CURVE SKETCHING – A DISCOVERY APPROACH USING GRAPHING SOFTWARE AND/OR GRAPHICS CALCULATORS

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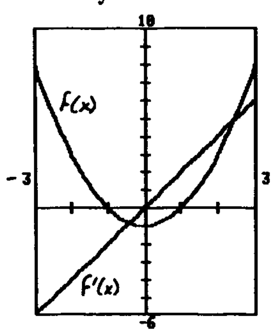
Mathematics students encounter curve sketching as an application of differentiation in pre-calculus and beginning calculus classes. The typical student usually will memorize the first and second derivative tests without fully comprehending the wealth of information each derivative provides. The advent of graphing calculators and computer software now permits students to compare visually and to describe relationships between graphic representation of the actual function and its corresponding derivatives. After examining several cases, students can make their own conjectures and write a set of guidelines to assist them in constructing accurate sketches of functions using this information.

Most graphics calculators (and software) allow graphs of two or more functions to be placed on the same axis simultaneously, or added to the same display, one at a time. The application presented here uses graphing software for class discussion and students use the Casio fx7000g graphics calculators at their desks. (An assumption is being made that you are familiar with the software and/or calculator operation for graphing functions.)

We first consider a simple function that students already are comfortable with, such as: $f(x) = x^2 - 1$

so $f'(x) = 2x$ and $f''(x) = 2$



Display the graphs of f(x) and f′(x).

Ask the students the following:

On what interval(s) is $f'(x) < 0$?

On what interval(s) is $f'(x) > 0$?

Where does $f'(x) = 0$?

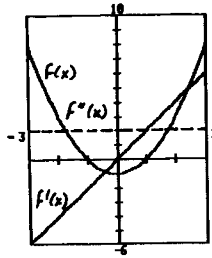
Compare the graphs of f(x) and f′(x).

How can you describe the behavior of f(x) on the interval where $f'(x) < 0$?

Describe how the behavior of f(x) changes for $f'(x) > 0$?.

What happens to f(x) at the point where $f'(x) = 0$?

(Recall that the value of the first derivative at any point represents the slope of the tangent to f(x) at that point.)



Add the graph of $f''(x) = 2$ to the same display.

When is $f''(x) < 0$?
When is $f''(x) > 0$?
When is $f''(x) = 0$?

Comparing the graph of $f''(x)$ and $f(x)$ – What conjectures (if any) can you make at this time?

Comparing the graph of $f''(x)$ and $f'(x)$ – Does the second derivative give any information about the first derivative?

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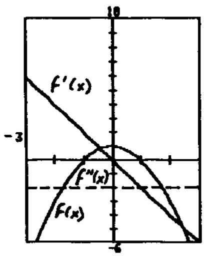
Keeping in mind what we have seen, what would change if we considered the graph of: $f(x) = -x^2 + 1$ $[f'(x) = -2x$ and $f''(x) = -2]$

Graphing all three simultaneously produces these results:



Where is $f'(x) < 0$?
Where is $f'(x) > 0$?
Where is $f'(x) = 0$?

Where is $f''(x) < 0$?
Where is $f''(x) > 0$?
Where is $f''(x) = 0$?

Again, what is the behavior of f(x) when the first derivative is negative? Positive? Zero?

Does the fact that the second derivative is always negative tell us anything about the behavior of f(x)? of f′(x)?

Considering both examples, write three or four statements describing apparent relationships between functions and their first and second derivatives.

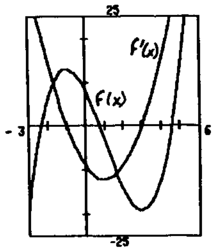
At this time, the first derivative contributions of increasing, decreasing, and relative extrema may be apparent. Due to the simplicity of the chosen functions, concavity and the second derivative test for relative maximum and minimum will not be so apparent. Have the students record their conjectures and list their generalizations on the board. Test these conjectures and expand upon them by examining a more complicated polynomial function.

Consider $f(x) = x^3 - 3x^2 - 9x + 8$

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

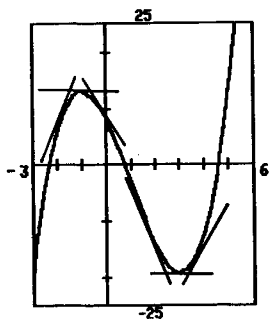
Graphing f(x) and f′(x) simultaneously produces these results:

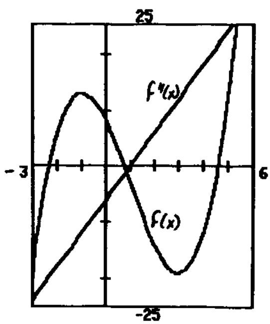


Ask the students to describe the behavior of the function in terms of the behavior of the first derivative. A typical student response is that f′(x) is positive from negative infinity to -1 and from +3 to positive infinity, and these are the intervals where f(x) is increasing. The first derivative is negative in the interval (-1,3) and f(x) is decreasing on this interval. The first derivative is equal to 0 when x = -1 and x = 3, and the corresponding graph of f(x) turns at these values.

It is apparent to the students that a relative maximum occurs when x = -1 and a relative minimum occurs when x = 3. Point out that the value of f′(x) changes from positive to negative at the relative maximum, but the values change from negative to positive at the relative minimum.

Before examining the graph of f″(x), explore the idea of concavity geometrically by looking at the graph of f(x) and determining in which intervals the curve would lie under its tangent lines (concave down) and where it would lie above its tangent lines (concave up). The display would look like this:





By inspection, the function is concave down from negative infinity to +1, and concave up from +1 to positive infinity. Add the graph of $f''(x)$ to the display and note the following.

$f''(x) < 0$ when $f(x)$ is concave down

$f''(x) > 0$ when $f(x)$ is concave up

When $f''(x) = 0$ the function $f(x)$ changes from concave down to concave up – this is known as a point of inflection.

** Returning to graphs of the two functions graphed earlier;

Note when $f(x) = x^2 - 1$, $f''(x) = 2$, which is positive everywhere. This implies that the graph of $f(x)$ should be concave up.

Similarly, when $f(x) = -x^2 + 1$, $f''(x) = -2$, which is negative everywhere, implying that the graph of $f(x)$ should be concave down. These results can be verified by inspection.

Further investigation at the relative extrema of these three functions shows $f''(x) < 0$ when a relative maximum occurs and $f''(x) > 0$ when a relative minimum occurs.

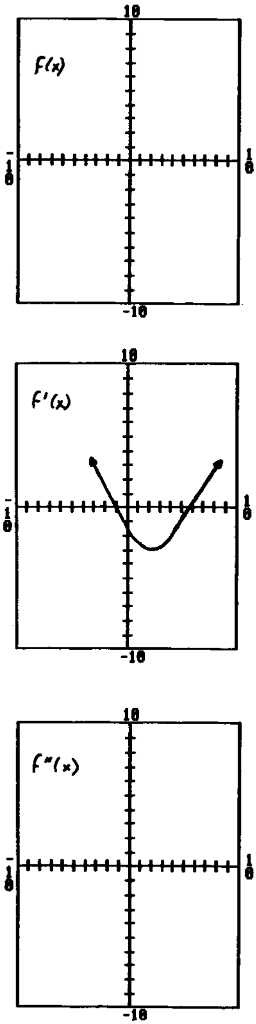
As a class assignment, have students write their own conjectures for first and second derivative tests. Then ask the following:

> If you were given only the graph of the second derivative, what would you know? Could you sketch a graph of the first derivative? Could you then sketch the original function? What are you unable to determine about the original function that is important in making an accurate sketch? Students quickly will see that there are many congruent possibilities for graphs of the first derivative which differ by only a constant and many possibilities for graphs of the original function. It is difficult to ascertain where $f(x) = 0$ without more information.

Allow students to test their conjectures with the following sample student exercises:

Sample Student Exercise–1

On the coordinate planes below you are given the graph of f′(x); use conjectures that you have made to sketch a possible graph of f(x) and f″(x). Be prepared to justify your sketch!



Sample Student Exercises–2

On the coordinate planes below, you are given the graph of f″(x). Sketch a <u>possible</u> graph of f′(x) and f(x). Be prepared to justify your sketch!