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TISTICAL ANALYSIS OF
GRAVITY ANOMALIES

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FOREWORD

This report was prepared ~~by Dr. R. A. Hirvonen, Research Associate,~~ of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF19(604)-6201, OSURF Project No. 1058, under the supervision of Dr. W. A. Heiskanen, Director ~~of the Institute~~. The contract covering this research is administered by the Geophysics Research Directorate, Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Engineers.

ABSTRACT

In the first part of this paper the author continues his research on the statistics of gravity anomalies. New and independent methods for computing the statistical functions E_s and G_s are described. E_s, the error of representation, can be obtained using a map with iso-anomaly curves. G_s, the root mean square anomaly, may be computed if the covariance function, C_d, is given empirically. The formulas are illustrated by some numerical examples.

The second part treats the continuation of the gravity anomaly field to different elevations. An earlier integral formula, expressing them in terms of gravity anomalies at ground level, can be simplified and adapted for electronic computing. For manual computation the statistical method described above is used to find a template which gives satisfactory accuracy with the least amount of labor.

The derived methods for the manual and high speed computation for the extension of the gravity field in unsurveyed areas and its continuation to high elevations have great theoretical and practical significance.

Comparison of different methods continues.

Statistical Analysis of Gravity Anomalies

The statistical analysis of gravity anomalies serves several purposes. First, when geodetic computations, e.g., applications of the formulas of Stokes and Vening Meinesz, are to be carried out, we want to know the precision obtainable. Second, we must plan the most economical method of computation warranting a given precision. Third, for the interpolation of anomalies between the observed stations, and especially for the hypothetical filling of the large unsurveyed regions, we must assume that the anomaly field complies with the general statistical pattern.

The statistical analyses require very extensive computations. In order to reduce the extra work, such methods should be used which are in close connection with the main routine of the geodetic computations.

When the gravimetric survey of a region has been completed, the routine of the geodetic computations usually starts with the evaluation of the mean anomalies within small compartments of a network of coordinate lines. We may assume here that the compartments are blocks with sides s. If trapezoids formed by the meridians and parallels have been used, we replace s by

$$s = \sqrt{ab} \tag{1}$$

where a and b are the average sides of the trapezoids.

Denoting always by M(x) the mean value of any variable x over the entire surveyed region and by m(x) the mean value over one individual square with side s, we define and compute several statistical quantities as follows: Mean anomalies of the squares

$$\Delta g_s = m(\Delta g) \tag{2}$$

Variance of the anomalies

$$G_o^2 = M(\Delta g^2) \tag{3}$$

Variance of the mean anomalies

$$G_s^2 = M(\Delta g_s^2) \tag{4}$$

If one individual point anomaly within a particular square is taken as the representative of the mean anomaly of the entire square, we make an error

(5) $$\epsilon = \Delta g - \Delta g_s$$

The error of representation, E_s, is defined by the formula

(6) $$E_s^2 = M(\epsilon^2)$$

Now we can use the formula

(7) $$G_o^2 = G_s^2 + E_s^2$$

as a check, or, instead of (3) for the computation of G_o^2.

Often the results of the gravimetric survey are given in the form of a map with isoanomaly curves. Suppose that the region has been divided into squares which are so small that the isoanomaly curves within most squares are parallel straight lines. Denoting by d the average distance and by δ_g the interval of the anomalies between the contour lines we have

(8) $$E_s^2 = \frac{\delta_g^2}{12} M\left(\frac{s^2}{d^2}\right) = \frac{s^2 \delta_g^2}{12} M\left(\frac{1}{d^2}\right)$$

Accordingly, a rapid estimation of E_s for the smallest values of s can be carried out by counting the number of curves in the following way. At various spots, chosen regularly and evenly over the entire region surveyed, we read the variation of the gravity

(9) $$\epsilon_s = \frac{s \cdot \delta_g}{d}$$

over a constant short distance s perpendicular to the contour lines. Taking the mean of ϵ_s^2 over all spots, we obtain

(10) $$E_s^2 = \frac{1}{12} M(\epsilon_s^2)$$

The next step is to compute the mean anomalies Δg_{ks} and the pertaining statistical quantities for a network of larger squares, each one consisting of k^2 small initial squares. Denoting by

(11) $$\epsilon_k = \Delta g_s - \Delta g_{ks}$$

the deviation of the mean anomaly of each small square from the new group mean, we have

(12) $$E_{ks}^2 = E_s^2 + M(\epsilon_k^2)$$

(13) $$G_{ks}^2 = M(\Delta g_{ks}^2)$$

(14) $$G_o^2 = E_{ks}^2 + G_{ks}^2$$

The last equation can be used as an approximate check only.

In the same way we may now proceed to <u>still larger squares.</u> Then we have a series of values of both G_s^2 and E_s^2 for different values of s, and two curves can be drawn showing G_s^2 and E_s^2 as continuous functions of s. Of course, an adjustment is required in order to maintain the condition

(7) $$G_s^2 + E_s^2 = G_o^2 = \text{constant.}$$

It is advisable to draw the curve for the function

(15) $$B = \frac{E_s^2}{G_o^2}$$

in order to make the curves obtained from different regions more comparable.

B is an increasing function of s. For small values of s we have, according to (9) and (10)

(16) $$B = \frac{s^2}{S^2}$$

where S can be considered as a constant. With increasing values of s, the function B approaches the limit 1. Therefore, the experimental function B which has been found by the statistical methods described above can be replaced by a mathematical function

(17) $$B = \frac{s^2}{S^2 + s^2}$$

or, preferably by

(18) $$B = \frac{ps^2 + s^4}{q + rs^2 + s^4}$$

Considering the fluctuations of the statistical values, this function with three parameters p, q, r may give a sufficient accuracy in all practical cases. Then these parameters, together with the fourth parameter G_o, give a general picture of the statistical behavior of the gravity anomalies in any region.

As another method of the statistical analysis, we can study the correlation of anomalies over a given constant distance. Suppose that we have a great sample of pairs of gravity stations with a constant distance d and evenly distributed over the surveyed region. Half of the pairs should be taken in north-south direction and half in the east-west direction, preferably along two sets of profiles. Then we can compute the covariance of anomalies

(19) $$C_d = M(\Delta g_1 \Delta g_2)$$

With increasing values of d, we obtain C_d as a decreasing function of d with the maximum value

(20) $$C_o = G_o^2$$

for d = 0. For small values of d we have

(21) $$C_d = C_o - cd^2$$

For large values of d, the empirical function C_d may fluctuate between small positive and negative values, but for all practical purposes C_d can be put zero when d is greater than a limit of correlation. We shall, however, replace C_d by a mathematical function which tends to zero asymptotically.

Relationship between the variance G_s^2 and covariance C_d

There is a theoretical connection between the functions G_s^2 and C_d. Suppose that the point anomalies represent mean anomalies of very small sides δ and that C_d has been determined empirically for distances

(22) $$d = k\delta \qquad k = 0, 1, 2, 3 \ldots$$

The value of G_d^2 can now be computed by the general formula of the accumulation of correlated errors. For example, if $d = 2\delta$, we have

(23) $$G^2_{2\delta} = \frac{1}{4} G^2_{\delta} + \frac{1}{2} C_{2\delta} + \frac{1}{4} C_{\sqrt{2}\delta}$$

In the general case we have

(24) $$G^2_{k\delta} = \frac{1}{k^2} G^2_{\delta} + \frac{4}{k^4} \sum_{h=0}^{k-1} \sum_{i=1}^{k-1} (k-h)\ (k-i)\ C_{\delta\sqrt{h^2+i^2}}$$

Let now k increase towards infinity and δ decrease towards zero so that

(25) $$k\delta = s$$

Denoting

(26) $$\frac{h}{k} = x = r \cos t$$

(27) $$\frac{i}{k} = y = r \sin t$$

we obtain

(28) $$G^2_s = 4 \int_0^1 \int_0^1 (1-x)\ (1-y)\ C_{s\sqrt{x^2+y^2}}\ dx\ dy$$

$$= 4 \int_0^{\sqrt{2}} rC_{sr}\ dr \int_{t_1}^{t_2} (1-r \cos t)\ (1-r \sin t)\ dt$$

The integration gives

(29) $$\int (1-r \cos t)\ (1-r \sin t)\ dt = t-r \sin t + r \cos t - \frac{1}{4} r^2 \cos 2t$$

When $0 < r < 1$, the limits of the integration are

$$t_1 = 0$$
$$t_2 = \frac{\pi}{2}$$

and when $1 < r < \sqrt{2}$, we have

$$t_1 = \text{arc cos } \frac{1}{r}$$
$$t_2 = \text{arc sin } \frac{1}{r}$$

In this way we can define the "weight function"

(30) $$W = 4r \int_{t_1}^{t_2} (1-r\cos t)(1-r\sin t)\,dt$$

which consists of two parts:

When $0 < r < 1$

(31) $$W = (2\pi - 8r + 2r^2)\, r$$

When $1 < r < \sqrt{2}$

(32) $$W = (2\pi - 4 - 2r^2 + 8\sqrt{r^2-1} - 8 \arctan\sqrt{r^2-1})\, r$$

The numerical values of this weight function W are given in Table I. Now we can compute

(33) $$G_s^2 = \int_0^{\sqrt{2}} W C_d \, dr$$

where

(34) $$d = rs$$

The practical computation of the integral in formula (33) can be carried out as follows. The curve representing C_d is replaced by a polygon:

When $d_{i-1} < d < d_i$

(35) $$C_d = C_{i-1} \frac{d_i - d}{d_i - d_{i-1}} + C_i \frac{d - d_{i-1}}{d_i - d_{i-1}}$$

Now the integration can be carried out within each interval and the integrals summed up. Denoting

(36) $$I_i = \int_{r_{i-1}}^{r_i} W\, dr$$

(37) $$J_i = \int_{r_{i-1}}^{r_i} r W\, dr$$

we can write

TABLE I

Weight function W and it's integrals I and J

r	W	I	J
0	0.000	0.0000	0.0000
0.05	0.294	0.0075	0.0002
0.1	0.550	0.0288	0.0019
0.15	0.769	0.0619	0.0061
0.2	0.953	0.1051	0.0137
0.25	1.102	0.1566	0.0253
0.3	1.219	0.2148	0.0413
0.35	1.305	0.2780	0.0619
0.4	1.361	0.3448	0.0869
0.45	1.390	0.4137	0.1162
0.5	1.392	0.4833	0.1493
0.55	1.362	0.5524	0.1856
0.6	1.322	0.6198	0.2243
0.65	1.253	0.6842	0.2646
0.7	1.164	0.7448	0.3054
0.75	1.056	0.8004	0.3457
0.8	0.930	0.8501	0.3842
0.85	0.789	0.8931	0.4197
0.9	0.633	0.9287	0.4508
0.95	0.464	0.9562	0.4761
1	0.283	0.9749	0.4944
1.05	0.169	0.9859	0.5056
1.1	0.101	0.9925	0.5127
1.15	0.057	0.9964	0.5173
1.2	0.029	0.9985	0.5195
1.25	0.013	0.9993	0.5208
1.3	0.004	0.9999	0.5212
1.35	0.001	1.0000	0.5214
$\sqrt{2}$	0	1.0000	0.5214

(38) $$\int_{r_{i-1}}^{r_i} W C_d \, dr = I_i \frac{d_i C_{i-1} - d_{i-1} C_i}{d_i - d_{i-1}} + J_i \frac{s(C_i - C_{i-1})}{d_i - d_{i-1}}$$

The values of I and J can be computed also by the exact formulas

When $0 < r < 1$

(39) $$I = \pi r^2 - \frac{8}{3} r^3 + \frac{1}{2} r^4$$

(40) $$J = \frac{2}{3} \pi r^3 - 2r^4 + \frac{2}{5} r^5$$

When $1 < r < \sqrt{2}$

(41) $$I = (\pi - 2) r^2 - \frac{1}{2} r^4 + \frac{4}{3} \sqrt{r^2 - 1}\,(2r^2 + 1) - 4r^2 \arctan \sqrt{r^2 - 1}$$

(42) $$J = \frac{2}{3} (\pi - 2) r^3 - \frac{2}{5} r^5 + \sqrt{r^2 - 1}\,(2r^3 + \frac{1}{3} r) - \frac{8}{3} r^3 \arctan \sqrt{r^2 - 1} + \frac{1}{3} \ln (r + \sqrt{r^2 - 1}) + \frac{2}{15}$$

Table I shows the values of I and J for different values of r. In practice we have

(43) $$r_i = \frac{d_i}{s}$$

and

(44) $$I_i = I(r_i) - I(r_{i-1})$$

(45) $$J_i = J(r_i) - J(r_{i-1})$$

This method is not convenient if the values of G_s^2 are given and we want to compute C_d. Therefore, we replace C_d by any mathematical function which can be represented by a series

(46) $$C_d = C_o - c_1 d^2 + c_2 d^4 - c_3 d^6 + \dots$$

Then formula (33) gives after lengthy computations, which will be omitted here

(47) $$G_s^2 = C_o - g_1 s^2 + g_2 s^4 - g_3 s^6 + \dots$$

with a general conversion formula

(48) $$\frac{g_n}{c_n} = \frac{2}{n+1} \frac{1}{n+2} \left\{ (2^{n+1} + \frac{1}{2n+3}) - 2 \sum_{m=0}^{n} \binom{n}{m} \frac{1}{2m+3} \right\}$$

where $\binom{n}{m}$ means the binomial coefficients

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

Table II gives the numerical values of this fraction for the small values of n.

A finite power series cannot give a good representation of C_d because for great values of d it tends to infinity instead of zero. But if we can represent C_d by a function

(49) $$C_d = \frac{C_o}{1+c^2d^2} = C_o \, (1 - c^2 d^2 - c^4 d^4 - c^6 d^6 + \dots)$$

then, according to Table II, G_s^2 may be replaced by function

(50) $$G_s^2 = C_o \frac{\arctan cs}{cs} = C_o \, (1 - \frac{c^2 s^2}{3} + \frac{c^4 s^4}{5} - \frac{c^6 s^6}{7} + \dots)$$

In practice, these mathematical functions with two parameters C_o and c may not give a satisfactory approximation to the real functions found by the statistical computations, but we can introduce four parameters in form

(51) $$C_d = \frac{x_1}{1+y_1^2 d^2} + \frac{x_2}{1+y_2^2 d^2}$$

(52) $$G_s^2 = x_1 \frac{\arctan y_1 s}{y_1 s} + x_2 \frac{\arctan y_2 s}{y_2 s}$$

If the empirical values of C_d are given, we pick up four characteristic values and solve four linear equations of type (18):

TABLE II

Conversion of the correlation function to the mean anomaly function.

n	$\frac{g_n}{c_n}$	$\frac{c_n}{g_n}$	n	$\frac{c_n}{g_n}$
0	1	1	6	8,181
1	$\frac{1}{3}$	3	7	7,128
2	$\frac{119}{630}$	5,294	8	5,831
3	$\frac{29}{210}$	7,241	9	4,527
4	$\frac{187}{1575}$	8,422	10	3,364
5	$\frac{3107}{27027}$	8,699		

(53) $$C_o + Xd^2 = C_d\,(1 + Yd^2 + Zd^4)$$

with respect to X, Y, Z, and C_o. Then we have

(54) $$R = \sqrt{Y^2 - 4Z}$$

(55) $$y_1^2 = \frac{1}{2}\,(Y+R)$$

(56) $$y_2^2 = \frac{1}{2}\,(Y-R)$$

(57) $$x_1 = \frac{y_1^2\,C_o - X}{R}$$

(58) $$x_2 = \frac{X - y_2^2\,C_o}{R}$$

Numerical Examples

The methods described in the previous chapter have been tested by the aid of the published anomaly maps of the State of Ohio. In order to study the stability of the statistical quantities, the entire area was divided into twelve blocks as shown ~~in Fig. 1:~~ [illegible]

φ				
41° 40'				
	1	2	3	4
40° 40'				
	5	6	7	8
39° 40'				
	9	10	11	12
38° 40'				

84° 40'	83° 40'	82° 40'	81° 40'	80° 40'

The size of each block is $1^\circ \cdot 1^\circ$. According to formula (1) we have in average

$$a = 111 \text{ km}$$

$$b = 85 \text{ km}$$

$$s = 97 \text{ km}$$

Each block was still divided into 9 squares of size $20' \cdot 20'$, into 36 squares of size $10' \cdot 10'$, and into 144 squares of size $5' \cdot 5'$. The corresponding values of s are respectively

32, 16, and 8 km.

The number of squares, of which more than half have been covered by gravity observations, is shown in Table III.

TABLE III

Number of Observed Squares

Square Size	$5' \cdot 5'$	$10' \cdot 10'$	$20' \cdot 20'$
Block			
1	144	36	9
2	142	36	9
3	120	30	6
4	143	36	9
5	144	36	9
6	144	36	9
7	144	36	9
8	138	34	9
9	119	29	7
10	144	36	9
11	124	32	8
12	33	8	2
Margins	179	32	
Ohio	1718	417	95

The mean free air anomaly of each block is given in Table IV under the heading Δg_1 (formula 2). The mean gradient ϵ_s (formula 9) for a distance of $s = 5'$ has been estimated in each square of size $10' \cdot 10'$. Table IV gives the results in form of E^2_s (formula 10) for each block under heading E^2_5. The values of G^2_5 and G^2_0 have been computed for the squares $5' \cdot 5'$ by formula (4). Formulas (11) and (12) have been used for the computation of E^2_{10}, E^2_{20}, and E^2_{60}.

TABLE IV

Ohio Mean Free Air Anomalies

Block	Δg_1	E_5^2	E_{10}^2	E_{20}^2	E_{60}^2	G_5^2	G_o^2
1	-23.5	6.6	18.9	63.4	274.7	820.3	826.9
2	-19.8	6.1	17.8	37.0	74.4	512.3	518.4
3	+ 9.9	3.9	10.4	36.8	164.2	245.6	249.5
4	+ 5.0	3.0	9.1	22.7	85.0	100.1	103.1
5	+ 4.3	8.2	23.7	81.0	158.9	168.9	177.1
6	- 3.6	15.4	41.2	113.6	341.4	338.9	354.3
7	- 2.9	5.1	11.9	24.1	130.6	134.1	139.2
8	- 8.6	6.0	14.1	36.2	99.9	164.2	170.2
9	- 3.7	13.4	32.7	89.6	264.3	270.9	284.3
10	- 0.2	14.2	37.5	90.7	443.0	409.8	424.0
11	-18.7	3.9	9.6	24.3	34.9	446.0	449.9
12	-29.6	5.7					
Margins	-13.5						
Ohio	- 6.7	7.7	20.4	56.5	197.3	329.2	336.9
E_s^2/G_o^2		0.023	0.061	0.168	0.586		
E_s/G_o		0.15	0.25	0.41	0.77		
s	km	9.3	16	32	96		

Method of covariances

The method of covariances can be recommended for the statistical analyses of very long profiles only. From short profiles we can not expect reliable results. First, the mean anomaly of the entire profile may deviate very much from zero--especially if Bouguer anomalies are used--and then the covariances systematically maintain large positive values. Second, the "waves" of positive and negative anomalies are so wide that within the entire profile the number of independent waves is too small for a statistical treatment. Nevertheless, the method has been applied to the free air anomaly field of Ohio in order to obtain a comparison with the E-values. Five profiles in N-S direction and five in S-W direction have been studied. The resulting values of C_d/C_o for each profile are given in Table V.

TABLE V

C_d/C_o for Free Air Anomalies of Ohio

Profile d	1	2	3	4	5	6	7	8	9	10
10 km	(0.98)	(98)	88	(74)	88	94	90	94	96	(55)
20	(94)	(93)	72	(34)	64	85	74	88	85	(32)
30	(86)	(84)	55	(-01)	49	75	61	84	70	(23)
40	(79)	73	38	(-12)	(18)	64	52	(74)	54	33
50	(74)	60	18	(-02)	(04)	53	44	(67)	37	31
60	(71)	47	(00)	18	(-08)	41	33	(63)	24	38
70	(70)	34	(-22)	34	(-14)	28	14	(57)	14	43
80	(68)	22	(-42)	52	(-13)	16	-03	(53)	04	25
90	(66)	09	(-55)	(56)	-02	04	(-14)	49	-06	04
100	(61)	-03	(-62)	43	13	-06	(-23)	(47)	-12	-10

TABLE V (cont'd)

d	Mean	C	C'	E'	E
10	0.917	0.29	0.24	0.14	0.15 for d = 9.3 km
20	780	.47	.45	.25	0.25 for d = 16
30	657	.59	.60	.37	0.41 for d = 32
40	523	.69	.71	.46	
50	405	.77	.78	.53	
60	335	.82	.83	.59	
70	278	.85	.87	.63	
80	193	.90	.89	.67	
90	097	.95	.91	.70	
100	042	.98	.93	.72	0.77 for d = 96

As can be seen from Table V many profiles show exceptionally great or small values which would not be possible if the entire region were large enough. Therefore, the mean values have been computed omitting two greatest and two smallest values on each row. Then the values of

$$C = \sqrt{1 - \frac{C_d}{C_o}}$$

have been computed.

The column C' shows the values of a mathematical function

$$C' = \sqrt{1 - \frac{1}{1+(\frac{d}{D})^2}} = \frac{d}{\sqrt{D^2+d^2}}$$

In this case, D = 40 km gives an excellent fit to the empirical values of C.

According to formula (50) we can test now the function

$$E' = 1 - \frac{D}{d} \arctan \frac{d}{D}$$

which should fit to the empirical values of

$$E = E_d / G_o$$

As shown on the bottom rows of Table IV and in Fig. 2, the actual fit is satisfactory.

Computation of the Gravity Anomalies at Higher Altitudes

When the values of a harmonic function are known at all points of a surface S, it is theoretically possible to compute the value of the function at any point P outside the surface. This problem of continuation will be applied here to the gravity field of the earth: we assume that the anomalies are known everywhere on the physical surface of the earth and we want to compute the anomaly at height h above the ground. Note that the reduction of gravity observations to sea level requires also a similar continuation but downwards.

Because the topographical surface of the earth can be very irregular, the practical solution is approximate only. In most cases, however, a satisfactory accuracy will be obtained by a simple integral formula

$$\Delta g_P = \frac{1}{4\pi} \int_\sigma \Delta g_s \, t^2 \, \frac{1-t^2}{D^3} \, d\sigma$$

where

σ solid angle around the center of the earth

$$t = \frac{r_s}{r_P}$$

r distance from the center of the earth

$$D = \sqrt{1-2t \cos \Psi + t^2}$$

Ψ angle between r_P and r_s

The distances from the fixed point P to points of s are

$$\Delta = r_P D$$

When the distances Δ are large with respect to the topographic heights of surface s itself, we can replace r_s by a constant R which corresponds to an average ground level below P. Denoting

$$h = r_P - R$$

we can write

$$\Delta = h^2 + 2r_P R \text{ vers } \Psi$$

$$\Delta g_P = \frac{h}{2\pi} \frac{r_P + R}{2r_P} R^2 \int_\sigma \Delta g_s \frac{d\sigma}{\Delta^3}$$

The integration should be carried out around the entire earth, but the distant anomalies have no practical significance because of the great values of the denominator Δ^3. Therefore, we can confine the integration to the nearest surroundings of the point O which is supposed to be the vertical projection of P at the average ground level. The curvature of the earth can be neglected, and we can use plane coordinates

$$x = R(\varphi_s - \varphi_o)$$

$$y = R(\lambda_s - \lambda_o) \cos \varphi_o$$

$$x = s \cos \alpha$$

$$y = s \sin \alpha$$

$$s = R \sin \Psi$$

$$\Delta = \sqrt{s^2 + h^2}$$

$$d\sigma = \sin \Psi \, d\Psi \, d\alpha = \frac{dx \, dy}{R^2}$$

Computation with Automatic Machines

Suppose that the mean anomalies Δg_s once and for all have been computed for squares of sizes

$\Delta\varphi, \Delta\lambda = 5' \cdot 5', 10' \cdot 10', 20' \cdot 20', 1^\circ \cdot 1^\circ$, etc. and punched on cards. The values of φ_s and λ_s for the center of the square (or for one corner) must be punched too. Also, the value of

$d\sigma = \Delta\varphi\Delta\lambda \cos\varphi$ (φ, λ in radians) is required for each card. Then the computation can be programmed according to formula

$$\Delta g_P = hK\Sigma \; \Delta g_s \frac{d\sigma}{\Delta^3}$$

where

$$K = \frac{R^2}{2\pi}$$

can be taken as a constant. In some cases, for the nearest surroundings still smaller squares than $5' \cdot 5'$ should be used but for the more remote areas a certain number of squares $20' \cdot 20'$ and $1^\circ \cdot 1^\circ$ will do. A special program can be prepared in order to pick up automatically the relevant cards. We shall return later to the question, what is the best size of squares for different distances.

Manual Computations

Instead of squares which once and for all have been fixed according to geographical coordinates φ and λ, we use for each point P a new set of squares according to coordinates s and α. For this purpose special templates must be prepared on a transparent paper. The template must be drawn in the scale of the gravity map which shows the observed anomalies and the iso-anomaly curves. We have then

$$R^2 d\sigma = s\Delta s \Delta\alpha$$

$$\Delta g_P = \Sigma k \Delta g_s$$

$$k = \frac{h}{n}\left(\frac{1}{\Delta_i} - \frac{1}{\Delta_o}\right)$$

where

$\Delta_i = \Delta$ for the inner radius of each zone

$\Delta_o = \Delta$ for the outer radius

n = number of equal compartments in each zone

For the computation of one point P, the mean anomaly Δg_s inside each compartment of the template must be evaluated anew by the aid of the map. Usually the mean anomaly can be approximated by the point anomaly at the center of the compartment. Even then the reading of the anomalies, which cannot be performed by the machines, is the most time consuming part of the work.

We devide each ring into n_i equal compartments which are as close to a square as possible. Now it is possible to select the differences $s_{i+1} - s_i$ so that we have

$$k_i = n_i k$$

In other words, the mean anomaly of each compartment will be multiplied by the same coefficient k. If the anomalies run smoothly within each compartment, the point value at the center of the compartment may give the mean anomaly with a sufficient accuracy except in the large outer zones.

For each height h the template must be drawn in a different scale. Templates for

$k = 0.01$
$k = 0.02$

are given with height h as the unit of distances.

In Table VI are given the templates for $k = 0.01$ and 0.02.

In Table VII these templates have been tested for two altitudes using an artificial gravity map. The results can be compared with those obtained with the templates of Peters, Tsuboi, and Tengstrom. In these particular examples the large-scale system of Henderson seems to be good, but in irregular anomaly fields our templates surely would give the most accurate results with the least effort.

TABLE VI

Template for k = 0.01

Zone	n_i	r_{i+1}	x and y	
0	1	0.1425	0.000	0.000
1	6	0.3952	0.269	0.000
			0.135	0.233
2	12	0.7240	0.560	0.000
			0.485	0.280
3	14	1.1080	0.916	0.000
			0.825	0.397
			0.571	0.716
			0.204	0.893
4	16	1.6866	1.397	0.000
			1.291	0.535
			0.988	0.988
5	14	2.5109	2.099	0.000
			1.891	0.911
			1.309	1.641
			0.468	2.046
6	14	4.2313	3.371	0.000
			3.037	1.463
			2.101	2.636
			0.752	3.286
7	10	7.627	5.93	0.00
			4.80	3.48
			1.83	5.64
8	8	19.975	13.80	0.00
			9.76	9.76
9	1	24.98	22.48	
10	1	33.32	29.15	
11	1	49.99	41.65	
12	1	100	75	
13	1	∞		
N = 100				

TABLE VI (cont'd)

Template for k = 0.02

Zone	n_i	r_{i+1}	x and y	
0	1	0.2031	0.000	0.000
1	6	0.5934	0.398	0.000
			0.199	0.345
2	10	1.1383	0.866	0.000
			0.701	0.509
			0.268	0.824
3	12	2.1608	1.650	0.000
			1.429	0.825
4	10	4.4341	3.30	0.00
			2.67	1.94
			1.02	3.14
5	6	9.95	7.19	0.00
			3.60	6.23
6	1	12.46	11.20	
7	1	16.64	14.55	
8	1	24.98	20.81	
9	1	49.99	37.5	
10	1	∞		

N = 50

TABLE VII

Test Computations

	f_1	Error	N	R/h	f_2	Error	N	R/h
Correct Value	0.8624	0/0			0.7438	0/0		
Hirvonen, k=0.01	.8530	- 1.1	95	13.8	.7503	+ 0.9	88	5.9
Hirvonen, k=0.02	.8522	- 1.2	45	10.0	.7355	- 1.1	45	10.0
Peters, Large	.8994	+ 4.3	65	11.2	.8132	+ 9.3	65	5.6
Peters, Dense	.9477	+ 9.9	65	5.6	.9156	+23.1	65	2.8
Tsuboi	.9802	+13.6	49	4.2	.9202	+23.7	49	4.2
Henderson, Large	.8585	- 0.4	73	25.0	.7372	- 0.9	73	12.5
Henderson, Dense	.8755	+ 1.5	73	12.5	.7723	+ 3.8	73	6.2

This time we are looking for a template in which the error caused by each compartment is of the same order. The error is a result of two factors. First, each point anomaly has been replaced by the mean anomaly of the compartment. Second, the coefficient $\frac{1}{\Delta^3}$ has been computed as a mean value over the compartment, but if in different parts of the compartment there are variations of the anomaly, the coefficients should be variable too. The combined effect of these variations is proportional to the product of them both. The error caused by the mean anomaly is identical with the error of representation, denoted by E_s in the previous chapters. As we have seen, E_s is proportional to s when s is small. Here it means that E_s is proportional to Δs which is the average diameter of a compartment.

The error of factor $\frac{1}{\Delta^3}$ is proportional to

$$\Delta s \frac{d}{ds}\left(\frac{1}{\Delta^3}\right) = -\Delta s \frac{3s}{\Delta^5}$$

hence, the error of the product

$$k\Delta g_s = \frac{(\Delta s)^2}{\Delta^3} \Delta g_s$$

is proportional to

$$(\Delta s)^4 \frac{s}{\Delta^5}$$

If we require that the error caused by each compartment must be of the same order, Δs must be proportional to

$$\sqrt[4]{\frac{\Delta^5}{s}} = \Delta \sqrt[4]{\frac{\Delta}{s}} = \delta$$

Table VIII gives the values of $\frac{1}{\Delta^3}$, $\frac{s}{\Delta^5}$ and δ as functions of s. In this table h is taken as unity.

For greater distances Δ and δ are approximately equal to s. Therefore, the width of the outer zones of the template is proportional to the mean radius of the zone. When the zones are divided into equal compartments which have the shape of a square as close as possible, the number n of compartments is the same for all outer zones, which circumstance makes the drawing of the templates easy.

According to this principle, templates for any desired degree of accuracy can be prepared. Table IX gives the numerical constants for a template, which gives a satisfactory accuracy with a reasonable amount of labor. The template can be drawn by the aid of the first two columns, remembering that the unit of radius is equal to h in the scale of the anomaly map. The coefficients of the third column should be written into the respective zones of the template. When the mean anomaly of each compartment of the template has been evaluated from the anomaly map, the sum of mean anomalies is computed separately for each zone and multiplied by the respective coefficient. The sum of the products is the gravity anomaly at height h above the average ground level.

The computer will soon realize that it is seldom necessary to use the outermost zones of the template.

For each altitude h a new template should be prepared. However, it is possible to use the same template for different altitudes if for each altitude a new set of coefficients

$$\frac{h}{n}\left(\frac{1}{\Delta_i} - \frac{1}{\Delta_o}\right) = \frac{h}{n}\left(\frac{1}{\sqrt{h^2+s_i^2}} - \frac{1}{\sqrt{h^2+s_2^2}}\right)$$

is computed where n is the number of compartments of each zone with inner radius s_i and outer radius s_o. Table IX gives extra columns of coefficients for the cases that our template is used for the double or for the half of the original height.

TABLE VIII

$\Delta = \sqrt{1+s^2}$ and related functions

s	Δ	$1:\Delta$	$1:\Delta^3$	$s:\Delta^5$	δ
0	1.0000	1.00000	1.00000	0.00000	
0.1	1.0050	0.99504	0.98518	.09754	1.79
0.2	1.0198	.98058	.94287	.18132	1.53
0.3	1.0440	.95783	.87874	.24185	1.43
0.4	1.0770	.92848	.80041	.27600	1.38
0.5	1.1180	.89443	.71554	.28622	1.37
0.6	1.1662	.85749	.63051	.27817	1.38
0.8	1.2806	.78087	.47614	.23226	1.44
1.0	1.4142	.70711	.35355	.17678	1.54
1.2	1.5690	.64018	.26237	.12903	1.67
1.4	1.7205	.58123	.19636	.09287	1.81
1.6	1.8868	.53000	.14888	.06691	1.97
1.8	2.0591	.48564	.11454	.04862	2.13
2.0	2.2361	.44721	.08944	.03578	2.30
2.5	2.6926	.37139	.05123	.01766	2.74
3.0	3.1623	.31623	.03162	.00949	3.21
3.5	3.6401	.27472	.02073	.00548	3.68
4.0	4.1231	.24254	.01427	.003357	4.15
4.5	4.6098	.21693	.01021	.002162	4.64
5.0	5.0990	.19612	.00754	.0014506	5.12
6	6.0828	.16440	.00444	.0007205	6.10
7	7.0711	.14142	.00283	.0003960	7.09
8	8.0623	.12403	.00191	.0002349	8.08
9	9.0554	.11043	.00135	.0001478	9.07
10	10.0499	.09950	.00099	.0000975	10.06
11	11.0454	.09054	.00074	.0000669	11.06
12	12.0416	.08305	.00057	.0000474	12.05
13	13.0384	.07670	.00045	.0000345	13.05
14	14.0357	.07125	.00036	.0000257	14.04
15	15.0333	.06652	.00029	.0000195	15.04
20	20.0250	.04994	.0001245	.0000062	20.03
25	25.0200	.03997	.0000638	.0000025	25.02

TABLE IX

Template Constants

Outer Radius	Number of Squares	Coefficients h = 1	h = 2	h = 0.5
0.4	1	0.07152	0.01942	0.21913
1.0	8	0.02767	0.01077	0.04171
1.8	12	0.01846	0.01259	0.01496
3.0	12	0.01412	0.01572	0.00860
4.5	16	0.00621	0.00928	0.00337
6.7	16	0.00433	0.00751	0.00225
10.0	16	0.00301	0.00562	0.00153
15.0	16	0.00206	0.00400	0.00104
22.0	16	0.00132	0.00260	0.00066
32.0	16	0.00089	0.00176	0.00044

Table IX gives, also, an idea about the size of the squares which should be used for automatic computers. Obviously, the smallest standard squares $5' \cdot 5'$, for which punch cards have been prepared, do not give the same accuracy as the template method when h is smaller than 20 km. In this case the contribution of the innermost squares should always be checked by the template method.

We have developed methods for the computation of the earth's gravity field at various altitudes which we believe to be most suitable for practical applications. There are, in fact, several different questions to be settled.

First, should the computation be based on the usual integration methods or on Taylor series as suggested by Tengström?

Second, should the computation be based on the original free air anomalies or on the "idealized disturbing coating" as suggested by Helmert, Lambert, Tengström, and Orlin? The coating already involves preliminary integrations but then it can be used for several different purposes. For our present problem, especially over restricted areas, it seems to be a laborious detour.

Third, should we compute several extensive fields of anomalies for coatings at different altitudes as suggested by Tengström? Then the move from one level to the next could be performed by inextensive integrations (or Taylor series) but the total amount of computations may be too great for restricted areas.

Fourth, which template of integration should be used? The solving of these problems has great theoretical and practical significance.