

HALL'S CONJECTURE ON EXTREMAL SETS FOR RANDOM TRIANGLES

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ABSTRACT. In this article, we discuss Hall's conjecture [17] about the distribution of random triangles. We consider the probability that three points chosen uniformly at random, in a bounded convex region of the plane, form an acute triangle. Hall's conjecture is the "isoprobabilistic inequality" which states that this probability should be maximized by the disk.

1. WHAT IS THE PROBABILITY THAT A RANDOMLY CHOSEN TRIANGLE IS ACUTE?

A classic question in geometric probability is understanding the distribution of shapes of random triangle.

This question dates back to 1861, when W.S.B. Woolhouse [27] asked for the probability that a random triangle is acute. This question was posed as a mathematical puzzler for *The Lady's and Gentleman's Diary*, and so Woolhouse also provided a solution. As noted in [6], Woolhouse did not specify how to choose a random triangle, and consequently his problem was ill-posed. However, his answer ($4\pi^{-2} - 1/8$) is correct if the three vertices are chosen uniformly at random in a disk.

The question was further popularized by Charles Dodgson (better known as Lewis Carroll), who posed the challenge in his lesser known book *Pillow Problems* [5] — a work containing 72 problems (with solutions) of mathematical olympiad caliber. However, his answer was different than Woolhouse because he used a different method to define a random triangle. He assumed that the longest side of the triangle is a known line segment, and then the third point is chosen uniformly at random

After some initial thought, one realizes that the answer depends on how random triangles are defined. Many methods have been proposed to define random triangles; such as randomly

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choosing the side lengths or angles, see [16], [6]. Many of the historical methods were ad hoc and involved arbitrary decisions, and as a result seem to contradict each other. There is an excellent paper by Stephen Portnoy [23], which explains how many of the natural methods for choosing random triangles are unsatisfactory.

The modern approach, originally due to D.G. Kendall [14], is to study the *shape space*, which is the moduli space of triangles modulo similarities and try to understand the natural geometry on this space. Kendall discovered that this space naturally can be associated with a hemisphere with its round geometry. This gave a natural way to understand the shape of random triangles and to answer probability questions in a way that did not depend on ad hoc decisions. For more history on the shape manifold and some modern results using Grassmannians, Cantarella et. al. [4] is a good resource. In this new formulation, the probability that a random triangle is acute is exactly $1/4$.

However, research is still being done when the vertices are picked according to some distribution, which is the focus of this paper. It is worth noting that people have studied the question when the vertices are chosen using different distributions. Explicit calculations have been done in the following cases: (i) the vertices are chosen from a Gaussian distribution (in which case the probability is exactly $1/4$)¹, [19]; (ii) uniformly at random in a rectangle [20], (iii) uniformly at random in a triangle [1].

1.1. Hall's conjecture. In 1982, Glen Hall [17] computed the probability that three points chosen uniformly at random in the n -ball would form an acute triangle. He did so by applying Baddeley's generalization of Crofton's differential equation [2] to simplify the relevant integrals. In his paper, he observed that the n -ball is a critical point for this probability and that as n increases, the probability of choosing an acute triangle increases and converges in the limit to 1. In light of this, he conjectured among convex domains in \mathbb{R}^n , the probability that three randomly chosen points form an acute triangle is maximized when the domain is the n -ball. In my work, we will consider a natural strengthening of this conjecture. Before doing so, I will

¹The fact that this agrees with the probability from the shape manifold is not a coincidence.

need to introduce some notation.

Notation: I will typically denote a convex region in \mathbb{R}^n by S . When considering a 1-parameter family of such regions, with parameter t , I will denote it by $S(t)$. In this context $S(0)$ will invariably denote the n -ball with center the origin. I will use $p(S)$ to denote the probability that $\triangle T$ is an acute triangle, where triangle $\triangle T$ is obtained by choosing three vertices in S i.i.d. uniformly. Furthermore, $F_S(\theta)$ will denote the cumulative distribution function (CDF) of the probability that the largest angle of $\triangle T$ is less than or equal to θ .

The Strong Hall Conjecture is the inequality:

Conjecture 1 (Strong Hall Conjecture).

$$(1) \quad F_S(\theta) \leq F_{n\text{-ball}}(\theta).$$

Note that in this new statement, Hall's paper uses $P(S)$ to be $F_S(\frac{\pi}{2})$ and so Hall's conjecture is the sub-conjecture that $F_S(\frac{\pi}{2}) \leq F_{n\text{-ball}}(\frac{\pi}{2})$. Heuristically, Hall's conjecture seems almost certain to be true. In order to maximize the probability, the first variation of the probability must vanish. From Hall's work, it appears that the only way for this to occur is for the domain to have $SO(n)$ -symmetry. Furthermore, increasing the dimension increases the probability, so there is no expected degeneracy (such as the probability being maximized on a lower dimensional subset). Nevertheless, the conjecture remains open since translating statements about probability into geometric data is a difficult problem. The Strong Hall's Conjecture fails if S is allowed to be non-convex. For instance, if one picks three points uniformly from three disks centered at the vertices of a large equilateral triangle, then there is a relatively large probability of forming a triangle that is very close to equilateral.

1.2. Some background on Crofton's differential equation and variational calculus. Although I do not use Crofton's differential equation for my work, it plays an important role in the history of the problem. It describes how a geometric probability evolves as a region expands and was originally published in the Encyclopedia Britannica in 1885 [10]. Roughly

speaking, it states that when some region $S(t)$ expands as time goes by, the probability $P(t)$ of some event evolves in the following way:

$$\frac{d}{dt}p = \frac{1}{V}(q - p)\frac{dV}{dt}$$

Here, $V(t)$ is the volume of S and q is the probability of the event when one picks a point on the boundary of S . Crofton's differential equation was originally designed as a tool to simplify complicated geometric integrals that appear in geometric probability. It can be view as an early prototype for modern variational techniques in geometric probability.

By modern standards, Crofton's differential equation is vague as neither p nor q are rigorously defined. There is also no consideration of smoothness. In 1977, Baddeley, having studied Crofton's work, wrote the formula down in modern notation and made the equation precise[2]. In practice, it's use is the following: if one knows a natural evolution for which $\frac{d}{dt}P = 0$ (such as homothetically expanding the region), then one can use this formula to show that $q = p$. It is generally easier to compute Q rather than P , so this greatly simplifies calculation.

The moving manifold approach has largely been replaced by variational techniques. The modern variational approach to this problem is to consider the probability as a functional and compute its variations. This is a much more flexible approach with considerably more power, but it lacks some of the elegance of the moving manifold approach. For this project, I needed the additional generality of the variational approach, though I tried to keep the notation consistent with the moving manifold approach, when possible.

In order to present the results, it is necessary to discuss the difference between local extrema and weak local extrema. Following the convention of [13], I say that the functional $p(S)$ has a weak local maximum for $S = S_0$ if there exists an $\epsilon > 0$ such that $p(S) - p(S_0) < 0$ for all S in the domain of p which satisfy the condition $d_1(S, S_0) < \epsilon$, where d_1 corresponds to distance in the C^1 norm. The advantage of studying weak local maxima is that is easy to prove that they exist; a sufficient condition is that the second variation of $p(S)$ is strongly negative at S_0 .

There is also a stronger version of a local extrema, which are simply called local maxima. We say that the functional $p(S)$ has a local maximum for $S = S_0$ if there exists an $\epsilon > 0$ such that $p(S) - p(S_0) < 0$ for all S in the domain of p which satisfy the condition $d_H(S, S_0) < \epsilon$. Here, d_H is the Hausdorff distance, which yields a coarser topology. It is much more natural to consider convex regions in terms of Hausdorff distance, but finding strong extrema is often much more difficult than finding weak ones.

1.3. The results.

Theorem 2. *Let S be a bounded convex subset of \mathbb{R}^2 and pick three points uniformly at random from S . Define $p(S)$ as the probability that three points picked uniformly from S form an acute triangle. Then the ball is a weak local maximum for P . More precisely, the second variation of the probability is strongly negative. An immediate consequence is that given a C^2 one-parameter family of convex regions $S(t)$ such that $S(0)$ is the disk and whose first variation modulo congruences is non-zero, then $p(S(t))$ has a local maximum at 0.*

The proof of Theorem 2 uses calculus of variations, but is directly inspired by Baddeley's generalization of Crofton's differential equation [2]. After calculating the second variation, I applied the Fourier transform to the variation, and used the Plancherel theorem and correlation theorem to show that the second variation is strongly negative. This theorem holds in weaker regularity than C^2 , and I discuss the specific regularity necessary for the variation. For the purpose of conciseness, all the calculations in my work are terms of Hall's original conjecture on acute triangles. However, all of the results generalize to prove a weak local version of the Strong Hall's Conjecture. Beautiful and unexpected patterns emerge from Fourier analysis used to consider the Strong Hall's conjecture, and this deserves its own treatment. I plan to discuss this topic in a follow up article.

I then extend this result to \mathbb{R}^3 . The proof of Theorem 3 is similar to that of Theorem 2 but more involved. I use the Plancherel theorem and correlation theorem for $SO(3)$ (which is compact but non-Abelian) to calculate the second variation. Furthermore, the calculations are too involved to do by hand, and so require Mathematica.

Theorem 3. *Let S be a bounded convex subset of \mathbb{R}^3 and consider p as before. Then the ball is a weak local maximum for p . More precisely, the second variation of the probability is strongly negative. Therefore, given a C^2 one-parameter family of convex regions $S(t)$ such that $S(0)$ is the ball and the first variation modulo congruences is non-zero, then $p(S(t))$ has a local maximum at 0.*

Theorem 4 removes the regularity assumptions and prove that the disk is a local maximum for the probability in the Hausdorff topology. The proof uses some algebraic topology to construct a particular time variation from D to S . Along this variation, I prove that the second derivative of the probability is uniformly Hölder-1/2 continuous along the variation. This can be thought of as a type of local $C^{2,1/2}$ estimate on the probability near the disk. By Theorem 2, the second derivative of the probability is strictly negative at 0, and so is strictly negative along the entire variation for some uniform t . I construct a super-solution to the ordinary differential equation for the probability and find an upper bound on the probability which is smaller than that of the disk.

Theorem 4. *Let S be a bounded convex subset of \mathbb{R}^2 and pick three points uniformly at random from S . Then disk is a local maximum for the probability that the three points picked uniformly from S formed an acute triangle. More precisely, there exists ϵ such that given a convex region S with $d_H(D, S) < \epsilon$, then $p(S) < p(D)$. Here d_H is the Hausdorff distance between the S and D .*

Theorem 5. *Let S be a convex subset of \mathbb{R}^2 whose isoperimetric ratio is greater than $\frac{2888}{9}$. Then $p(S) < p(D)$. In particular, the probability is $O(R^{-1})$ where R denote the isoperimetric ratio by R .*

Theorem 5 is well-known in the literature, but we were not able to find any explicit estimates. Convex regions with large isoperimetric ratio are very long and thin, so the three points are nearly collinear with large probability and thus have very small probability of forming an acute triangle. The numeric estimate is helpful for the last section. Since the only non-compact parts of the moduli space of convex figures (in the C^0 sense) correspond to blow-ups of the

isoperimetric ratio, this shows that the supremum is achieved on a compact subset in the C^0 topology. Theoretically, this reduces the proof of the full conjecture to a finite computation, although it is not currently tractable.

These results will be published in a separate volume, so we do not include the proofs in this report.

1.4. Isoperimetric inequalities and other applications. Hall's conjectured inequality falls into a class of inequalities that appear to be probability versions of the isoperimetric inequality. The classic isoperimetric inequality states that of all unit volume convex regions, the measure of the boundary is minimized when the convex region is the disk. In the probabilistic inequalities, there is some geometric function, and the expected value of that function realizes an extrema when the domain is the disk. We refer to these inequalities as *isoperimetric inequalities*, although they have also been called extremal problems [9].

The most famous version of such an inequality is the Rayleigh-Faber-Krahn inequality. This states that given a bounded domain in \mathbb{R}^n , its first Dirichlet eigenvalue is no less than the corresponding Dirichlet eigenvalue of a Euclidean ball with the same volume. If we interpret the first Dirichlet eigenvalue of a domain in terms of exponential rate of exit times of Brownian motion, then the Dirichlet eigenvalue quantifies an aspect of probability for a geometric process. In this interpretation, the Rayleigh-Faber-Krahn inequality states that the ball realizes an extrema. A similar phenomena occurs in the most famous problem in geometric probability: Sylvester's 4-point problem [22]. In that case, the disk is known to maximize the probability modulo affine transformations [8].

Another natural conjecture is that of among unit volume domains, the expected value of the distance to the boundary is maximized when the domain is the ball. To prove this, it suffices to obtain an estimate on the total integral of the solution to the eikonal equation with Dirichlet conditions. Using recent work of Domokos and Lángi [11], a proof using the eikonal abrasion flow may be feasible, but we will not consider that here as it leads us too far astray from our original project.

At first appearance, all these problems seem unrelated. However, all of them seem to satisfy probabilistic versions of the isoperimetric ratio. It is worth noting that these isoprobabilistic inequalities are not universal. There are natural geometric problems, such as the so-called “grass-hopper problem,”² for which the disk is not a maxima [14]. It is of interest to determine conditions on a geometric problem so that an isoprobabilistic inequality holds.

Understanding random triangles is an interesting problem in itself, and has appeared in various spatial processes and applications. One classic example is from Broadbent’s 1980 paper “Simulating the Ley-Hunter,” [3] which statistically tested the hypothesis that ancient megalithic sites were built along ley-lines. That is to say, giant stone structures in England such as Stonehenge were purposely placed collinearly. As Broadbent was unable to calculate precise distributions of random triangles, he instead used computer simulations. His results suggested that although there was greater collinearity than expected, it was probably due to the clustering of the megalithic sites.

D.G. Kendall [18] further studied the problem, and used a method similar to Crofton’s differential equation to understand the distribution of the shape of the triangles when three points are chosen from a convex region. He was able to calculate this distribution explicitly when the region is a disk and show that the distribution of shapes is very close to being uniformly distributed. The importance of this is that it allows collinearity tests to be applied assuming a uniform prior, without introducing too much error. As one deviates from a disk, the assumption of uniformity introduces more error, and may fail to be useful in those cases.

One other interesting result on the subject is due to David Mannion, who considered a sequence of random triangles by choosing successively the three vertices of one triangle at random in the interior of its predecessor. He showed that almost surely the three points converge to a collinear pattern. In fact, he was able to calculate the exact Lyapunov exponent for this process [21].

²This problem was featured in a recent [FiveThirtyEight Riddler](#) [24].

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