by another method, already been found by Hirvonen [6]. He used it for constructing a template in which each compartment, on the average, gives the same error, i.e. μ(88) is the same for each compartment. Therefore, Δr must be proportional to

$$D_0 \sqrt[4]{\frac{D_0}{r_0}} . \tag{89}$$

If, for instance, we take the template given by Hirvonen ([6], Table IX - the radii are expressed with H as unit)

TABLE I

Outer Radius r_2	Number of Compartments n	Coefficient k
0.4	1	0.07152
1.0	8	0.02767
1.8	12	0.01846
3.0	12	0.01412
4.5	16	0.00621
6.7	16	0.00433
10.0	16	0.00301
15.0	16	0.00206
22.0	16	0.00132
32.0	16	0.00089

we have

$$\Delta r = \frac{1}{2} D_0 \sqrt[4]{\frac{D_0}{r_0}} \, ,$$

$$\frac{r_0}{D_0^5} (\Delta r)^4 = \frac{1}{16} \, .$$

Hence, in this case,

$$\mu = \frac{\alpha}{64\pi\sqrt{2}} H \, ,$$

and with the numerical values (39),

$$(90) \qquad \mu = 0.0016 \, H_{(km)} \, \text{mgal}.$$

This is the influence of one compartment. The total effect of all compartments can be computed by

$$(91) \quad m^2_{H,integr.} = \frac{H^2}{4\pi^2} \int_{\varphi=0}^{2\pi} \int_{r=0}^{\infty} \int_{\varphi'=0}^{2\pi} \int_{r'=0}^{\infty} F(r) F(r') C(s) \, r r' dr dr' d\varphi d\varphi'$$

where s is given by (87) and F(r) by (84), $\frac{1}{D^3}$ being different for each compartment. This formula is, of course, the analogue to (66).

As one easily sees, the errors of interpolation and of integration must somehow be correlated. But it seems hardly possible to set up this correlation mathematically. In any case, the influence of the error of integration will not be very great.

Of course, also the reading or estimating of the mean gravity anomaly is subject to errors. Studies on this <u>reading error</u> are yet to be done.

4. Accuracy of Formulas

Summarizing all possible errors arising in the computation of gravity in higher elevations we find:

a) Observation errors of gravity and topographic height.

b) Error of representation in the point value method and error of interpolation (plus reading error) in the template method, respectively.

c) Integration errors arising from replacing the integration by a summation.

d) Errors caused by using the simplified plane formula (14).

e) Errors arising from neglecting the influence of distant zones.

Observation errors according to a) are supposed to be negligible. Errors b) and c) have been treated in detail in section 3. Hence there remains the consideration of errors d) and e).

4.1. Various Formulas

A formula for computing the gravity at a point Q outside the earth which is theoretically correct would have the form

$$g_Q = \iint_S g_n \, G(S) \, dS \, . \tag{92}$$

The integral is to be performed over the physical surface S of the earth, g_n is the component of gravity normal to S and G(S) would be the derivative of Green's function of second kind with respect to the height H of Q.

Apart from the fact that this function G(S) is unknown, formula

(92) is unsuitable for practical computation because one must integrate over the complicated surface of the earth.

Therefore, various simplifications have been proposed. In the first place, gravity g is replaced by the gravity anomaly Δg, thereby dealing with smaller numbers. From the surface gravity g we compute free air anomalies Δg which always refer to the physical surface of the earth.

Second, we may replace S by a sphere σ, thereby neglecting the topography. So we get

$$(93) \qquad \Delta g_Q = \frac{\rho^2 - a^2}{4\pi\rho} \iint_\sigma \frac{\Delta g}{D^3}\, d\sigma$$

where

$$D^2 = a^2 + \rho^2 - 2a\rho \cos\psi ,$$

$$\rho = a + H ,$$

a being the radius of the earth [11].

If, finally, the elevation H of Q is not too large, we can replace the sphere by its tangential plane, getting the simple formula (14):

$$(94) \qquad \Delta g_Q \equiv \Delta g_H = \frac{H}{2\pi} \iint \frac{\Delta g}{D^3}\, dx\, dy .$$

In both formulas, the influence of topography has been neglected, in (94) also the curvature of the earth. This gives rise to new errors.

4.2. Influence of Topography

We are now developing a formula which takes account of topography. Again, we replace the geoid by its tangential plane, this time, however, not neglecting the topography. The physical surface S of the earth is assumed to be lying at elevation h over the plane (Fig. 7).

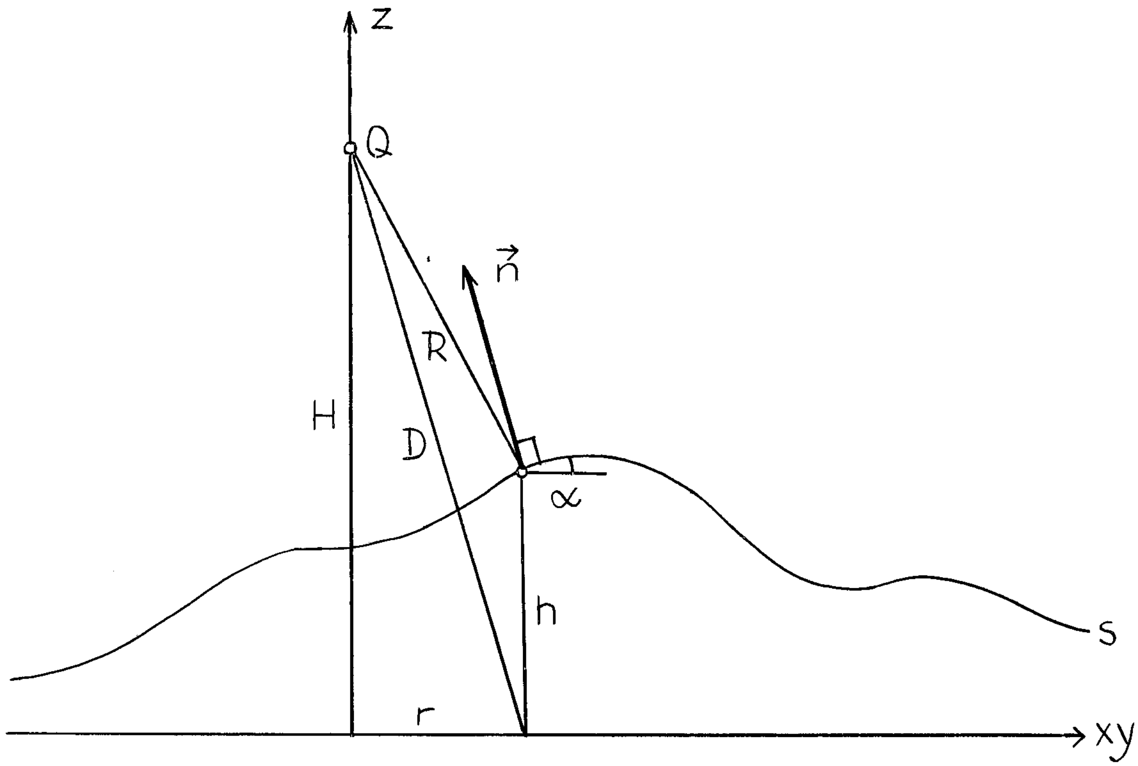

Figure 7

According to Green's Theorem we have

$$(95) \qquad T_Q = \frac{1}{4\pi} \iint_S \left[T \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial T}{\partial n} \right] dS.$$

Differentiating with respect to H we get

$$(96) \qquad \left.\frac{\partial T}{\partial H}\right)_Q = \frac{1}{4\pi} \iint_S \left[T \frac{\partial^2}{\partial H \partial n}\left(\frac{1}{R}\right) - \frac{\partial}{\partial H}\left(\frac{1}{R}\right) \frac{\partial T}{\partial n} \right] dS.$$

From Fig. 7 we see

$$R^2 = r^2 + (H-h)^2 = x^2 + y^2 + (H-h)^2.$$

Differentiating with respect to H gives

$$\frac{\partial}{\partial H}\left(\frac{1}{R}\right) = -\frac{H-h}{R^3} \quad .$$

If $\vec{n}$ is the unit vector normal to S and grad R the vector

$$\operatorname{grad} R = \left(\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}\right)$$

we have

$$\frac{\partial}{\partial n}\left(\frac{1}{R}\right) = -\frac{1}{R^2}\frac{\partial R}{\partial n} = -\frac{1}{R^2}\,\vec{n}\cdot\operatorname{grad} R\,.$$

For grad R we get easily

$$\operatorname{grad} R = \left(\frac{x}{R}, \frac{y}{R}, -\frac{H-h}{R}\right)$$

and for $\vec{n}$,

$$\vec{n} = (\sin\alpha\cos\psi, \sin\alpha\sin\psi, \cos\alpha),$$

α being the inclination of the terrain and ψ the azimuth of $\vec{n}$. Setting

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

we find

$$\frac{\partial}{\partial n}\left(\frac{1}{R}\right) = -\frac{\cos\alpha}{R^3}\left[r\,\mathrm{tg}\,\alpha\cos(\varphi-\psi) - (H-h)\right]$$

and

$$\frac{\partial^2}{\partial H\,\partial n}\left(\frac{1}{R}\right) = 3\cos\alpha\,\frac{H-h}{R^5}\left[r\,\mathrm{tg}\,\alpha\cos(\varphi-\psi) - (H-h)\right] + \frac{\cos\alpha}{R^3}.$$

All these expressions can be developed into a series with respect to h, retaining only terms linear in h. This is introduced into (96), giving

$$\left.\frac{\partial T}{\partial H}\right)_Q = \frac{1}{4\pi}\iint\limits_S\left(-\frac{3H^2}{D^5}+\frac{1}{D^3}\right)T\cos\alpha\, dS + \frac{1}{4\pi}\iint\limits_S \frac{H}{D^3}\frac{\partial T}{\partial n}dS +$$

$$(97) \qquad + \frac{1}{4\pi}\iint\limits_S\left(\frac{9H}{D^5}-\frac{15H^3}{D^7}\right)T\, h\cos\alpha\, dS +$$

$$+ \frac{1}{4\pi}\iint\limits_S \frac{3Hr}{D^5}\operatorname{tg}\alpha\cos(\varphi-\psi)T\cos\alpha\, dS + \frac{1}{4\pi}\iint\limits_S\left(\frac{3H^2}{D^5}-\frac{1}{D^3}\right)\frac{\partial T}{\partial n}h\, dS.$$

In sufficient approximation

$$(98) \qquad \left.\frac{\partial T}{\partial H}\right)_Q = -\Delta g_Q\,,\quad \frac{\partial T}{\partial n} = -\Delta g\cos\alpha$$

and exactly

$$(99) \qquad dS\cos\alpha = dx\,dy\,.$$

If $h \equiv 0$ we get

$$\Delta g_Q = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{3H^2}{D^5}+\frac{1}{D^3}\right)T\, dx\, dy + \frac{H}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta g}{D^3}dx\, dy,$$

on the other hand we have by (94)

$$\Delta g_Q = \frac{H}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta g}{D^3}dx\, dy.$$

Therefore, both expressions must be equal and we finally get

$$(100)\quad \begin{aligned} \Delta g_Q = {} & \frac{H}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta g}{D^3}\,dx\,dy - \frac{3H\gamma}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{3}{D^5}-\frac{5H^2}{D^7}\right)h\zeta\,dx\,dy - \\ & - \frac{3H\gamma}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\zeta r}{D^5}\,\mathrm{tg}\,\alpha\,\cos(\varphi-\psi)\,dx\,dy + \\ & + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{3H^2}{D^5}-\frac{1}{D^3}\right)h\,\Delta g\,dx\,dy. \end{aligned}$$

Here we have introduced the undulation ζ by Bruns' Formula $T=\gamma\zeta$ (γ is the theoretical gravity). The last three terms of the formula express the effect of topography, their sum is the error caused by computing Δg_Q by the simple formula (94).

Numerical studies on these terms will follow. They will decide when it is necessary to take them into account. These correction terms are suited for template computation, an approximate value of undulation ζ being sufficient.

4.3. Effect of Neglecting the Outer Zones

Let us assume that the gravity anomalies are known--or are considered--only in a limited area A, surrounding point Q. The remainder of the earth's surface is denoted by B. Then by (93)

$$\Delta g_H = \frac{\varrho^2-a^2}{4\pi\varrho}\iint_A \frac{\Delta g}{D^3}\,d\sigma + \frac{\varrho^2-a^2}{4\pi\varrho}\iint_B \frac{\Delta g}{D^3}\,d\sigma.$$

The second integral is neglected, giving rise to an error

$$\varepsilon_H = \frac{a^2(\rho^2-a^2)}{4\pi\rho} \iint_B \frac{\Delta g}{D^3} \cos\varphi \; d\varphi \, d\lambda .$$

The mean square error is, therefore, given by

$$(101) \quad m_H^2 = \left(\frac{a^2(\rho^2-a^2)}{4\pi\rho}\right)^2 \iint_B \iint_B C(s) \frac{\cos\varphi \; d\varphi \, d\lambda}{D^3} \frac{\cos\varphi' d\varphi' d\lambda'}{D'^3} ,$$

C(s) being defined by (8) and s by (5). The evaluation of this integral and numerical studies are reserved for future work.

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