and five as special cases we have a general divisibility rule for all possible divisors.

The author is not presumptuous enough to feel that the general divisibility rule is something of great mathematical significance. Perhaps those interested in number theory will find this to be more informative than will those not so inclined. In any case, if you have mastered this algorithm then the next time you are stopped on the street and asked if 67 is a factor of 134483807 you will at least be able to provide an accurate answer without resorting to division.

DEMOCRACY AND GENETICS: TWO EXAMPLES OF MATHEMATICAL MODELS

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Mathematical modeling has been introduced into the college undergraduate curriculum relatively recently. In many colleges and universities, a course in mathematical modeling is still non-existent. And, the topic is even more rarely included in the high school curriculum. Yet, there are many advantages to the early teaching of mathematical model building in the behavioral sciences as well as the physical sciences. First, experience in modeling is an excellent way to develop mathematical creativity. Second, a demonstration of a wide variety of mathematical applications is an appropriate way to motivate the study of mathematics and the consideration of a career related to mathematics. Third, the students can be made aware of the enormous number of contributions which mathematics has made to modern civilization.

What is mathematical modeling and how does this approach differ from a more traditional classroom treatment of the applications of mathematics? The answers to these questions provide another strong argument in favor of teaching mathematical modeling. The usual classroom treatment of the applications of mathematics involves the presentation of a mathematical concept, including an explanation

of definitions and a proof of theorems, followed by a set of examples of "practical problems" whose solutions are tied to the specific concept under discussion. Mathematical modeling, on the other hand, begins with a presentation of a real life problem. The conditions under which the problem can be solved and the mathematical tools for solving it are open questions. The students must invent an appropriate mathematical model. This process involves clarifying the problem, defining the variables, listing the assumptions, applying mathematical procedures, interpreting the results, and analyzing the extent to which the model fits the problem. Hence, mathematical modeling provides students with a different type of learning experience; it is frequently more creative, less structured, and more rewarding. Certainly, it serves as a valuable example of the process of doing mathematics.

During the academic year 1978-79, the authors taught a course in mathematical modeling to talented high school seniors. The project was funded by the National Science Foundation, as part of the Student Science Training program. Classes were held on Saturday mornings. Textbooks for the course were Introduction to Finite Mathematics, 3rd edition, by Kemeny, Snell, and Thompson and An Introduction to Mathematical Models in the Social and Life Sciences by Olinick. Also, a great number of reference books were used and a wide variety of topics were studied.

Several topics were introduced by presenting a problem to the students. The class was divided into groups of five students. Each group was asked to design a mathematical model, use it to solve the problem, and interpret the results. We shall present two examples, and we hope that the readers will be motivated to use similar exercises to introduce their students to mathematical modeling.

The first topic concerned a mathematical model of a democratic procedure for determining the outcome of an election, in which there were more than two choices. The major questions to be answered were: What are the axioms which define a democracy? Can a democracy exist?

After a brief discussion of the National Labor Relations Act,

the following problem was assigned:

The sixteen hundred faculty members at a public university in Ohio are scheduled to vote in a collective bargaining election. Two professional associations, Brand X and Brand Y, would like to win the right to represent the faculty. Hence, the voters will have three choices: Brand X, Brand Y, and No Agent. Since there is no enabling legislation for public employees in Ohio, representatives of Brand X and Brand Y must negotiate the election rules with the University Administration. The following sets of rules have been proposed:

1. Hold a two-stage election. First, vote for or against collective bargaining. If a majority votes for collective bargaining, hold a second election, in which the choices are Brand X and Brand Y. Again, the outcome will be determined by a majority.
2. Hold an election in which there are three choices: Brand X, Brand Y, and No Agent. If none of these receives a majority of votes, hold a run-off election between the two choices with the greatest number of votes. The choice with the majority of votes will be the winner.

The students were directed to invent a set of axioms defining democratic procedures; to list the assumptions inherent in the model; to determine which voting mechanism satisfied the axioms for a democracy (or to devise another mechanism which would satisfy the axioms); and to comment on any difficulties which were encountered, such as insufficient information.

Although the students, at this stage, had no previous experience with mathematical modeling, their responses were creative and their interest was high. They were particularly intrigued by the idea that this topic might properly be labeled "mathematics". A formal discussion of Arrow's axioms and a proof of Arrow's Impossibility Theorem followed. Armed with this new knowledge, the students were asked to reconsider the original problem. The second attempt, of course, resulted in more sophisticated solutions.

A second problem in mathematical modeling given to the class was the following:

Albinism is a trait characterized by milky skin color, very

light hair color, and pinkish eye color. The relative frequency of albinos in the United States is 1 person in 20,000. Whether a person is an albino or a non-albino depends on a single pair of genes which we will denote by A and a. Persons who are non-albino can have genotype AA or Aa. Albinos always have genotype aa. Using the dominant-recessive terminology, A is a dominant gene and a is a recessive gene. For convenience, we shall refer to A as the non-albino gene and to a as the albino gene:

The students were, again, divided into small groups and asked to design a mathematical model to answer the following questions:

1. Can albino children be born to non-albino parents?
2. In view of the dominance of the non-albino gene, A, will the proportion of albinos in the population become smaller as time goes on?
3. If 1 person out of 20,000 is an albino (aa), what is the relative frequency of carriers (Aa)?

A three-hour introduction to elementary probability concepts preceded this assignment. Also, most of the students had studied high school biology. Once again, their solutions to the problem were creative, and their interest level was high. Assumptions about the population, however, were naive and incomplete. This time, after an introduction to Markov chains, the problem was discussed in detail, in class, and a complete solution was provided.

Problems involving further applications of probability and applications of game theory and graph theory were presented, later in the year, in the same format. As the students progressed through the course, the problems became more difficult and the hints became more subtle. Students were encouraged to take advantage of simulation techniques and reference books as well as their own ingenuity.

Generally, the students enjoyed working in small groups and found mathematical modeling to be a stimulating intellectual challenge. Occasionally, of course, they would flounder and get discouraged by "not knowing where to start." A few hints and a discussion of the frequently encountered real-life dilemma of not knowing where to start were sufficient help in generating ideas.

It must be emphasized that the participants in this project were extremely talented. Students with less mathematical maturity would, perhaps, need a gentler introduction to mathematical model building, using easier problems and more specific hints. But, just as mathematical proof can be taught at various levels, so mathematical modeling can be adjusted to the class. It is a topic that has both mathematical and pedagogical significance. A high school curriculum would be greatly enriched by the addition of an introduction to mathematical modeling. Try it!

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