Mathematical Morsels: A Monthly Voyage of Problem Solving Goodness

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In the last journal, monthly "morsels" were presented to promote open-ended problem solving. Below are the last set of problems and some solution strategies.

Problem 1: A 5, 12, 13 triangle is a planar figure where its perimeter and area are numerically equal (both equal to 30). Can you find more triangles that have this curious property? How about other shapes, such as rectangles?

Solution: You can view a video of the solution process at https://vimeo.com/86904412.

Possibly the easiest polygon might be a square of side length *s*. We require $4s = s^2$ so $s = 4$ units.

Next consider a rectangle with length *l* and width *w*. We need $2l + 2w = lw$ and thus $l = \frac{2w}{w-2}$ for the perimeter and area to be equal. For any width $w > 2$, we can find the associated length. For example, if $w = 6$, then $l = \frac{2(6)}{6-2} = 3$ and a 6 by 3 rectangle has its perimeter numerically equal to its area.

In a similar fashion, consider a right triangle with legs *a, b* and hypotenuse $c = \sqrt{a^2 + b^2}$. We require $\frac{1}{2}ab = a + b + \sqrt{a^2 + b^2}$ or $ab - 2a - 2b = 2\sqrt{a^2 + b^2}$. Upon squaring both sides and simplifying, $a^2b^2 + 8ab = 4a^2b + 4ab^2$. Dividing by *ab* since the product is not zero, $ab + 8 = 4a + 4b$, so $a = \frac{4(b-2)}{b-4}$. We have now found infinitely-many triangles whose perimeter equals its area: $a = \frac{4(b-2)}{b-4}$, b, $c = \sqrt{a^2 + b^2}$ where $b > 4$. For example, letting $b = 6$, we find the lengths 8, 6, and 10.

What other plane figures can you experiment with?

Problem 2: Insert + and – signs as many times as necessary in the string 123456789 to find a calculation that yields exactly 100. For example, 12 + 3 – 4 – 5 + 6 + 78 +9 = 99 (so close!). Can you find them all?

Solution: There are 12 possible ways to get 100. This was determined using a Java program, which iterated through all possible placements of + and – signs.

123 + 45 – 67 + 8 – 9 = 100
123 + 4 – 5 + 67 – 89 = 100
123 – 45 – 67 + 89 = 100
123 – 4 – 5 – 6 – 7 + 8 – 9 = 100
12 + 3 + 4 + 5 – 6 – 7 + 89 = 100
12 + 3 – 4 + 5 + 67 + 8 + 9 = 100
12 – 3 – 4 + 5 – 6 + 7 + 89 = 100
1 + 23 – 4 + 56 + 7 + 8 + 9 = 100
1 + 23 – 4 + 5 + 6 + 78 – 9 = 100
1 + 2 + 34 – 5 + 6 +78 – 9 = 100

$1 + 2 + 3 - 4 + 5 + 6 + 78 + 9 = 100$
$-1 + 2 - 3 + 4 + 5 + 6 + 78 + 9 = 100$

Problem 3: At a faculty meeting, each teacher was given one ticket numbered by consecutive integers starting with 1. Andrew noticed that the sum of the ticket numbers less than his number equaled the sum of the ticket numbers above his number. If there are less than 100 total people at the meeting what was Andrew's ticket number?

Solution: Let Andrew's ticket number be n and the total number of people at the meeting be $p < 100$. Then $1 + 2 + ... + n - 1 = n + 1 + n + 2 + ... + p$. The arithmetic series formula, $\frac{(n-1)n}{2} = \frac{p(p+1)}{2} - \frac{n(n+1)}{2}$ simplifies to $n^2 = \frac{p(p+1)}{2}$. Testing values of p from 2 to 100 yields two possible solutions: $(n, p) = (6, 8)$ or $(35, 49)$. Checking, Andrew's ticket number is either 6 with 8 people at the meeting (since 1 + 2 + 3 + 4 + 5 = 15 and 7 + 8 = 15) or Andrew's ticket number is 35 (since 1 + 2 + … + 34 = 595 and 36 + 37 + … + 49 = 595).

Problem 4: Cara received a check and went to her bank to cash it. The bank teller, without adequate coffee, accidentally interchanged dollars and cents; that is, what was written for cents was given in dollars, and vice versa. It was not until Cara bought a piece of candy for five cents did she realize the teller's error, at which point she had twice the amount of money written originally on the check. How much money was the check made out for?

Solution: Suppose Cara started with d dollars and c cents. The teller gives her c dollars and d cents or $100c + d$ cents. After buying candy, she has $100c + d - 5$ cents in which case $100c + d - 5 = 2(100d + c)$. Simplifying, $98c - 199d = 5$. By the Euclidean Algorithm, $(199)(33) + 98(-67) = 1$ so $199(165) + 98(-335) = 5$. Introducing a variable t, we have $199(165 - 98t) + 98(199t - 335) = 5$. To mimic our relationship among c and d, $98\underbrace{(199t - 335)}_{c} - 199\underbrace{(98t - 165)}_{d} = 5$. Since $98t - 165 > 0$ and $199t - 335 > 0$, we see that $t \geq 2$.

However, if $t > 2$, then c has three or more digits which is not practical. So $t = 2$, $c = 63$ and $d = 31$. Cara's original check was for $31.63.

Problem 5: There are 9 sticks, which have unique lengths of 1 inch, 2 inches, 3 inches, … up to 9 inches. Three sticks are chosen at random without replacement. What is the probability the three sticks will form a triangle?

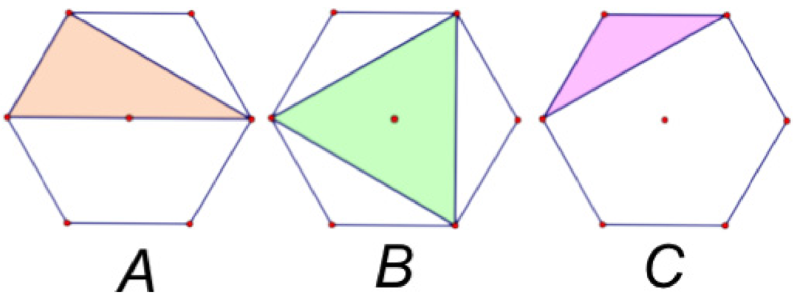
Solution: There are ${}_9C_3 = 84$ ways to select three different lengths disregarding order. By the Triangle Inequality, the sum of any two lengths must exceed the third. Enumerating each possibility, we find the following 34 possibilities that yield triangles.

{2, 3, 4}, {2, 4, 5}, {2, 5, 6}, {2, 6, 7}, {2, 7, 8}, {2, 8, 9}, {3, 4, 5}, {3, 4, 6}, {3, 5, 6}, {3, 5, 7}, {3, 6, 7}, {3, 6, 8}, {3, 7, 8}, {3, 7, 9}, {3, 8, 9}, {4, 5, 6}, {4, 5, 7}, {4, 5, 8}, {4, 6, 7}, {4, 6, 8}, {4, 6, 9}, {4, 7, 8}, {4, 7, 9}, {4, 8, 9}, {5, 6, 7}, {5, 6, 8}, {5, 6, 9}, {5, 7, 8}, {5, 7, 9}, {5, 8, 9}, {6, 7, 8}, {6, 7, 9}, {6, 8, 9}, {7, 8, 9}

The probability of forming a triangle is then $\frac{34}{84} \approx 40.5\%$

Problem 6: Consider a regular hexagon of side length 1. Consider the set *T* of all triangles formed by connecting any three vertices of the hexagon by line segments. What is the average of all the areas contained in *T*?

Solution: There are three types of triangles that can be formed from the vertices as shown below.



Type A: There are ${}_6C_1 \cdot 2 = 12$ possible triangles of area $\frac{1}{2}(1)(\sqrt{3}) = \frac{\sqrt{3}}{2}$.

Type B: There are 2 possible triangles of area $\frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 = \frac{3\sqrt{3}}{4}$.

Type C: There are 6 possible triangle of area $\frac{1}{2}(1)(1)\sin 120° = \frac{\sqrt{3}}{4}$.

The sum of all areas is $12\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{3\sqrt{3}}{4}\right) + 6\left(\frac{\sqrt{3}}{4}\right) = 9\sqrt{3}$. The average of the areas is then $\frac{9\sqrt{3}}{20}$.

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