

## The Cream in My Polytope

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### Abstract

This guided discovery project provides high school math students an opportunity to solve a real-world problem by applying tools they have learned in geometry and, optionally, calculus. The task is to determine the volume, surface area, and other properties of a certain form of folded paper container, such as those used by Wendy's for sour cream. Students first construct the containers, and then are challenged to find a geometric way to determine the volume of these objects, which do not fit conventional volume formulas. They next find surface area and, optionally, angles of the container. In the process, 2- and 3-dimensional visualization skills are exercised. Calculus students are asked to find the volume by the method of slabs, and subsequently to determine the dimensions that optimize the surface-to-volume ratio. Several additional extensions are suggested, and core curriculum standards are listed. Diagrams and calculations are provided. While much support material is provided, it is hoped that the teacher will encourage students to explore, discover and invent as much as possible on their own.

A song from 1928 begins: "You're the cream in my coffee; you're the salt in my stew." I have occasionally seen individual containers for coffee creamer and sour cream – and, in England, for milk – constructed from a paper cylinder pinched perpendicularly at the ends. More recently, they have become popular containers among crafts and scrapbook enthusiasts.

These humble little packages provide a plethora of mathematical explorations: find the volume; find the surface area; find the dimensions that will minimize the surface-to-volume ratio for a given volume; etc. Geometry, algebra, calculus, and Wendy's baked potatoes rolled into one package; can life get any better?



perpendicular to both the axis and the first end (see photos). Ignore excess material needed to seal the ends. Crease along the segments joining the ends to form four congruent triangular faces. Call the length of the axis  $a$ , which turns out

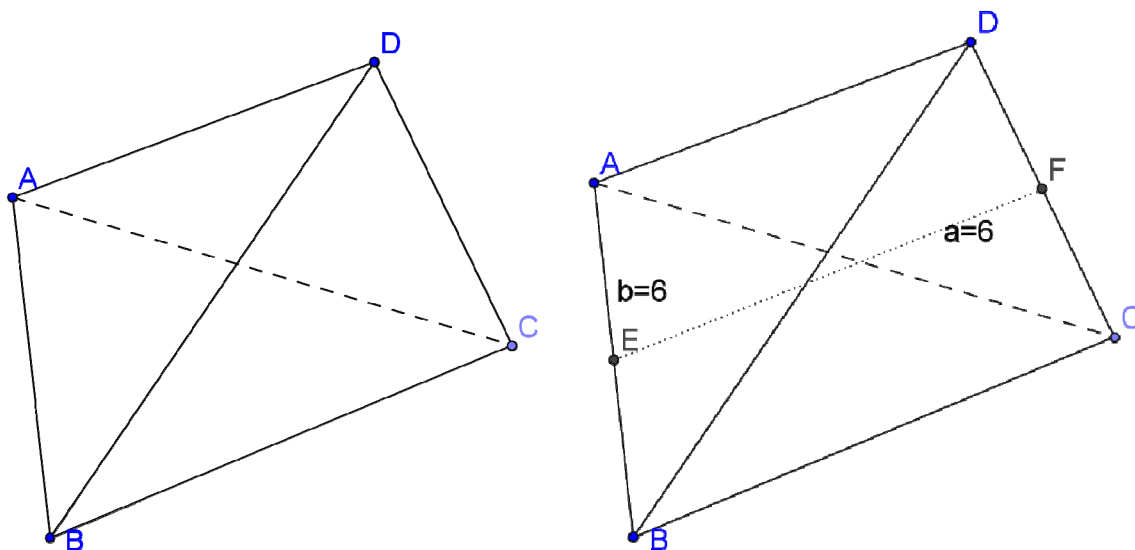
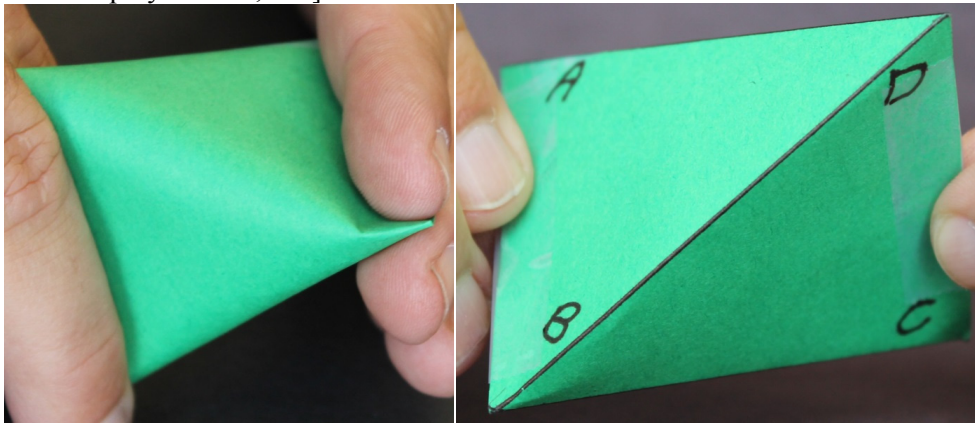
I approach this as an inquiry-based lesson. Students start out by actually constructing a container, then brainstorming approaches to calculating the various dimensions. Much of this can be done with geometry only, but it is also a very appropriate project for calculus students – who can take things farther than geometry allows.

### Activity Description

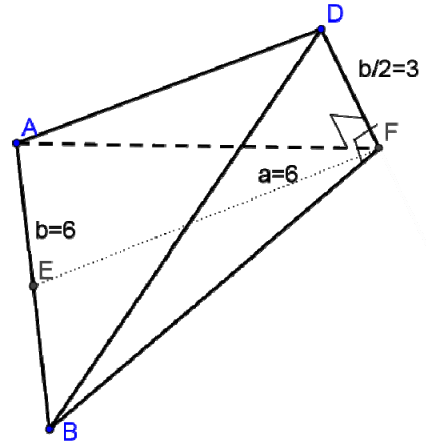
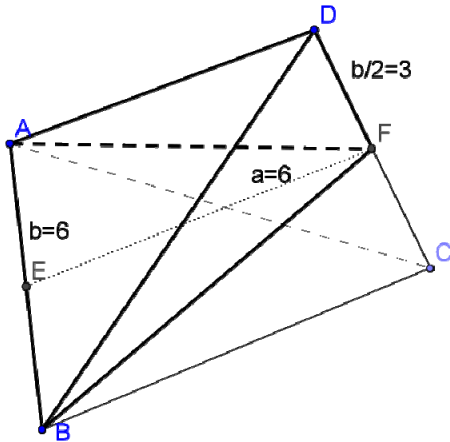
To start, take a geometric approach to a very specific example. Your students should construct a paper cylinder with *circumference* 12 cm and length  $3\sqrt{5} \sim 6.71$  cm, which may sound strange, but it makes things much clearer, as you will see. Call it the length, not height, for practical reasons: the containers



to be 6 cm if the cylinder is  $3\sqrt{5}$  cm in length (there's the connection: nice lengths to work with). The width of the pinched ends call  $b$ , which here is also 6 cm. Yes, it is a polyhedron, but it is also a polytope, and that will give your students a new math term to explore. [A polytope is any  $n$ -dimensional geometric object with flat edges or faces. Two-dimensional polytopes are called polygons; three-dimensional polytopes are called polyhedrons; etc.]



Finding the volume of this solid geometrically is easier than it may look at first. Slice it in half, along the axis and through one end, and perpendicular to the other end. This step is the most difficult for students to visualize. I suggest cutting a model container from a Styrofoam block before class, then slicing it in half with students observing. Note that angles BFD and AFD are right angles.



Polyhedron ABFD is just a triangular-base right pyramid, so the volume can be found by:

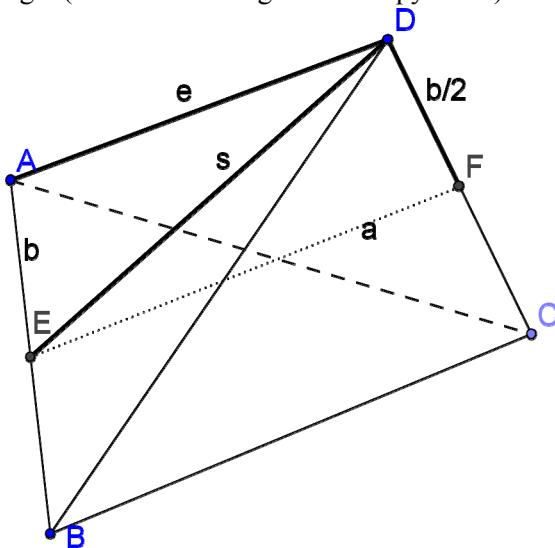
$$V_{ABDF} = \frac{Bh}{3} = \left(\frac{1}{3}\right)\left(\frac{6 \cdot 6}{2}\right)(3) = 18. \text{ But this is only one half of the total volume: } 18 \text{ times } 2 = 36 \text{ cm}^3.$$

Now let's generalize this for any  $a$  and  $b$ :

$$V_{ABDF} = \frac{Bh}{3} = \left(\frac{1}{3}\right)\left(\frac{a \cdot b}{2}\right)\left(\frac{b}{2}\right) = \frac{ab^2}{12}, \text{ then times } 2, \text{ as above, giving a total volume of } V_{ABCD} = \frac{ab^2}{6}.$$

$$\text{Finally, in the special case where } a = b, V_{ABCD} = \frac{a^3}{6} = \frac{b^3}{6}$$

While working with geometry, find the surface area and edge lengths. [From this point on, just deal with general formulas, not just the specific  $6 \times 6$  (i.e. “ $a$  by  $b$ ”) cm container.] Looking at the right triangular pyramid that we used for volume (from the top face, triangle ABD); we can see that the altitude of that triangle (i.e. the slant height  $s$  of the pyramid) can be found as follows.



Note that angle EFD is a right angle.

$$\text{Since } s = \sqrt{a^2 + \frac{b^2}{4}}, \text{ the area of one face } A = \frac{b\sqrt{a^2 + \frac{b^2}{4}}}{2}, \text{ and the total surface area, } SA = 2b\sqrt{a^2 + \frac{b^2}{4}}.$$

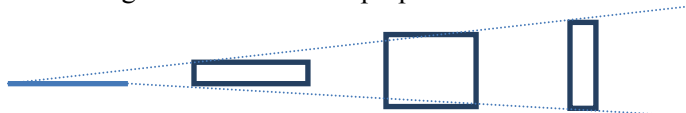
Are you surprised that, for the  $6 \times 6$  ( $a$  by  $b$ ) cm model,  $s = 3\sqrt{5}$  cm? Or that the  $SA = 36\sqrt{5}$  cm<sup>2</sup>?

To find the container edge length, find the length of a non-base edge,  $e$ , of the (isosceles) triangle ABD.

$$e^2 = \left(\frac{b}{2}\right)^2 + \left(\sqrt{a^2 + \frac{b^2}{4}}\right)^2 \rightarrow e = \sqrt{a^2 + \frac{b^2}{2}} \rightarrow \text{sum of all edge lengths} = 2b + 4\sqrt{a^2 + \frac{b^2}{2}}.$$

### Calculus Approach

Now look at the calculus approach to volume, and start out with the special case where  $a = b$ . Consider some of the rectangular cross-sections perpendicular to the axis:



The total volume of the container is the sum of the volumes of the rectangular slabs with thickness  $dx$ , each with area  $x(a-x)$ , as  $x$  goes from 0 to  $a$ :

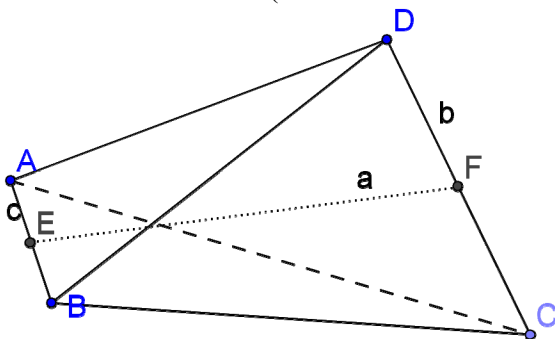
$$V_{ABCD} = \int_0^a x(a-x) dx = \int_0^a (ax - x^2) dx = \left(\frac{ax^2}{2} - \frac{x^3}{3}\right) \Big|_0^a = \frac{a^3}{6}, \text{ as from the geometric approach above.}$$

Now, let's generalize that to the case where  $a \neq b$  (but still starting with a cylinder):

$$V_{ABCD} = \int_0^a \left(\frac{b}{a} \cdot x\right) \left(b - \left(\frac{b}{a} \cdot x\right)\right) dx, \text{ in which the first factor of the integrand varies from 0 to } b \text{ as } x \text{ varies}$$

from 0 to  $a$ , while the second factor varies from  $b$  to 0. The integration yields  $V_{ABCD} = \frac{ab^2}{6}$ , from the geometric approach above.

As an extension for students to explore, consider the case where the original paper roll is a frustum of a cone instead of a cylinder. This implies that the widths of the ends of the container will differ; use  $c$  to symbolize the shorter end (but that does not matter).



Geometrically finding the volume by slicing the container in half, as we did above in the cylindrical case:

$$V_{ABDF} = \left(\frac{1}{3}\right) Bh = \left(\frac{1}{3}\right) \left(\frac{a \cdot c}{2}\right) \left(\frac{b}{2}\right) \text{ for one-half of the solid, then multiplying by } 2 = \frac{abc}{6}, \text{ the total volume.}$$

It is easy to see that the cylindrical formulas are just special cases of this. Developing this formula from calculus is just a generalization of the approach made above where

$$V_{ABCD} = \int_0^a \left(\frac{b}{a} \cdot x\right) \left(c - \left(\frac{c}{a} \cdot x\right)\right) dx, \text{ as the first factor of the integrand varies from 0 to } b \text{ while } x \text{ varies from}$$

0 to  $a$ , while the second factor varies from  $c$  to 0. The integration yields  $V_{ABCD} = \frac{abc}{6}$ .

So far, we have been able to approach things using both geometry and calculus, the geometric approach often being the easier. Now we must depend on the power of calculus: using a cylindrical tube, and for a given volume, what values of  $a$  and  $b$  minimize the surface-to-volume ratio? (This is an important packaging consideration.) Again, we will ignore the excess material needed to seal the ends, although that is a practical packaging consideration. Combining previously determined formulas in ratio form:

$$\frac{SA}{V} = \frac{2b\sqrt{a^2 + \frac{b^2}{4}}}{\frac{ab^2}{6}} = \frac{6\sqrt{4a^2 + b^2}}{ab}. \text{ Since } V = \frac{ab^2}{6}, b = \sqrt{\frac{6V}{a}} \rightarrow \frac{SA}{V} = \frac{6\sqrt{4a^2 + \frac{6V}{a}}}{a\sqrt{\frac{6V}{a}}}.$$

The final fraction can be tidied-up a bit, but it's hardly worth the trouble. Finding the derivative of  $\frac{SA}{V}$  with respect to  $a$ , and solving for the minimum point yields  $a = \sqrt[3]{3V}$ . [If you have lots of free time, you could do that work by hand. I used a TI-*n*spire CX CAS to do it.] Our first example, the  $6 \times 6$  cm container, had a volume of  $36 \text{ cm}^3$  and a surface area of  $36\sqrt{5} \text{ cm}^2$ , yielding  $\frac{SA}{V} = \sqrt{5} \sim 2.236$ . By choosing  $a = \sqrt[3]{3 \cdot 36} \sim 4.762$  (instead of 6),  $\frac{SA}{V} \sim 2.182$  (and  $b \sim 6.735$ ), a modest improvement.

Not surprisingly, the optimal  $\frac{SA}{V}$  ratio is not constant; it depends on the volume. [This may be easier for students to visualize with a sphere. The geometry concepts of surface area ratio and volume ratio for similar solids are fundamental here.]

However, since these polyhedrons would appear to be similar solids, you might guess that the ratio  $\frac{b}{a}$  is constant - and you would be right. As a clue, let's consider the optimized  $V = 36 \text{ cm}^3$  example two paragraphs above,  $\frac{b}{a} = \frac{6.735}{4.762} \sim 1.414$ , which looks suspiciously like  $\sqrt{2}$ . Combining the formulas for the volume,  $V = \frac{ab^2}{6}$ , and the optimum value of  $a$ ,  $a = \sqrt[3]{3V}$ , we get:

$$a = \left(3 \cdot \frac{ab^2}{6}\right)^{\frac{1}{3}} \rightarrow a^3 = \frac{ab^2}{2} \rightarrow \frac{b^2}{a^2} = 2 \rightarrow \frac{b}{a} = \sqrt{2}.$$

## Summing Up

Many other explorations are possible, some easy, some not so. A few suggestions:

- Determine the various angles formed at each of the four vertices and between the pairs of opposite faces.
- Determine the optimal — for the conical case, or when allowing for the additional material needed to seal the ends.
- We creased the edges to make flat faces – to make it easier to study. What happens if you don't crease them, allowing the faces to bulge?
- There's a *much* easier way to find surface area, especially for the cylindrical case. What is it?
- Clearly, if the end seams were parallel, the volume would be zero. However, does making them perpendicular *necessarily* maximize the volume?



High school CCSS content: HSA-SSE.A.1b – Interpret complicated expressions  
HSA-CED.A.2 – Create equations in two or more variables

HSG-SRT.B.5 – Use congruence and similarity criteria  
 HSG-GMD.A.3 – Use volume formulas  
 HSG-GMD.B.4 – Identify cross-sections of 3-dimensional objects

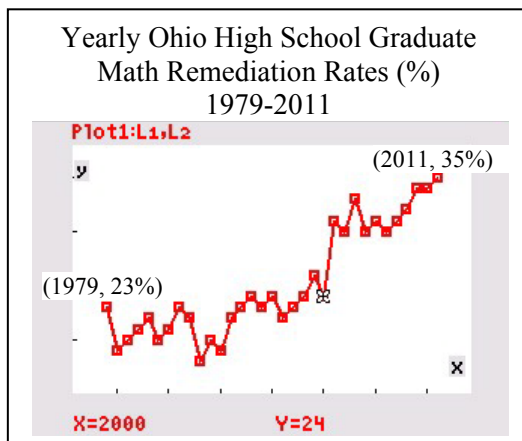
Container construction photos were taken by **Emma Siewny**.

Wendy's sour cream container photo source: <http://ericsaltchemistry.blogspot.com/>  
 Crafts video for making sour cream container: <http://www.youtube.com/watch?v=0rEvUkqqXnI>  
 or <http://www.youtube.com/watch?v=QCo3rvYV2og>

**Mr. Kinner** teaches honors and AP math courses at Fenwick High School in Franklin, OH. His focus is finding ways to breathe life into textbook mathematics for his students through explorations with technology and outside-the-box projects.



## Food for Thought



**Source:**  
 1979 to 2003 data from:  
[http://qry.regents.state.oh.us/cgi-pub/site\\_map?name=rem\\_rep&jsflag=yes](http://qry.regents.state.oh.us/cgi-pub/site_map?name=rem_rep&jsflag=yes)  
 2003 to 2007 data from:  
[http://regents.ohio.gov/perfrpt/hs\\_transition2003-2007/HS\\_Report\\_2003-07.pdf](http://regents.ohio.gov/perfrpt/hs_transition2003-2007/HS_Report_2003-07.pdf)  
**See:**  
[http://regents.ohio.gov/perfrpt/hs\\_2008/hs\\_trans\\_HS\\_rpt\\_AU08.pdf](http://regents.ohio.gov/perfrpt/hs_2008/hs_trans_HS_rpt_AU08.pdf) for links to various remediation reports

“Relative to learning algebra, one widely used strategy is to preserve the current approaches to teaching algebra and then address shortcomings in student outcomes with remediation.”

Stroup, W., Carmona, L., & Davis, S. M. (2013). “Improving on Expectations: Preliminary Results from Using Network-supported Function-based Algebra” in Oldknow, A. & Knights C. eds. (2013). *Mathematics education with digital technology: Education and digital technology*, 67. London: Bloomsbury Publishing Plc.