THE GEOMETRY OF THE TRANSLATED NORMAL CURVE.

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Introduction. In curve tracing the graphic representation is constructed from the equation. Due largely to the requirements of statistics the converse, namely, to find the equation of the curve when the distribution of points is given, has become of interest. This problem is very different from the exercises of analytical geometry in which a given law of distribution of points is to be translated into algebraic language. For the presence in the statistical data of accidental irregularities makes it undesirable as well as practically impossible to obtain a curve passing *through* the points. Instead, a curve is "fitted" to the points, that is, a curve is passed *among* the points in accordance with some generally accepted principal such as that of least squares or the agreement of moments.

Aside from the straight line and the parabolas, the curves proposed by Pearson* have found acceptance. In order to derive curves which can be fitted to widely varying distributions of points, Professor F. Y. Edgeworth† has proposed to modify, to *translate*, the normal probability curve with unit standard deviation,

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad .$$

In this article we shall discuss the geometry of the curves which Edgeworth obtains by this transformation and derive a method for an approximate solution of the two equations, one of the fourth and the other of the sixth degree, which arise in the fitting of a curve of this class.

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* Pearson, Karl:—"Skew Variation in Homogeneous Material;" Phil. Trans. 1895, Vol. CLXXXVI, A, pp. 253 et seq.

"On the Systematic Fitting of Curves to Observations and Measurements," Biometrika, I, pp. 265 et seq. and Biometrika II, pp. 1 et seq.

Elderton:—"Frequency Curves and Correlation," pp. 1-105; C. & E. Layton, 1906.

† Edgerton, F. Y.:—"On the Representation of Statistics by Means of Analytical Geometry," Jour. Roy. Stat. Soc., 1914, Feb., Mar., May, June and July.

In order that the final curve may be written in terms of the co-ordinates x and y the equation of the base or generating normal probability curve is written:

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

where t denotes abscissas and z ordinates.

Let the abscissas of the transformed curve be functions of the corresponding abscissas of the base curve. Then it may be assumed that x can be developed in powers of t, and hence we may write on omitting fourth and higher powers,

$$x = a(t + \kappa t^2 + \lambda t^3),$$

where a, κ and λ are constants to be determined in "fitting" the curve.

Since x denotes the value of a measurement and y the frequency of x, that is, the number of individuals possessing that value of x, the magnitude of an element of area denotes the number of individuals between two values of x. Obviously, therefore, if the transformation is to be of concrete value the magnitude of an element of area must not be altered, though of course the shape will be changed. Hence

$$y\,dx = z\,dt,$$

and

$$y = z\,dt/dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cdot \frac{1}{a(1+2\kappa t+3\lambda t^2)}$$

The formulas of transformation are thus:

$$x = a(t + \kappa t^2 + \lambda t^3),$$

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cdot \frac{1}{a(1+2\kappa t+3\lambda t^2)}$$

Maximum and Minimum Points. Since only curves with one maximum point or mode are practically useful it is desirable to determine what values of the constants a, κ and λ give unimodal curves.

We have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cdot \frac{(3\lambda t^3 + 2\kappa t^2 + (1+6\lambda)t + 2\kappa)}{a(1+2\kappa t+3\lambda t^2)^2}$$

From the vanishing of the numerator of dy/dx there must result either one or three real modes for each pair of values for λ and κ, that is, for each translated curve. To determine what values of λ and κ give uni-modal curves and what tri-modal it is convenient to consider the plane of λ and κ.

The discriminant of the equation

$$3\lambda t^3+2\kappa t^2+(1+6\lambda)t+2\kappa=0$$

is

$$16\kappa^4-\kappa^2(1+66\lambda+117\lambda^2)+3\lambda(1+6\lambda)^3=0$$

This fourth degree curve crosses the horizontal or λ-axis at $\lambda=0$ and at $\lambda=-1/6$ and when $\lambda=0$ its equation reduces to $16\kappa^4-\kappa^2=0$ or $\kappa=\pm0$, $\kappa=\pm\frac{1}{4}$. There is thus contact with the vertical or κ-axis at the origin and that axis is crossed at the points $(0, \pm\frac{1}{4})$. At the point ($\lambda=-1/6$, $\kappa=0$) there is a cusp with the λ-axis for tangent. The other two intersections with the line $\lambda=-1/6$ are imaginary, indicating the presence of two branches to the curve.

The discriminant of the denominator of dy/dx is the parabola (in λ and κ),

$$\kappa^2-3\lambda=0$$

The evident close geometrical connection between the two discriminants suggests arranging the discriminant of the cubic curve in the following form:

$$(\kappa^2-3\lambda)\,(16\kappa^2-117\lambda^2-18\lambda-1)-27\lambda^3(1-24\lambda)=0$$

From the equation in this, the well known uv+kws=0 form, numerous elementary geometrical facts can be derived. The relations to the hyperbola, $16\kappa^2-117\lambda^2-18\lambda-1=0$, and to the parabola, $\kappa^2-3\lambda=0$, premit of the ready plotting of the curve with sufficient accuracy. The general shape of the curve is shown in Figure 1.

It is to be noted that one branch of the curve is within the parabola, almost coinciding with it, while the other crosses it at $\lambda=1/24$. From the original form of this equation it appears that the two branches of this discriminant meet just inside the parabola in the end points with approximate co-ordinates $(0.043, \pm0.360)$. The geometry of the cusp and end-points on the discriminant curve is suggestive of interesting development in detail.

Values of λ and κ for points on the discriminant give curves with two modes coinciding. All points on one side of the discriminant have three real and distinct modes, and all on the other have one real and two imaginary modes. To determine on which side the points giving three real modes lie we examine a point inside the discriminant. When $\kappa=0$ the modal equation becomes

$$3\lambda t^3+(1+6\lambda)t=0$$

Hence the roots are $t=0$ and $t=\pm\sqrt{-\frac{1+6\lambda}{3\lambda}}$. The quantity under the radical is positive for values of λ between 0 and $-1/6$. Therefore, all points within the discriminant curve yield tri-modal curves and all without uni-modal curves.

The plane of λ and K

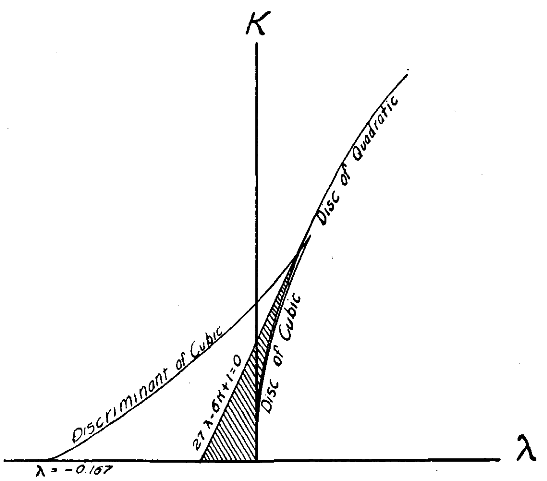


Fig. I
(The horizontal scale is twice the vertical scale)

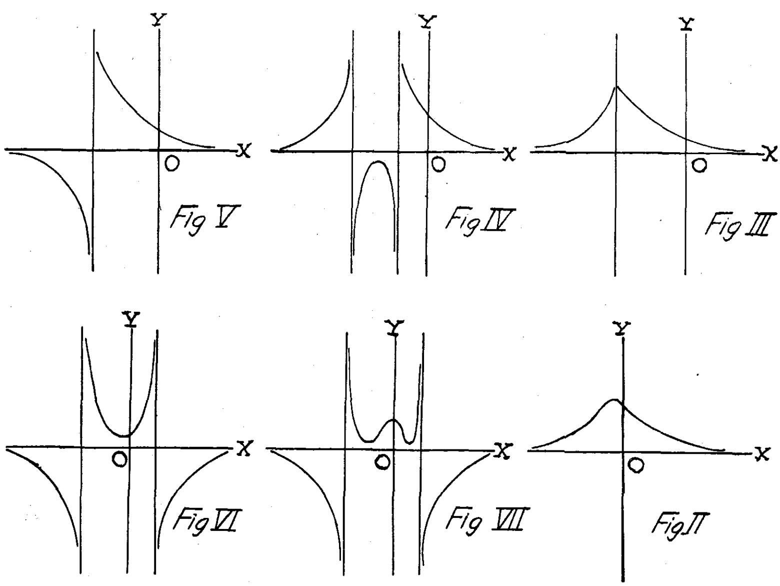
The infinite values of dy/dx arise from zero values of the quadratic, $1+2\kappa t+3\lambda t^2$. The greatest possible number of modes for any one curve is therefore five, three from the cubic and two from the quadratic. Since for infinite values of dy/dx the corresponding ordinates are infinite, it is advisable to study the location of the infinite points of the curve, rather to the neglect of the idea of maximum values at such points.

Infinite Ordinates. The infinite points on a curve are given by the values of t satisfying the equation

$$3\lambda t^2+2\kappa t+1=0$$

Except under certain limited conditions to be determined later a curve with infinite ordinates can not be of great statistical value.

The parabola, $\kappa^2-3\lambda=0$, obtained by equating the discriminant of this quadratic to zero separates the points on the (λ, κ) plane which correspond to curves of *no* infinite points from those corresponding to curves of *two* infinite points.



Types of Curves

Therefore, all pairs of values of λ and κ within the parabola, with the exception of the very narrow region also within the first discriminant curve, give uni-modal curves without infinite ordinates.

Types of Curves. Without entering into detailed proofs we will now investigate the general shape of the curves corresponding to values of λ and κ in each of the distinct regions of the plane of λ and κ.

In the region beneath the parabola and to the right from the shaded area of Fig. I the curve is essentially of the shape shown in Fig. II. This type includes the most common skew curves and hence is of great importance in statistics.

As the point (λ, κ) moves from the λ-axis the crest rises until the parabola is reached when the infinite ordinates appear as two coincident lines, shown in Fig. III.

After the parabola is passed, the infinite ordinates separate and the curve apparently separates into three branches as in Fig. IV.

In crossing the κ-axis to the left one asymptote moves off to infinity giving a curve of the type shown in Fig. V.

Then the asymptote reappears giving a curve of the type shown in Fig. VI.

This general shape is preserved as the point moves toward the λ-axis and when the point reaches the discriminant curve the middle branch is flattened at the minimum point.

For points within the discriminant curve two minimum points appear and the central branch now shows a maximum with a minimum point on either side as in Fig. VII.

The Tri-modal Curves. The curves corresponding to values of (λ, κ) within the discriminant, because of the requirement that an element of area under the translated curve must always be equivalent to the corresponding element under the base or generating curve, can be of statistical value only under the following conditions.

The area between the two ordinates corresponding to $t = \pm 3$ is 0.99998 of the total area under the curve, so that when neither of the minimum points corresponds to points closer than three units to the origin of the base curve the curve may be practically valuable. A moment's consideration will show that the abscissas of the two minimum points must be practically the same as that of the corresponding infinite ordinates. The roots of the quadratic

$$3\lambda t^2 + 2\kappa t + 1 = 0$$

are numerically greater than 3 for all pairs of values of (λ, κ) lying above the line

$$27\lambda - 6\kappa + 1 = 0$$

As statistically promising within the discriminant of the cubic we then have the shaded area of the (λ, κ) plane.

The Origin. The generating curve is the symmetrical normal probability curve with origin at its center. Since $x = 0$ when $t = 0$, the origin of the translated curve coincides with that of the base or generating curve. The translated curve may not be symmetrical so that the mean ordinate may not coincide with the modal ordinate. Because of the relation between correponding areas the ordinate at the origin must continue to divide the area under the curve into equal parts, that is, the origin and median always coincide.

Determination of the Constants. Since the exact position of the median can not ordinarily be determined by inspection or direct computation there are in reality four constants to be determined: the distance between the median and the mean, a, κ and λ.

In determining the constants it is usual to compute the value of the first four moments. The third and fourth moments are extensions of the idea of the well known formulas for the first and second moments. Denoting the moments about the median by μ, we have

$$\mu_1' = \frac{1}{N}\int_{-\infty}^{+\infty} xydx$$

$$\mu_2' = \frac{1}{N}\int_{-\infty}^{+\infty} x^2ydx$$

$$\mu_3' = \frac{1}{N}\int_{-\infty}^{+\infty} x^3ydx$$

$$\mu_4' = \frac{1}{N}\int_{-\infty}^{+\infty} x^4ydx$$

where N is the total area under the curve.

The values of the μ's are computed from the data* and equated to the corresponding integrals which of course involve the four constants. In this way four equations are obtained from which the values of the constants may be determined. Since it is our present object to discuss the solution only of these equations, merely the principal results will be given.

*Elderton, 1. c.

The general form for the moments about the median of the area under the translated curve is

$$\mu_n' = \frac{1}{N}\int_{-\infty}^{+\infty} x^n y dx$$

$$= \frac{1}{\sqrt{2\pi}N}\int_{-\infty}^{+\infty} \frac{a^n(t+\kappa t^2+\lambda t^3)^n}{a(1+2\kappa t+3\lambda t^2)} e^{-\frac{t^2}{2}} a(1+2\kappa t+3\lambda t^2)dt$$

$$= \frac{1}{\sqrt{2\pi}N}\int_{-\infty}^{+\infty} a^n(t+\kappa t^2+\lambda t^3)^n e^{-\frac{t^2}{2}} dt$$

On applying the two well known formulas:

$$\int_{-\infty}^{+\infty} x^{2n+1}e^{-x^2}dx = 0$$

$$\int_{-\infty}^{+\infty} x^{2n+2}e^{-x^2}dx = \frac{2n+1}{2}\int_{-\infty}^{+\infty} x^{2n}e^{-x^2}dx,$$

the determination of μ_1', μ_2', μ_3' and μ_4' is reduced to a matter of algebraic detail. Then on transferring to the arithmetic mean as origin the values of μ_2, μ_3, and μ_4 can be determined in terms of a, κ and λ. It is most convenient however, to make use of the quantities $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ or rather $\beta = \beta_1/8$ and $\epsilon = (\beta_2 - 3)/12$ and express the constants in terms of these quantities. It is to be noted that both ϵ and β are zero for a normal distribution, that is, for $\lambda = \kappa = 0$.

Omitting the detailed reduction* which is straightforward and direct, we have

(1) $\mu' = a\kappa$

(2) $\mu_2 = a^2(1+6x+15x^2+2\kappa^2)$

(3) $\beta = \dfrac{2\kappa^2(2\kappa^2+Q)^2}{(2\kappa^2+R)^3}$

(4) $\epsilon = \dfrac{4\kappa^4+4\kappa^2 S+T}{(2\kappa^2+R)^2}$

where the symbols, S, R, Q and T are defined as follows:

$S = 1+18\lambda+90\lambda^2,$
$R = 1+6\lambda+15\lambda^2,$
$Q = 1.5+18\lambda+135/2\lambda^2,$
$T = 2\lambda+36\lambda^2+270\lambda^3+810\lambda^4.$

* Compare Edgeworth, "A Method of Representing Statistics by Analytical Geometry," Proceedings Fifth International Congress of Mathematicians, Cambridge, 1912.

Obviously no algebaric solution can be obtained from equations (3) and (4) for κ and λ in terms of the computed values β and ϵ, and hence a resort to tables is necessary. The values of β and ϵ for values of κ from 0 to 0.0335 and of λ from –0.040 to +0.100 have been computed.* The process of determining the constants of the translated normal curve consists first in computing β and ϵ from the given data, and then in entering the table and interpolating for the corresponding values of κ and λ.† On substituting these values in (2) the value of a can be found and thence on multiplying a by κ the position of the median of the distribution is obtained.

The sign of κ is determined by the sign of the third moment about the mean μ_3, that is, by the direction of the skewness or asymetry. For positive skewness the mean must lie to the right of the median and hence μ_1', the first moment about the mean, must be positive which necessitates a positive sign for κ. Therefore, the sign of κ is the same as that of the skewness.

To fit a curve to the given data, after the constants have been determined it is necessary to find, by solving a cubic equation for each value, the values of t corresponding to the x's of the respective classes. The cubic is

$$a\lambda t^3+a\kappa t^2+at-x=0$$

Any of the various methods of approximating to the solution of a cubic may be used in solving these equations.

The area of each class can now be obtained by computing the corresponding areas under the standard normal curve from a table of the probability integral.

The Method of Interpolation. The actual fitting of the curve can now be readily accomplished.‡ The distinctively geometrical operation is the interpolation for the values of λ and κ for a given pair of values of β and ϵ.

Within the limits of the table§ the curves resulting from the assignment of a constant value to β are practically straight

*Only a part of the original table appears in the accompanying table. The original values were computed to four places of decimals, but three place numbers are sufficient to illustrate the method of approximating to the solution.

†Compare "Tables for Statisticians and Biometricians," Cambridge University Press, 1914.

‡For the statistical details see Elderton, l. c.

§As may be seen on examining the Table.

lines, $\beta=0$ is the λ—axis; $\beta=1$ is a line parallel to the λ—axis. Hence we may safely assume that the variation from one column to the next and from one line to the next is linear for values of β. That is, ordinary first difference interpolation methods are applicable.

As regards the system of ϵ curves we have for instance $\epsilon=.128$ at ($\lambda=.050$, $\kappa=0$); again, at approximately (.045, .060) and (.40, .085). We are therefore warranted in assuming the applicability of first difference methods to interpolation between the ϵ curves.

As an illustration let us find the values of λ and κ for $\epsilon=0.112$ and $\beta=0.044$. On inspection of the table it is seen that λ lies between 0.30 and .035 and κ between .090 and .095. When $\kappa=.090$, $\lambda=.033$ for $\epsilon=.112$. When $\kappa=.095$, $\lambda=.031$ for $\epsilon=.112$. For $\beta=.042$ and $\kappa=.090$, $\lambda=.033$ and for $\beta=.046$ and $\kappa=.095$, $\lambda=.031$, $\epsilon=.112$ in each case. Hence, to first differences, $\lambda=.032$ and $\kappa=.093$ for $\epsilon=.112$ and $\beta=.044$. For interpolation in parts of the table showing more rapid variations appropriate methods will suggest themselves.

Taken geometrically the table represents two distinct systems of curves, with each curve of one system intersecting all the curves of the other system. Therefore, a pair of values for λ and κ can always be found for values of ϵ and β within the range of the table.

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TABLE OF ϵ AND β.

(ϵ is the first and β the second number of each pair.)

κ \ λ	−040	−035	−030	−025	−020	−015	−010	−005	000	005	010	015	020	025	030	035	040	045	050
000	−061	−056	−050	−043	−035	−027	−019	−010	000	010	021	033	045	057	071	084	098	113	128
	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
005		−055	−049	−042	−035	−027	−019	−010	000	010	021	033	045	057	071	084	098	113	128
		000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000	000
010		−055	−049	−042	−035	−027	−018	−010	000	011	022	033	045	058	071	085	099	113	128
		000	000	000	000	000	000	000	001	001	001	001	001	001	001	001	001	001	001
015			−049	−042	−035	−027	−018	−009	001	011	022	034	046	058	071	085	099	114	129
			001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001	001
020			−048	−041	−034	−026	−017	−008	002	012	023	034	046	059	072	086	100	115	130
			001	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002
025			−047	−040	−033	−025	−016	−007	003	013	024	035	047	060	073	087	101	116	131
			002	002	002	003	003	003	003	003	003	003	003	003	003	003	003	003	003
030			−046	−039	−032	−024	−015	−006	004	014	025	036	049	061	074	088	102	117	132
			003	003	004	004	004	004	004	004	004	004	005	005	005	005	005	005	005
035			−045	−038	−031	−023	−014	−005	005	015	026	038	050	063	076	089	104	118	133
			004	005	005	005	005	005	006	006	006	006	006	006	006	007	007	007	007
040				−037	−030	−022	−013	−004	006	017	028	039	052	064	077	091	105	120	135
				006	006	007	007	007	007	007	008	008	008	008	008	008	009	009	009
045				−036	−028	−020	−011	−002	008	019	030	041	053	066	079	093	107	122	137
				008	008	008	009	009	009	009	010	010	010	010	011	011	011	011	011
050				−034	−026	−018	−009	−000	010	021	032	043	055	068	081	095	109	124	139
				009	010	010	011	011	011	012	012	012	012	013	013	013	013	014	014
055				−032	−024	−016	−007	002	012	023	034	045	057	070	083	097	111	126	141
				011	012	012	013	013	013	014	014	015	015	015	016	016	016	016	017
060					−022	−014	−005	004	014	025	036	048	060	073	086	100	114	129	144
					014	015	015	016	016	017	017	017	018	018	019	019	019	019	020
065					−020	−012	−003	006	017	028	039	050	062	075	089	102	116	131	146
					016	017	018	018	019	019	020	020	021	021	022	022	022	023	023
070					−018	−009	000	009	019	030	041	053	065	078	091	105	119	134	149
					019	020	020	021	022	022	023	024	024	025	026	026	026	026	027
075					−015	−007	002	012	022	033	044	056	068	081	094	108	122	137	152
					022	023	023	024	025	026	026	027	028	028	029	029	030	030	031
080					−013	−004	005	015	025	036	047	059	071	084	097	111	125	140	155
					025	026	026	027	028	029	030	031	031	032	033	033	034	034	035
085						−001	008	018	028	039	050	062	075	088	101	115	129	144	159
						029	030	031	032	033	034	034	035	036	037	037	038	039	039
090						002	011	021	032	043	054	066	079	092	105	119	133	148	163
						032	033	034	036	037	038	039	039	040	041	042	042	043	044
095						005	015	025	035	046	058	070	083	096	109	123	137	152	167
						036	037	038	039	041	042	043	044	045	046	046	047	048	049
100						009	018	028	039	050	062	074	087	100	113	127	141	156	171
						039	041	042	044	045	046	047	048	049	050	051	052	053	054