HEXAGON NOTATION.

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(1) Salmon, in the "Notes" at the end of his Conic Sections designates by $\begin{Bmatrix} ab \\ de \end{Bmatrix}$ the point of intersection of the lines ab, de; by $\begin{Bmatrix} ab, & cd, & ef \\ de, & fa, & bc \end{Bmatrix}$ the Pascal line which contains the three points indicated by the vertical columns; and by the following a g-point and an h-point, respectively:

$$\left\{\begin{matrix} ab, & de, & cf, \\ cd, & fa, & be, \\ ef, & bc, & ad, \end{matrix}\right\} (g); \qquad \left\{\begin{matrix} ab, & ce, & df \\ cd, & bf, & ae \\ ef, & ac, & bd \end{matrix}\right\} (h)$$

The lines in (g) (h), taken in pairs, indicate the Pascal lines which meet in a point, *but the lines do not give the hexagons.* In (h) only two Pascals are indicated, since ce, ac, do not meet on the Pascal line, nor do df and bd.

To get the hexagon indicated by the first and second lines of (g) start with ab of the first line; look up the letter with b in the second line, giving abe: then the letter with e in the first line, abed; then that with d in the second line, abedc; then that with f in the first line, getting, finally, abedcf.

Treating the pairs of lines in (g), (h) in this way we get, rather tediously, the hexagons:

$$\left\{\begin{matrix} abedcf \\ cdafeb \\ efcbad \end{matrix}\right\} (g); \qquad \left\{\begin{matrix} abfdce \\ cdbfea \\ efdbac \end{matrix}\right\} (h)$$

(2) I give here a notation *which indicates the hexagons by horizontal lines, as also their Pascal lines, when horizontal lines are taken in pairs.* The points (g), (h) above are:

$$\left\{\begin{matrix} ab, & ed, & cf \\ cd, & af, & eb \\ ef, & cb, & ad \end{matrix}\right\} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} (g'); \qquad \left\{\begin{matrix} ab, & fd, & ce \\ dc, & ae, & fb \\ ef, & db, & ac \\ ba, & ec, & df \end{matrix}\right\} \begin{matrix} (1) \\ (2) \\ (3) \\ (1'). \end{matrix} \; (h')$$

In (g′) line (2) is formed from line (1) by writing under each segment of (1) its opposite segment in (1), *reversing the hexagon order of letters in (1):* (3) from (2) as (2) from (1); (1) and (2) give the Pascal line of (1); (2) and (3) that of (2); (3), (1) that of (3).

In (g′) there is also cyclic permutation of the initial letters of the segments in one direction (to the right) and of the final letters in the other direction (to the left). This offers the easiest way of writing (g′).

In (h′), line (2) is formed from line (1) by setting under each segment of (1) its opposite segment, *retaining the hexagon order of letters in (1) in one column* (here the first) *and reversing it in the other two;* (3) from (2) as (2) from (1); (1′) from (3) as (3) from (2); (1′) is (1). The lines (1) and (2) give the Pascal of (1); (2), (3) that of (2); (3), (1′) that of (3).

(3) When only the hexagons which enter into (g) and (h) are desired, they may be written thus:

$$\left\{\begin{matrix} \text{a b c d e f} \\ \text{a f c b e d} \\ \text{a d c f e b} \end{matrix}\right\} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \ (g''); \qquad \left\{\begin{matrix} \text{c e a, b f d} \\ \text{e a c, d b f} \\ \text{a c e, f d b} \end{matrix}\right\} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \ (h'')$$

In (g″) one set of alternate letters (here a, c, e) is held fixed; the other set permuted cyclically (in either direction).

In (h″), line (1) is divided into two groups by a comma. Each group is permuted cyclically, the one in the opposite direction of the other. If the hexagons in (h″) are to indicate the same point as (h′), set astride the comma the segment of the first line of (h) in the column which was to hold the hexagon order (here ab) in (h″).

These notations lend themselves, as will be seen in the following, most readily to determine the whole geometry of the hexagon configuration.

(4) The g-point of any hexagon is the center of perspective of its two triangles of alternate sides, the axis of perspective being the Pascal line of the hexagon.

To write the g-point of any hexagon, interchange any pair of its alternate letters for the first line and proceed as in (g′) or (g″).

For a b c d e f (interchanging c and e) it is:

$$\left\{\begin{matrix} \text{ab, ed, cf} \\ \text{cd, af, eb} \\ \text{ef, cb, ad} \end{matrix}\right\} (g_1) \qquad \text{or} \qquad \left\{\begin{matrix} \text{a b e d c f} \\ \text{a f e b c d} \\ \text{a d e f c b} \end{matrix}\right\} (g_2).$$

(5) *The conjugate g-point of any given g-point*

The g-point of any hexagon of a g-point is one and the same g-point (the conjugate g-point).

For (g_1) of (4) it is g'_1 or g'_2 following:

$$\left\{\begin{matrix} \text{ab, cd, ef} \\ \text{ed, af, cb} \\ \text{cf, eb, ad} \end{matrix}\right\} (g'_1); \qquad \left\{\begin{matrix} \text{a b c d e f} \\ \text{a f c b e d} \\ \text{a d c f e b} \end{matrix}\right\} (g'_2).$$

Of a pair of conjugate g-points, one is inside the conic; the other outside; the line joining them is divided harmonically by the conic (Steiner).

The g-point *on* the Pascal of any hexagon is conjugate to that *of* the hexagon.

(6) *To find three hexagons for which a given g-point is the g-point.*

Write the conjugate g-point.

(7) *The two ordinary h-points of any hexagon and how to write them.* For any hexagon, as a b c d e f (1), write the triangle of alternate letters:

ac, ce, ea.

Write under each letter its opposite letter in (1) giving,

$$\left\{\begin{matrix} \text{ac, ce, ea} \\ \text{df, fb, bd} \end{matrix}\right\} (T).$$

The triangle, T, whose vertices are indicated by vertical columns here, is in perspective with each of the triangles of alternate sides of (1), giving for (1) two h-points. (T) may be called the Pascal triangle of the given hexagon.

To write the h-points of (1), write (1) forward and backwards as in,

a b c d e f (1)
a f e d c b (2)

Group the alternate letters of (1), (2) in two groups, in opposite directions, giving for (1) and (2) respectively:

a c e, f d b; a e c, b d f,

as the first lines of the desired h-points. Then complete as in section (3) giving:

$$\left\{\begin{matrix} \text{ace, fdb} \\ \text{cea, bfd} \\ \text{eac, dbf} \end{matrix}\right\} (h_1); \qquad \left\{\begin{matrix} \text{aec, bdf} \\ \text{eca, fbd} \\ \text{cae, dfb} \end{matrix}\right\} (h_2).$$

We call these *the two ordinary h-points of the corresponding hexagon* (1), to distinguish them from the unique *h-point of the same hexagon* (see section (9)).

(8) *The two ordinary hexagons of any h-point.*

While the hexagons of a g-point give only one g-point, they give, not six h-points, but only three.

Any two hexagons related as

ab, cd, ef
ba, dc, fe

will have one of their ordinary h-points in common. This relation is the same as,

ab, cd, ef
ab, ef, cd

The hexagons of any g-point show this sort of relation in pairs.

For a given h-point, like

$$\left\{\begin{matrix} \text{ace, fdb} \\ \text{cea, bfd} \\ \text{eac, dbf} \end{matrix}\right\} \text{(C)}$$

its two ordinary hexagons are given by reading the first and last letters of each line, in regular order, down the lines (or last and first) giving,

ab, cd, ef
ba, dc, fe;

also straddling the comma.

In the notation like h′ of Section (2), the two hexagons are gotten by reading zig-zag, the column which retained the hexagon order. For h′, it was the first column, giving ab, cd, ef and ba, de, fe.

(9) *The unique h-point of any hexagon.*

In any h-point, as (C) in (8), there are only nine of the fifteen hexagon lines. In (C) they are ac, ce, ef, fd, db, ba, ea, bf, cd.

The remaining six lines form a hexagon related thus uniquely to the given h-point.

To write the first line of the unique h-point of any hexagon, a b c d e f (1), write the alternate letters in two groups; ace, bfd; the second group begins with the letter adjacent the initial letter of the first group on the side in the direction of the first grouping and is taken in the direction opposite the first.

This gives for a b c d e f, the unique h-point.

$$\left\{\begin{matrix} \text{ace, bfd} \\ \text{cea, dbf} \\ \text{eac, fdb} \end{matrix}\right\} \text{(D)}$$

The geometric relation of a hexagon and its unique h-point will appear later. (See section 40).

(10) *To write the unique hexagon of any given h-point.*

Set the fourth letter of *any line* between the first and second; the sixth between the second and third,

That for (C) is a f c b e d;
That for (D) is a b c d e f.

(11) *Relation between the unique hexagon of an h-point and its two ordinary hexagons.*

The two ordinary hexagons of (D) in (9) are, by (8),

ad, cf, eb
da, fc, be

and the unique hexagon is, in (10), a b c d e f, but these three hexagons are those of the g-point.

$$\left\{\begin{matrix} \text{a b c d e f} \\ \text{a f c b e d} \\ \text{a d c f e b} \end{matrix}\right\}$$

(12) *To write a g-point and two h-points on a straight line.*

It follows at once from the definition of these points that the g-point and the two ordinary h-points of any hexagon are on a straight line.

(13) *To write a g-point and three h-points on a line.* (Salmon's G-line).

The hexagons of any g-point will give g', h_1, h_2 on a line; also g', h_2, h_3 on a line; also g', h_3, h_1 (See sec. (8), (11), (12)).

Therefore, g', h_1, h_2, h_3 are on a line, where g' is the conjugate g-point of the given g-point. (See (5)).

Therefore, *the g-point and three unique h-points of the hexagon of a g-point are collinear.* And what is the same thing and the same line, *the g-point and three pairs of ordinary h-points of the hexagon of a g-point are a g-point and three h-points on a line.*

(14) *The number of g-points.*

In forming g' in section (2) we reversed the hexagon order of segments in each column. This can be done in only one way.

Therefore each hexagon enters into only one g-point. Therefore twenty g-points.

(15) *The number of h-points, and how to write three h-points on any Pascal line.*

In forming (h') of section (2) we held the hexagon order in one column and reversed in two. This can be done in three ways, therefore each hexagon enters into three h-points, therefore, three times as many h-points as g-points. Therefore sixty h-points.

This is also shown in the notation of section (3), since the line (1) can be grouped in three ways as indicated.

To write three h-points on a Pascal line, proceed as in h' of (2), retaining the hexagon order in columns 1, 2, 3, in order; or, if using the notation in (3), use as initial lines, abc, def; bcd, efa; cde, fab.

(16) *The number of G-lines.*

By (13) each g-point gives a G-line through its conjugate g-point.

Therefore, twenty G-lines.

(17) *Given a g-point to write the three h-points on a G-line with it.*

Write the conjugate g-point (5) and the three h-points unique to its hexagons ((9), (13)). Also on any Pascal line there is one g-point and three h-points, ((14, (15)).

(18) *Given one h-point of a G-line to write the g-point and two remaining h-points.*

Write the hexagon unique (10) to the given h-point; then the two hexagons which enter with this hexagon into a g-point ((2) or (3)); then their two unique h-points, and the conjugate g-point. Thus, through any h-point goes only one g-line.

(19) *To write three g-points on a line.*

Write the g-points of the hexagons of any h-point. The triangles of alternate sides of the hexagons of an h-point are but three in perspective in pairs at three g-points (4), with axes of perspective concurrent in the h-point; therefore the three centers of perspective are collinear.

(20) *To write three lines of three g-points each, with a g-point in common.*

Write the three lines of g-points of three h-points *on any Pascal line*, (15). The g-point in common to the three lines is the g-point conjugate to that on the common Pascal line. ((19) and (5)).

(21) *The nine h-points on three Pascal lines which meet in a g-point establish the same three lines noted in (20).* (See (11)).

If the g-point lines in (20) are,

$$g', g_2, g_3$$
$$g', g_4, g_5$$
$$g', g_6, g_7$$

those noted in (21) will be (in addition), easily tested by writing them in full:

$$g', g_2, g_8;$$
$$g', g_4, g_9;$$
$$g', g_6, g_{10};$$
$$g', g_3, g_8;$$
$$g', g_5, g_9;$$
$$g', g_7, g_{10};$$

which are the same lines as in (20), since two points fix a line. Here g' is the conjugate of the g-point of the three given Pascal lines.

(22) By (21) *the nine h-points on three Pascal lines meeting at a g-point establish three lines of four g-points each through the conjugate g-point.* (Salmon's I-lines).

These lines are established by different sets of three points, *the conjugate g-point always included.*

(23) *Through the g-point where three Pascal lines meet goes also one G-line (17), with three h-points. These three h-points establish the same I-lines noted in (22), by sets of three points, the conjugate g-point always excluded.*

The lines will be (if written out):

$$g_2, g_3, g_8;$$
$$g_4, g_5, g_9;$$
$$g_5, g_7, g_{10}.$$

(24) By (22 and (23) it follows that the twelve h-points of the four lines of h-points (three Pascal lines and one G-line) passing through a g-point establish three I-lines of four g-points each through the conjugate g-point.

(25) By (24) there are fifteen I-lines.

(26) By (24) there must be four h-points grouped about a g-point (one on each of its Pascal lines and one on its G-line) which establish a single line of four g-points through the conjugate g-point; and for each g-point three such h-point quadrangles.

(27) To write such a quadrangle of h-points as noted in (26),

a b c, d e f (1)
a c e, b d f (2)
a e d, c b f (3)
a d b, e c f (4)

(1), (2), (3), (4) form the initial lines, properly grouped for such a quadrangle. Line (2) is formed from (1) by taking alternate letters, in regular order, in two groups as indicated; (3) from (2) as (2) from (1); (4) from (3) as (3) from (2). The h-points with (1), (2), (3), (4) as initial lines are:

$$\left\{\begin{matrix}\text{abc, def}\\ \text{bca, fde}\\ \text{cab, efd}\end{matrix}\right\} (h_1); \qquad \left\{\begin{matrix}\text{ace, bdf}\\ \text{cea, fbd}\\ \text{eac, dfb}\end{matrix}\right\} (h_2).$$

$$\left\{\begin{matrix}\text{aed, cbf}\\ \text{eda, fcb}\\ \text{dae, bfc}\end{matrix}\right\} (h_3); \qquad \left\{\begin{matrix}\text{adb, ecf}\\ \text{dba, fec}\\ \text{bad, cfe}\end{matrix}\right\} (h_4).$$

Now write by 4, the g-points of the hexagons in h_1, h_2, h_3, h_4, and they will be respectively:

g_1, g_2, g_3;
g_3, g_4, g_2;
g_2, g_1, g_4;
g_4, g_3, g_1.

∴ g_1, g_2, g_3 and g_4 are collinear.

(28) *To write three such quadrangles of h-points* (as in 27) *grouped about the g-point containing abcdef* (1).

Group (1) in the three ways:

abc, def; bcd, efa; cde, fab,

and form from each grouping a set as in (27).

(29) *The three g-points conjugate to those on the Pascals of an h-point are collinear.*

They are conjugates by 5 and collinear by 19.

(30) *The three g-points of the hexagons of an h-point are collinear with the g-point on the Pascal of the hexagon, unique to the h-point.*

In 27, (h_1) gave the g-points g_1, g_2, g_3.

The second hexagon in h_2 gives:

$$\left\{\begin{matrix}\text{c e b f a d}\\ \text{c f b d a e}\\ \text{c d b e a f}\end{matrix}\right\} (g_4).$$

where the middle line is the hexagon unique to h_1 (10).

(31) *To write four g-points on an I-line.*

Use (30), which shows there are 15 I-lines.

(32) *The four g-points of an I-line are on the Pascals of the hexagons unique to the four h-points of the corresponding h-point quadrangles as given in (27); as also on the Pascals of the hexagons ordinary.*

The hexagons unique to the four h-points in (27) are:

$$\left\{\begin{array}{l} \mathrm{a\ d\ b\ f\ c\ e}\ (1) \\ \mathrm{a\ b\ c\ f\ e\ d}\ (2) \\ \mathrm{a\ c\ e\ f\ d\ b}\ (3) \\ \mathrm{a\ e\ d\ f\ b\ c}\ (4) \end{array}\right\} (A).$$

and (1) is in g_4; (2) in g_1; (3) in g_3; (4) in g_4.

[Note that line (2) of (A) is formed from line (1), by writing first the alternate letters of (1), abc; then the second set of alternate letters of (1), beginning with the fourth letter from the initial letter of the first set, the second set being taken in the same direction as the first set. The set is unique].

By (11) the same g-points fixed by (A) are also on the Pascals of the hexagons ordinary, one set of which for the h-points of (27) is:

$$\left\{\begin{array}{l} \mathrm{af,\ be,\ cd}\ (\mathrm{I}) \\ \mathrm{af,\ cd,\ eb}\ (\mathrm{II}) \\ \mathrm{af,\ eb,\ dc}\ (\mathrm{III}) \\ \mathrm{af,\ dc,\ be}\ (\mathrm{IV}) \end{array}\right\} (B).$$

(33) Set (A) of (32) shows there are 15 I-lines, since any line of (A) uniquely determines all the rest. There are thus only 15 such complete quadrilaterals as (A).

In (A) there are 4 Pascal lines meeting in six Pascal points, and the triangle of any three of the lines is the Pascal triangle (7) of the fourth hexagon. These fifteen quadrilaterals determine the most important features of the hexagon geometry. (See second paper.)

(34) *To write three I-lines through any g-point.*

Form from each line of the g-point a group like (A) of (32), for the initial lines of the four g-points on each of the three I-lines. Therefore through each g-point pass three I-lines of four g-points each.

For the g-point $\left\{\begin{array}{l} \mathrm{a\ b\ c\ d\ e\ f} \\ \mathrm{a\ f\ c\ b\ e\ d} \\ \mathrm{a\ d\ c\ f\ e\ b} \end{array}\right\} (g_1).$

The three lines of g-points are (by initial lines)

$$\left\{\begin{array}{l} a\ b\ c\ d\ e\ f\ (g_1) \\ a\ c\ e\ d\ f\ b\ (g_2) \\ a\ e\ f\ d\ b\ c\ (g_3) \\ a\ f\ b\ d\ c\ e\ (g_4) \end{array}\right\};\quad \left\{\begin{array}{l} a\ f\ c\ b\ e\ d\ (g_1) \\ a\ c\ e\ b\ d\ f\ (g_5) \\ a\ e\ d\ b\ f\ c\ (g_6) \\ a\ d\ f\ b\ c\ e\ (g_7) \end{array}\right\};\quad \left\{\begin{array}{l} a\ d\ c\ f\ e\ b\ (g_1) \\ a\ c\ e\ f\ b\ d\ (g_8) \\ a\ e\ b\ f\ d\ c\ (g_9) \\ a\ b\ d\ f\ c\ e\ (g_{10}) \end{array}\right\}$$

In a set of g-points like the ten above no two are conjugate to each other.

(35) *To write two sets of conjugate g-points, of ten points in each set, each set having a point in it conjugate to a point in the other set.*

Treat two conjugate g-points as was (g_1) of (34).

(36) *To write four G-lines concurrent in an i-point.* (Salmon's notation).

Write four g-points on a line (by (31) or (32)); then the three h-points unique to the hexagons of each of the collinear g-points. This will give four lines containing three h-points each, one line passing through each of the g-points conjugate to the four collinear g-points.

These four g-lines are concurrent in an i-point.

The triangles of alternate sides of the hexagons of a g-point are but three (A, B, C). Their Pascal triangles (7) are three (P_1, P_2, P_3). By (4) A, B, C are in perspective at the conjugate g-point (g_1); by 7 and 8, A, P_1, P_2 are in perspective at h_1; B, P_2, P_3 at h_2; C, P_3, P_1 at h_3. Therefore, g', h_1, h_2, h_3 are collinear (a G-line). It follows easily that for four collinear g-points, the resulting four G-lines are concurrent.

(37) *There are three i-points on each G-line.*

Through each g-point there are three I-lines of four g-points each (34). Each gives an i-point on the G-line through the conjugate g-point, as in (36).

(38) *The three h-points unique to the hexagons of an h-point are all on the Pascal line of the hexagon unique to the given h-point.*

The h-point unique to a b c d e f (1) is, by (9),

$$\left\{\begin{array}{l} ace,\ bfd\ (2) \\ cea,\ dbf\ (3) \\ eac,\ fdb\ (4) \end{array}\right\}\ (h_1).$$

The three h-points unique to (2), (3), (4) are:

$$\left\{\begin{array}{l}\text{aef, cdb}\\ \text{efa, bcd}\\ \text{fae, dbc}\end{array}\right\}\begin{array}{l}(5)\\ (1)\ (h_2);\\ (6)\end{array}\quad \left\{\begin{array}{l}\text{cab, efd}\\ \text{abc, def}\\ \text{bca, fde}\end{array}\right\}\begin{array}{l}(7)\\ (1)\ (h_3);\\ (8)\end{array}\quad \left\{\begin{array}{l}\text{ecd, abf}\\ \text{cde, fab}\\ \text{dec, bfa}\end{array}\right\}\begin{array}{l}(9)\\ (1)\ (h_4)\\ (10)\end{array}$$

These all contain (1).

(39) *The ten hexagons in (38) are a Veronese group.*

In such a group there are ten h-points, on ten Pascal lines, three h-points on each Pascal line, three Pascal lines through each h-point; at each h-point two triangles in perspective, their vertices h-points, their sides Pascal lines; the axis of perspective being for each point the Pascal of the hexagon *unique to the center of the perspective.* It is a group of ten Pascal lines and their ten unique h-points.

(40) *Geometric relation between an h-point and its unique hexagon.*

None of the hexagons, 2, 3, 4, in h_1 in 38 enter into the h-points on the Pascal of abcdef (1), unique to (2, 3, 4). Thus h_1 is the center of perspective and the Pascal of (1) the axis of perspective for the two h-point triangles whose corresponding sides are 5, 6; 7, 8; 9, 10.

In the accompanying figure, the unique h-points and Pascals are numbered as II, 2; IV, 4; etc., giving ten centers of perspective, ten axes of perspective; for each center a pair of triangles in perspective.

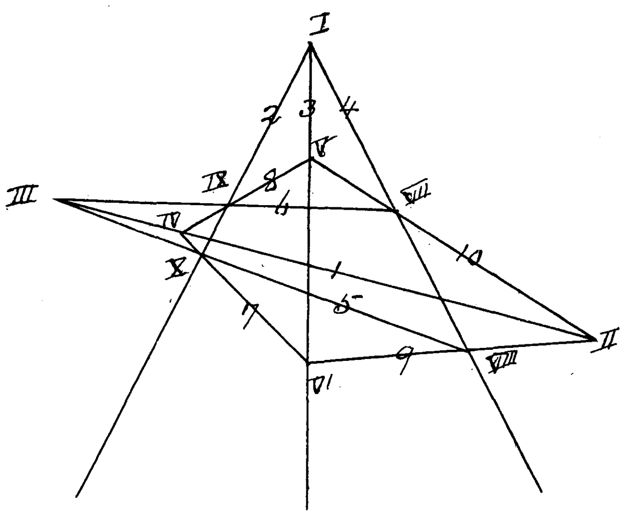

Fig. 1. A Veronese group as in 38 and 40. Ten such groups.

(41) *To write the six Veronese groups.*

Select the six hexagons of two conjugate g-points and treat each as was (1) in (38).

(42) A Veronese group may also be sorted out thus:

Start with any hexagon a b c d e f (1). Hold a, c fixed, permute all the other letters cyclically (forward); then hold c, e fixed; then e, a, giving:

$$\left\{\begin{array}{l} \text{a b c d e f (1)} \\ \text{a f c b d e (2)} \\ \text{a e c f b d (3)} \\ \text{a d c e f b (4)} \end{array}\right.$$
$$\left\{\begin{array}{l} \text{c b e d f a (5)} \\ \text{c a e b d f (6)} \\ \text{c f e a b d (7)} \end{array}\right.$$
$$\left\{\begin{array}{l} \text{e d a f b c (8)} \\ \text{e c a d f b (9)} \\ \text{e b a c d f (10)} \end{array}\right.$$

(43) The six Veronese groups may be formed as in (42), by starting with the six lines of two conjugate g-points.

(44) No two h-points of a Veronese group and no g-point and h-point of such a group are connected by a G-line. These lines tie the different groups, one h-point from each of three groups, and a g-point from a fourth.

(45) *The ten g-points in any Veronese group lie on five I-lines, four points on each line, two lines through each h-point. (The arrangement of the ordinary five point star, four point star, or three point star).*

Write a b c d e f forwards and backwards as in

ab, cd, ef (F)
af, ed, cb (B)

Form from (F) and (B) two groups as follows:

$$\left.\left\{\begin{array}{l} \text{ab, cd, ef (F)} \\ \text{ab, ef, dc (2)} \\ \text{ab, dc, fe (3)} \\ \text{ab, fe, cd (4)} \end{array}\right.\right\} (A_1)\ ; \quad \left.\left\{\begin{array}{l} \text{af, ed, cb (B)} \\ \text{af, cb, de (5)} \\ \text{af, de, bc (6)} \\ \text{af, bc, ed (7)} \end{array}\right.\right\} (A_2)$$

Here are seven hexagons in the Veronese group of (42).

On writing out the g-points *on* the Pascals of (A_1), they will be g_1, g_5, g_6, g_7 of 34; those on the Pascals of (A_2) are g_1, g_{10} g_9, g_8.

Thus any hexagon treated as in (A_1), (A_2) will give two lines of four g-points each, *inside the same Veronese group*. (The third line of g-points runs to different groups).

Now reverse (2), (3), (4) and form similar groups to (A_1).

$$\left\{\begin{array}{l}\text{ac, df, eb(2)}\\ \text{ac, eb, fd(8)}\\ \text{ac, fd, eb(9)}\\ \text{ac, eb, df(6)}\end{array}\right\}\text{(C)};\quad \left\{\begin{array}{l}\text{ae, fc, db (3)}\\ \text{ae, db, cf (5)}\\ \text{ae, cf, bd(10)}\\ \text{ae, bd, fc (9)}\end{array}\right\}\text{(D)};\quad \left\{\begin{array}{l}\text{ad, ce, fb (4)}\\ \text{ad, fb, ec (8)}\\ \text{ad, ec, bf (7)}\\ \text{ad, bf, ce(10)}\end{array}\right\}\text{(E)}.$$

Here is a total of ten hexagons, all in the Veronese group of (42).

Each set, A_1, A_2, C, D, E indicates a line of four g-points; and each hexagon occurs twice. Therefore the g-points within any Veronese group are on five lines, four points on each line and two lines through each point.

(46) *The star of g-points in each Veronese group.*

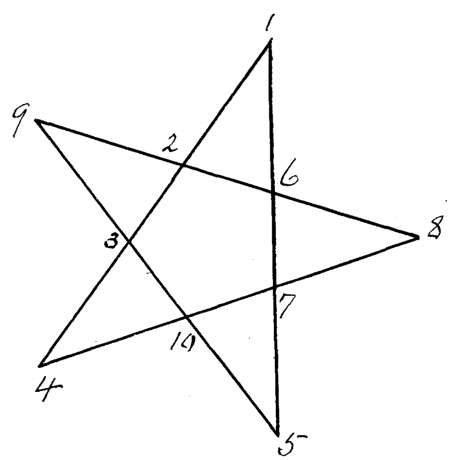

Fig. 2. The g-point star in a Veronese Group. (As it might be).

As shown by A_1, A_2, C, D, E; 2 in Fig. 2 indicates the g-point on the Pascal of hexagon (2).

A different arrangement of the points will give a three point or four point star.

(47) *Each pair of Veronese groups has an I-line of g-points in common.*

Any two hexagons related as (see 8),

ab, cd, ef (1)
ab, ef, cd (2)

give, when used as the leading lines of a set like (A_1) in (45), the same I-line; (2) will lead to the line g_1, g_7, g_6, g_5 on the Pascals of (A_1) in (45).

$$\left.\begin{matrix} \text{ab, ef, cd} \\ \text{ab, cd, fe} \\ \text{ab, fe, dc} \\ \text{ab, dc, ef} \end{matrix}\right\} \begin{matrix} (2) \\ (F). \end{matrix}$$

But (A_1) and (F) belong to different Veronese groups.

Thus the six stars of the I-lines in the six Veronese groups are linked by having one line in common between each pair; each star has one line in common with each of the five other stars. See (48), which shows that all the stars could not be five pointed.

(48) *Numerical table for the I-lines of g-points.*

Selecting any g-point as

$$\left\{\begin{matrix} \text{ab, cd, ef} \\ \text{ed, af, cb} \\ \text{cf, eb, ad} \end{matrix}\right\} (1)$$

the collinear groups of I-lines through (1) have for initial lines of their g-points:

$$\left\{\begin{matrix} \text{ab, cd, ef (1)} \\ \text{ab, ef, dc (2)} \\ \text{ab, dc, fe (3)} \\ \text{ab, fe, cd (4)} \end{matrix}\right\}; \quad \left\{\begin{matrix} \text{ed, af, cb (1)} \\ \text{ed, ch, fa (5)} \\ \text{ed, fa, bc (6)} \\ \text{ed, bc, af (7)} \end{matrix}\right\}; \quad \left\{\begin{matrix} \text{cf, eb, ad (1)} \\ \text{cf, ad, be (8)} \\ \text{cf, be, da (9)} \\ \text{cf, da, eb (10)} \end{matrix}\right\}$$

Denote by 2 the g-point on the Pascal of hexagon (2), etc. and write the above lines thus:

	1	
2	5	8
3	6	9
4	7	10

(a) The vertical columns are I-lines through 1.

(b) The line joining 2 and 5 meets that joining 3 and 6 at 10′ (the conjugate of 10); and so, in general, the line joining any two points in a horizontal line meets the line joining any other two, in another horizontal line and in the columns of the first two, in the conjugate of the point not in these columns nor lines (2, 8 meets 4, 10 at 6′; 7, 10 meets 6, 9 at 2′, and so on).

(c) Any two points in a horizontal line are in a line with the conjugates of those not in the line nor columns of this selection (2, 5, 9′ 10′ is a line; 3, 9, 5′, 7′ is a line, etc.)

d The conjugates of the horizontal lines are in line with 1′ (1′, 2′, 5′, 8′ is a line, etc.)

Proof of (c) and (b):

Collinear groups with (2) (in addition to that given) are (using other hexagons in a g-point with (2)):

$$\left.\begin{matrix} \text{ac, eb, df (2)} \\ \text{ac, df, be (5)} \\ \text{ac,be,fd (10')} \\ \text{ac, fd, eb(9')} \end{matrix}\right\} \text{(L)} ; \quad \left.\begin{matrix} \text{af, ec, db (2)} \\ \text{af, db, ce (6')} \\ \text{af, ce, bd (8)} \\ \text{af, bd, ec (7')} \end{matrix}\right\} \text{(M)}$$

as tested by sections 4 and 5.

And collinear groups with (3) are:

$$\left.\begin{matrix} \text{fc, ae, db (3)} \\ \text{fc, db, ea (8')} \\ \text{fc, ea, bd (6)} \\ \text{fc,bd,ae (10')} \end{matrix}\right\} \text{(N)} ; \quad \left.\begin{matrix} \text{de, fb, ac (3)} \\ \text{de, ac, bf (5')} \\ \text{de, bf, ca (9)} \\ \text{de, ca, fb (7')} \end{matrix}\right\} \text{(O)}$$

(L), (M), (N), (O) prove (C), while (L) and (M) show that line 2, 5 meets line 3, 6 at 10′, and so in same way for other statements.

Taking the g-point conjugate to 1, and treating it the same way as 1, we get

	1′	
6′	5′	7′
3′	2′	4′
9′	8′	10

as is easily tested.

Thus the fifteen I-lines given by the numerical table are:

1, 2, 3, 4	6, 9, 2′, 4′
1, 5, 6, 7	4, 7, 8′, 9′
1, 8, 9, 10	4, 10, 5′, 6′
2, 5, 9′, 10′	7, 10, 2′, 3′
2, 8, 6′, 7′	1′, 2′, 5′, 8′
5, 8, 3′, 4′	1′, 3′, 6′, 9′
3, 6, 8′, 10′	1′, 4′, 7′, 10′
3, 9, 5′, 7′	

as illustrated on the following diagram.

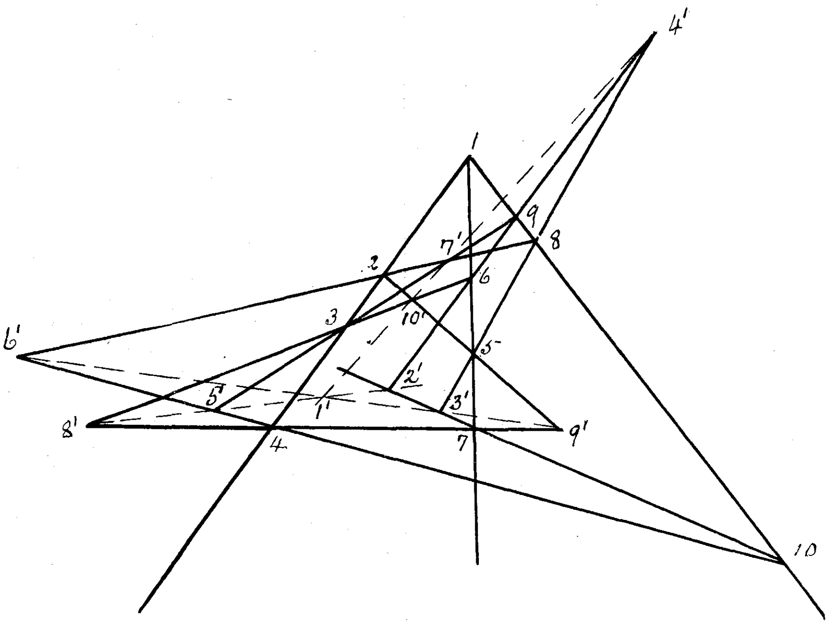

Fig. 3. The 15 I lines and 20 g points. 10 g points and conjugates.

(49) *The g-points not connected by I-lines.*

In the Veronese group in (46), no line runs from the g-point 1 on a b c d e f (1) to 8, 9, 10, but this

$$\left\{\begin{array}{ll} \text{ace, bfd} & (8) \\ \text{cea, dbf} & (10) \\ \text{eac, fdb} & (9) \end{array}\right\} \text{(h)}$$

is the point unique to (1) (See section 9).

Thus in any Veronese group, no g-point on the Pascal of any hexagon is connected by I-lines to the g-points on the Pascals of the hexagons of the h-point unique to the given hexagon.

Through each point pass three Pascal lines (of different Veronese groups), and thus the g-point 1, is not connected to any of the g-points on any of the Pascal lines of any of the nine hexagons unique to the hexagons of the given g-point; and no g-point is connected by an I-line to its conjugate g-point.

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