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Report No. 186

DETERMINATION OF SURFACE DENSITIES FROM A COMBINATION OF GRAVIMETRY AND SATELLITE ALTIMETRY

by

John F. Isner

Prepared for
Air Force Cambridge Research Laboratories
Air Force Systems Command
United States Air Force
Bedford, Massachusetts 01730

Contract No. F19628-72-C-0120
Project No. 8607
Task No. 8607-01
Work Unit No. 86070101

Scientific Report No. 5

Contract Monitor: Bela Szabo
Terrestrial Sciences Laboratory

The Ohio State Universrity
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Abstract

The utilization of satellite altimetry by itself, and in combination with existing gravity material is considered for the determination of the gravity field of the earth. This is done by developing equations that relate surface density values defined in discrete blocks to geoid undulations and gravity anomalies. The use of a higher order reference field defined by a set of spherical harmonics is considered and truncation errors are computed when the contribution of an area outside a spherical cap is obtained from a spherical harmonic expansion of the anomaly field. A suggested solution to recover 5° equal area blocks is made with specific recommendations made on the ordering of these blocks so that structured sets of normal equations will result. The determination of a more local field (such as 1°) is discussed using the global 5° field as a basis.

Foreword

This report was prepared by John F. Isner, a graduate student in the Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-72-C-0120, The Ohio State University Research Foundation Project No. 3368B1, which is under the direction of Professor Richard H. Rapp. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, L.G. Hanscom Field, Bedford, Massachusetts, with Mr. Bela Szabo, Project Scientist.

This report was also presented to the Graduate School of The Ohio State University in partial fulfillment of the requirements for the M.Sc. degree.

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1. Background

The science of geodesy is concerned with the determination of the size and shape of the earth or, equivalently, its external gravity field. Before the advent of artificial earth satellites, three techniques were available to the geodesist: gravimetry, geodetic surveying, and astronomical methods. The most important technique developed was an implementation of Stoke's formula: by covering the earth with a network of gravity measurements, the shape of the equipotential surface of gravity at sea level, the geoid, could be determined. The painstakingly slow progress of terrestrial and shipborne gravimetry progress of terrestrial and shipborne gravimetry programs, the huge computational task of gravity reduction required by the "classical" theory, and the prospect of extensive unsurveyed and even unsurveyable areas dimmed the hopes of geodesists that they would ever see a refined gravimetric geoid.

With the advent of artificial earth satellites, geodesists gained a means of collecting data at an unprecedented rate and extent. The technique of orbital analysis attacks the problem of geoid determination by comparing an observed orbit with an orbit predicted from an earth gravity model; the perturbations of the predicted orbit are ascribed to the residual gravity field, whose parameters are usually taken as spherical harmonic coefficients. The parameters are then determined by least squares analysis. In the early years of satellite geodesy there was much cause for excitement. Geodesists soon knew the broad features of the geoid worldwide.

If it were possible to determine the harmonic coefficients up to any desired degree, the geoid could be resolved in infinite detail. The difficulty with orbital analysis is that, while the low-frequency, long-wavelength coefficients are determined with good accuracy, the high-frequency, short-wavelength coefficients produce only slight perturbations in the orbit and, given the present accuracy of observations, are thus difficult to separate out in the solution. The problem is intensified by the phenomenon of attenuation with altitude of the influence of the spherical harmonics. The attenuation (in per cent) of a harmonic coefficient of degree n at an altitude h is given by:

$$A = \left(\frac{a}{a + h}\right)^{n+1} \times 100$$

Thus, for example, the effect of the 21st degree harmonics is reduced to 10% and 42nd harmonics to 1% of their terrestrial perturbing influence at 710 km (Greenwood et al, 1969a).

In consequence of such factors as these, the ultimate spatial resolution of the geoid obtainable by conventional orbital analysis appears to be limited to geoid wavelengths of 1000 km and greater. For the shorter wavelengths, two independent estimates of a given coefficient may differ more inter se than the posteriori accuracy estimate of either.

Beginning in the late middle Sixties with Kaula, a number of investigators proposed solutions which combine gravimetric and orbital data; such solutions are capable of achieving greater geoid detail than orbital analysis alone, but are hampered by the uneven distribution of gravimetry, particularly by the unsurveyed areas. (For a brief survey of combined solutions, see Koch, 1968).

About the same time, oceanographers were exploring the possible applications of satellite-borne altimeter systems to the measurement of surface features of the oceans. It was reasoned that since the geoid is the shape of the ocean surface would take if freed from dynamic effects (tides, currents, surface waves, etc.), an independent knowledge of the geoid (from geodesy) would enable altimetry to determine deviations from the "level surface" and would thus be invaluable in the study of ocean dynamics. Unfortunately the geodesists could not provide a geoid of the needed accuracy. To illustrate, it is estimated that the Gulf Stream produces a surface feature only 50 cm higher than the geoid, whereas the current estimates of the geoid itself differ by as much as 20 m in this area (Greenwood et al, 1969b). While such oceanographic applications seemed only remotely feasible, the geodetic value of satellite altimetry was immediately apparent: it provided a means of determining the geoid directly, at least over the oceans, thus solving the primary problem of geodesy. More correctly, altimetry measures instantaneous sea-surface topography which, if averaged over a period of time, yields steady-state sea surface topography. The difference between the latter surface and the geoid itself is so much smaller than the current estimated geoid accuracy -- on the order of 10 m (rms) -- that the difference could be neglected. If the total error of altimetry can be made sufficiently less than 10 m (rms) -- say 5 m (rms) -- then altimetry will contribute measurably to our knowledge of the geoid over the oceans.

The beauty of the altimetry concept is that it combines the speed of data collection of a satellite with the sensitivity to the short wavelengths of gravimetry; moreover it measures the function -- geoid shape -- directly rather than the parameters of the function (harmonic coefficients). This is always the most favorable way to proceed as far as error propagation is concerned. The ultimate refinement of the geoid obtainable by altimetry is limited only by coverage and by the size of the signal "footprint".

Several feasibility studies have been done to determine the optimum altimeter design and orbit characteristics (Frey, 1965; Godbey, 1969; Stanley, 1971). All of the suggested specifications can easily be met by available technology. The following specifications are taken from Stanley (1971) and should give an idea of the type of system required:

A. Orbit

1. Near polar ($i = 99.1^\circ$)
2. Near circular ($e = 0.001$)
3. Apogee = 500 nautical miles
4. Period = 103 minutes.

These parameters were chosen in order to gain maximum coverage of ocean areas. Consecutive arcs shift westward about 24° so that adjacent ground tracks are separated by roughly 1.3°. Complete geographical coverage will thus occur once every 18 days (24/1.3) if only North-South passes are used, or once every 9 days if all passes are used. This design offers certain problems insofar as tracking station and calibration station location are concerned.

B. Altimetry system

1. Small (34" diameter) parabolic antenna
2. Stabilization to within 0.5° of the vertical
3. Narrow beam width (1.75°)
4. Split-gate type receiver
5. Pulse repetition frequency 1000 hz.

A small antenna is easier to point and stabilize in a downward direction. It can thus be mounted on a conventional satellite. Range is measured by means of a split-gate receiver which effectively determines the centroid of the rising portion of the return waveform and hence the distance along the normal to the sea surface. Because the area illuminated by the signal is finite, the altitude measured by the system is actually a weighted average over this area; this tends to eliminate the effects of surface irregularities such as waves. The pulse repetition frequency is the number of transmitted pulses in one second. These are electronically integrated with a time constant of, say, 1/10 second.

The purpose of pulse repetition is to reduce the random error per pulse by a factor of roughly $1/\sqrt{n}$. Thus if a single pulse has a random error of 3 m (rms) and 100 pulses are averaged by integration, the random error is reduced to 0.3 m (rms) (the actual reduction is probably less than this). Due to the forward motion of the satellite, the final integrated altitude represents the mean altitude along a finite portion of the satellite ground track. The size of this "footprint" determines the size of the smallest geoidal feature resolvable by the system.

C. Errors

Several types of errors contribute to the total error of the altimetry geoid. For example there are errors in the altimeter itself, errors due to the propagation of the measuring signal through the atmosphere, and errors in the geocentric radius of the altimeter-bearing satellite at any given time. For an altimeter of the type under consideration, altimeter and propagation errors would have a combined rms error of well under one-half meter. The most serious errors are those in the determination of the orbit; the chief factors here are uncertainty in tracking station coordinates and uncertainty in the gravity model used to predict the orbit. For the purpose of this paper, a more detailed discussion of the various error sources is unnecessary. We are only interested in a final estimate of the accuracy of the geoid as determined by satellite altimetry. Quoting the conclusion of Stanley (1971):

> The above studies demonstrate that an orbiting altimeter with an instrumentation accuracy of better than 50 cm and noise levels of 20 cm can be built and flown within a relatively short time frame. They also indicate that with proper pre-mission analysis and careful scheduling, an $1^\circ \times 1^\circ$ global geoidal map of 1-2 m resolution can be provided.

Instead of working with geocentric radius, we follow the usual approach in physical geodesy and introduce a reference surface. Then the height anomaly ζ is the distance of the ocean surface above this reference surface; from Figure 1:

$$\zeta = r - r_0 - h$$

The reference surface is a well-defined mathematical figure so that the accuracy of r_0 depends only on the accuracy of satellite positioning; if r_0 is errorless, Stanley's conclusions hold for the height anomaly as well.

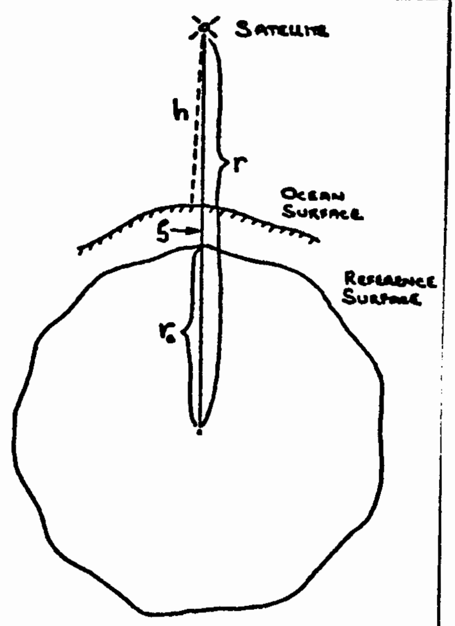


FIGURE 1

2. The Problem

One important consequence of Stokes' theorem is that the size and shape of an equipotential surface completely enclosing its masses, its rotational velocity and quantity of mass enclosed completely determine the gravity field external to that surface. If the earth were entirely convered by oceans, the conditions of the theorem would be met and the external gravity field could be derived from altimetry observations alone.

In reality the geoid does not contain all of its masses; this fact necessitates the various gravity reductions of the classical theory which is based on Stokes' theorem. These attempt to computationally "remove" the masses above the geoid. The geoid corresponding to the regularized earth model ("co-geoid") is different from the true geoid, the difference being the so-called "indirect effect". Molodensky proved that the method of mass removal is unimportant so long as the indirect effect is rigorously taken into account (Molodensky et al, 1962 Ch. III), but rigorous computation of the indirect effect is so laborious that it is never done in actual practice. Instead, a regularization method having a small indirect effect is chosen and this indirect effect is subsequently ignored. The free-air geoid is an example of a co-geoid which is known to be rather close to the true geoid.

Molodensky points to another weakness in the classical theory. He shows that regardless of how rigorously the indirect effect is handled, the true geoid can never be determined exactly. The assumptions we are forced to make concerning the nature of topographic masses (such as the assumption of homogeneous density) do not account for the "high-frequency" effects of local mass anomalies in the topography. Molodensky demonstrates the seriousness of this problem in mountainous areas ("Model Studies," Ch. VIII).

In 1945 Molodensky demonstrated that the shape of the physical surface of the earth could be determined to any desired accuracy from surface measurements alone. Unlike the classical approach, there is no reliance on Stokes' theorem. The boundary value problem is formulated directly for the physical surface of the earth. In abandoning the geoid in favor of the physical surface as the "boundary" we lose the convenience of equipotentiality but acquire a surface which does indeed enclose all of its masses and which is entirely accessible to measurements of various kinds. The situation is depicted in Figure 2. The physical surface of the earth has both oceanic and continental components. Over the oceans, altimetry determines the physical surface directly, while on the continents the physical surface is covered by gravity measurements. There may also be occasional gravity measurements at sea, chiefly in the form of profiles. Determination of the external field is thus a "mixed boundary value problem" (Young, 1970), one in which a heterogeneous mixture of "values" is known

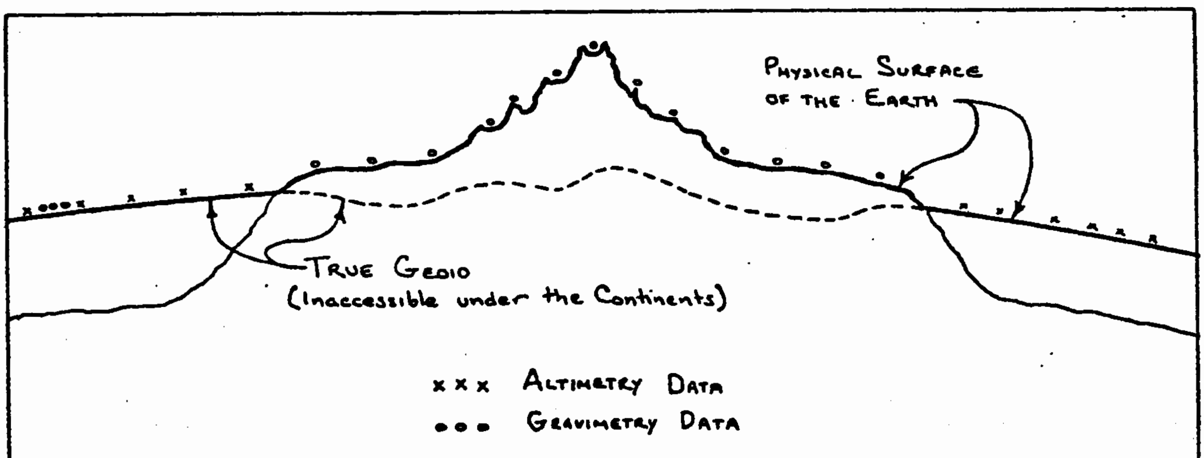


FIGURE 2

for a surface. The purpose of this paper is to examine some of the theoretical and practical considerations involved in formulating and solving such a problem.

Sections 3 to 9 are devoted to theoretical considerations which culminate in two simultaneous linearized integral equations [21, 22] or [24, 25] or [26, 27] (depending on the degree of approximation allowed). A key step along the way is the introduction of the surface density parameterization of the anomalous potential. Molodensky introduced the surface density parameter as a mathematical tool in the intermediate stages of his solution by Successive Approximation of his Basic Integral Equation (Molodensky et al, 1962 pp. 118-124). In a mixed boundary value problem, however, the surface density can be an efficient computational tool, as suggested by Koch (1970). This is accomplished by exploiting the analogy between a linear integral equation and a system of linear algebraic equations; two simultaneous linear integral equations therefore become a mixed system of linear algebraic equations in the common unknown parameters x_1, x_2, ... x_n, which may then be solved by methods of linear algebra. This mixed system is represented symbolically by the equation set [32, 33].

Sections 10 to 16 are devoted to various practical considerations in the solution of the system [32, 33] for the surface densities. The recovery of useful information from the surface densities is discussed in Section 17. In Section 15, [32, 33] serve as the basis for normal equations in a global solution for mean surface densities of 5° "blocks". In Section 16, [32, 33] are again the basis for normal equations, but this time in a non-global solution for mean surface densities of 1° "blocks". Throughout these sections, the question of truncation angles for [32] and [33] is of paramount importance: how many unknowns need be carried in each equation? The basis for the answer is provided in Appendices "A" and "B", both based on the statistical investigations of Meissl (1971a, b).

3. Formulation of Boundary Value Problems

A boundary value problem in potential theory may be formulated in either of two equivalent forms:

1. As a partial differential equation (Laplace's equation) along with the associated boundary conditions;
2. As an integral equation.

For most of the problems of physical geodesy, the first approach leads to equations for which rigorous solutions have not been found. A notable exception is in Stokes' derivation of the famous equation bearing his name; it can easily be solved using standard partial differential equation techniques (Caputo, 1967 pp. 66-69). The reason that the solution can be found so easily is that the "boundary" of the classical approach is an equipotential surface.

The integral equation technique is a much more useful tool in modern physical geodesy. This is due to the fact that integral equations may be solved for surfaces which are not equipotential. (For example we again choose Stokes' equation. We can arrive at Stokes' equation via the integral equation technique without the assumption of equipotentiality. See Caputo, pp. 71-75). The modern theory of Molodensky formulates the boundary value problem for the physical surface of the earth, a boundary which is definitely non-equipotential. Unfortunately the analytical solution of the resulting integral equation is a sum of infinitely many integral terms; the practical evaluation of the modern formulas is thus somewhat more complex than that of Stokes' formula.

A critical step in the formulation of any boundary value problem in physical geodesy is that of linerization. Without linerization the integral equations themselves would be nonlinear and unsolvable. The current best estimates of the physical constants of the earth are used to establish a reference surface which is centered at the earth's center of mass and rotates about the same axis. The degree of complexity of this surface may vary from that of a simple, four-parameter mean earth ellipsoid to more complex, many-parameter models (see Section 4). The reference (normal) field set up by this reference figure will be only slightly different from the gravity field of the earth, the difference being the so-called residual field. The boundary value problem is then formualted in terms of parameters which describe the residual field such as T, ζ, Δg, etc.. Since these are always small quantities, second and higher order terms are neglected resulting in linear integral equations. The error introduced by this approximation depends on the size of the residual field. If great accuracy is required, a high order reference surface must be adopted.

4. Use of a Higher Order Reference Surface

Until only recently, an ellipsoid was used as the source of the normal potential U. In his 1968 paper, Koch abandoned the usual ellipsoid in favor of a closer approximation of the geoid (Koch, 1968). This has only become feasible in the last few years due to the fact that satellite observations have yielded the lower harmonic coefficients with good accuracy. Koch suggested use of a reference surface computed from all (non-forbidden) harmonics of degree p less than or equal to 4.

In the case of a higher order reference surface, the disturbing potential is smaller, meaning that the linear approximation will be more accurate. Statistically speaking, the elements of this field behave better than those of the usual reference field (Meissl, 1971a).

Higher order reference surfaces have disadvantages as well. Virtually all existing gravity anomalies have been computed with respect to a reference, for example by the application of the International Gravity Formula. In order to make these anomalies compatible with the new field, each anomaly (or mean anomaly) would have to be corrected by adding the quantity:

$$|\text{grad } U_I| - |\text{grad } U|$$

where U_I is the potential of gravity of the International Ellipsoid and U is the potential of the new reference surface. The resulting anomalies are called "residual anomalies" and are somewhat smaller than the anomalies computed with respect to a reference ellipsoid.

A second disadvantage of higher order reference surfaces is their irregular geometry. The normal potential at a point P(r, Φ, λ) is given by:

$$U = \frac{k^2 M}{r}\left[1 + \sum_{n=2}^{p} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n P_{nm} \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right)\right] + \frac{\omega^2 r^2}{2} \cos^2 \Phi$$

The figure of the reference surface is the equipotential surface obtained by setting U equal to a constant W_0 , the earth's potential at sea level. Since W_0 is not known precisely enough for this purpose at the present time, Koch suggests a way of computing an approximate value U_0* (Koch, 1968). Then the radius vector of any point on the reference surface may be determined iteratively from the above equation, a formidable computation. This will never actually be necessary, however; just as the reference ellipsoid is replaced by a sphere in the derivation of Stokes' equation, the higher order reference surface may be replaced by an ellipsoid or even a sphere whenever an element of its geometry appears as the multiplier of a small quantity.

* The term $U_0 - W_0$ will occur repeatedly in our equations. Determining its value will improve our knowledge of the size of the earth (Heiskanen and Moritz, p. 103).

This is just another benefit resulting from the process of linerization. Normal gravity (or equivalently the above-mentioned correction to existing anomalies) will, of course, be computed without approximation.

5. Surface Density Parameterization of the Anomalous Potential

With the steady increase of both terrestrial gravity data and data from orbital analysis, and in view of the vast amount of data eventually to be acquired by satellite altimetry, an adjustment method which can accommodate a variety of kinds of observations in a simultaneous solution becomes increasingly desirable.

Beginning with Kaula in 1966, a number of authors have addressed themselves to the problem. Most of the methods suggested solve for potential coefficients as the unknown parameters of the earth's gravity field.

In 1968 Koch proposed that the distrubing potential be represented in terms of a simple layer of variable density distributed over the surface of the earth (Koch, 1968). He showed that the surface density parameterization, although it lacks physical significance, results in formuals which are much more economical to work with and more flexible than those based on spherical harmonics.

Basically Koch's proposal is summarized in the following two equations:

$$W = U + T$$

$$T = \iint_S \frac{\Phi}{\ell} \, dS$$

where Φ is the density function, ℓ is the distance from the computation point to the element dS and S denotes integration over the physical surface of the earth. In practice mean densities of finite-size elements are used; their values are determined by the least-squares solution of a system of several types of observation equations (to be derived). From the adjusted values, all elements of the anomalous field may be determined from formulas which relate those elements to the surface density function (see Section 17).

Besides economy, an important advantage of Koch's parameterization is that the density values are associated with specific areas of the earth's surface. Potential coefficients, since they represent integrals over the whole earth, tend to give poor local representation because of the influence of the vast unsurveyed areas on the higher harmonics. If the gravity field is known in greater detail in a particular area, surface densities will parameterize the field in that area much better than an equal number of potential coefficients.

The flexibility of the surface density parameterization makes it an ideal tool for attacking the present problem. The basic integral equations of both classical and modern physical geodesy will now be derived in terms of surface density for later application to the problem.

6. Derivation of the Basic Integral Equations in Terms of Surface Density

The basic integral equations of modern physical geodesy result when the boundary value problem is formulated for the physical surface of the earth (PSE). We begin by differentiating the definition of the disturbing potential:

$$[1]\ T = W - U$$

with respect to ν where $d\nu$ is a differential of the normal plumbline (see Figure 3). The resulting equation is evaluated at a point on the PSE:

$$[2]\ \left.\frac{\partial T}{\partial \nu}\right|_p = \left.\frac{\partial W}{\partial \nu}\right|_p - \left.\frac{\partial U}{\partial \nu}\right|_p$$

The right-hand terms may be rewritten as:

$$[3]\ \left.\frac{\partial U}{\partial \nu}\right|_p = \nabla U \cdot \hat{\nu}_p = -\gamma_p$$

$$[4]\ \left.\frac{\partial W}{\partial \nu}\right|_p = \nabla W \cdot \hat{\nu}_p = -g_p \cos\theta = -g_p\left(1 - \frac{\theta^2}{2}\right)$$

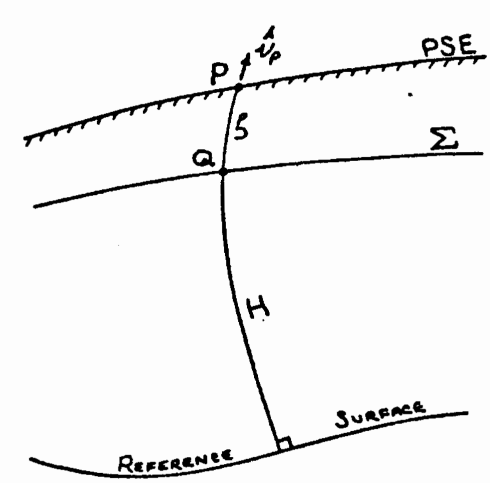


FIGURE 3

where θ is the deflection of the vertical. To a first order approximation γ_P is obtained by expanding normal gravity about the point Q where the normal plumbline intersects the telluroid, Σ (defined in Heiskanen and Moritz, p. 292):

$$[5] \quad \gamma_P = \gamma_Q + \zeta \left.\frac{\partial \gamma}{\partial U}\right|_Q$$

The height anomaly ζ is not known until after solution of the boundary value problem. By Bruns' formula applied to an arbitrary reference surface (Heiskanen and Moritz, p. 100)

$$[6] \quad \zeta = \frac{1}{\gamma_Q} (T_Q + U_0 - W_0)$$

so that [5] becomes:

$$[7] \quad \gamma_P = \gamma_Q + \frac{1}{\gamma_Q} (T_Q + U_0 - W_0) \left.\frac{\partial \gamma}{\partial \nu}\right|_Q$$

Combining [7], [4] and [3] with [2] we obtain:

$$[8] \quad \left.\frac{\partial T}{\partial \nu}\right|_P = -g_P\left(1 - \frac{\theta^2}{2}\right) + \gamma_Q + \frac{1}{\gamma_Q} (T_Q + U_0 - W_0) \left.\frac{\partial \gamma}{\partial \nu}\right|_Q$$

The famous "linear approximation is now made:

$$[9] \quad \left.\frac{\partial T}{\partial \nu}\right|_P = \left.\frac{\partial T}{\partial \nu}\right|_Q$$

enabling us to write the "fundamental boundary condition" as:

$$[10] \quad \left.\frac{\partial T}{\partial \nu}\right|_Q - \left.\frac{\partial \gamma}{\partial \nu}\right|_Q \frac{T_Q}{\gamma_Q} = \Delta g - G$$

where:

$$[11]\quad \Delta g = g_P - \gamma_Q$$

is the "residual anomaly" (defined in Section 4) and:

$$[12]\quad G = g_P \theta_P^2/2 + \frac{1}{\gamma_Q} \left.\frac{\partial \gamma}{\partial \nu}\right|_Q (U_0 - W_0)$$

The "linear approximation" is necessary because the physical surface of the earth is initially unknown in the formulation of the boundary value problem: we are trying to determine it. Strictly speaking, therefore, the "boundary value problem of physical geodesy" is not a "third boundary value problem of potential theory", since the latter involves a known boundary surface. We do, however, know the telluroid Σ from the normal height H which is determined by dividing the observed geopotential number by normal gravity (Heiskanen and Moritz, p. 170). Furthermore the telluroid is very close to the PSE. This is the essence of the linear approximation: by approximating an unknown surface (the PSE) by a known surface very close to it (the telluroid) our problem becomes a third boundary problem of potential theory which can then be attacked by conventional means. This explains why the telluroid is often called the "earth of the first approximation" (Molodensky et al, p. 118): after solving for height anomaly ζ we may proceed to re-solve the problem using an "earth of the second approximation" obtained by adding ζ to H, and so on. In practice, however, this iteration is unnecessary since the error introduced by the linear approximation is generally small, especially when a higher order reference surface is used (Molodensky et al, p. 105).

Next consider the disturbing potential T as arising from a density layer spread over the surface of the PSE, in accordance with the proposal of Koch:

$$[13]\quad T_P = \iint_{PSE} \frac{\Phi}{\ell_P}\, dS$$

Due to the fact that the PSE is unknown, the linear approximation again becomes necessary; the PSE is replaced by the telluroid as the surface of integration:

$$[14] \quad T_Q = \iint_\Sigma \frac{\Phi}{\ell_Q} \, d\Sigma$$

This definition of T is next inserted into the fundamental boundary condition ([10]):

$$[15] \quad \frac{\partial}{\partial \nu_Q} \iint_\Sigma \frac{\Phi}{\ell_Q} \, d\Sigma - \frac{1}{\gamma_Q} \frac{\partial \gamma}{\partial \nu} \bigg|_Q \iint_\Sigma \frac{\Phi}{\ell_Q} \, d\Sigma = -\Delta g + G$$

The first term is the derivative of the surface layer potential with respect to the direction of the normal plumbline at Q, evaluated at the surface of the telluroid. The potential itself is continuous at the telluroid, but its derivative has a discontinuity, a fact well known from elementary potential theory (Heiskanen and Moritz, p. 6):

$$[16] \quad \frac{\partial}{\partial \nu_Q} \iint_\Sigma \frac{\Phi}{\ell_Q} \, d\Sigma = -2\pi \Phi_Q \cos(\nu, n)_Q + \iint_\Sigma \frac{\partial}{\partial \nu_Q} \left(\frac{1}{\ell_Q}\right) \Phi \, d\Sigma$$

where $(\nu, n)_Q$ denotes the angle between the telluroid surface normal and the normal plumbline at Q, which will henceforward be denoted as α_Q. In practice, α_Q would be taken as the local ground slope at P, an approximation justified by the near parallelism of the telluroid and the general terrain surface. Inserting [16] into [15] we get the famous "Basic Integral Equation" of Molodensky (Molodensky et al, p. 104):

$$[17] \quad 2\pi \Phi_Q \cos\alpha_Q - \iint_\Sigma \frac{\partial}{\partial \nu_Q} \left(\frac{1}{\ell_Q}\right) \Phi \, d\Sigma + \frac{1}{\gamma_Q} \frac{\partial \gamma}{\partial \nu} \bigg|_Q \iint_\Sigma \left(\frac{1}{\ell_Q}\right) \Phi \, d\Sigma = \Delta g - G$$

The following change of variable is usually made

$$[18a] \quad dS = d\Sigma \cos\alpha$$

$$[18b] \quad X = \Phi / \cos\alpha$$

where dS is the projection of $d\Sigma$ on the local horizon (perpendicular to the normal plumbline). Finally we may apply Bruns' formula to [14] so as to transform the expression for the disturbing potential T into an expression for the height anomaly. The following set of integral equations results:

$$[19] \quad \zeta = \frac{1}{\gamma_Q} \iint_S \frac{X}{\ell_Q} \, dS + \frac{U_0 - W_0}{\gamma_Q}$$

$$[20]\quad \Delta g - G = 2\pi X_Q \cos^2\alpha_Q - \iint_S \frac{\partial}{\partial \nu_Q}\left(\frac{1}{\ell_Q}\right) X\, dS + \frac{1}{\gamma_Q}\frac{\partial \gamma}{\partial \nu}\bigg|_Q \iint_S \frac{1}{\ell_Q} X\, dS$$

7. Ellipsoidal Approximation of the Reference Surface

As discussed earlier, the reference surface chosen as the source of the normal gravity field will generally be more complicated than an ellipsoid. Except for the computation of normal gravity (for which purpose the equations of the higher reference surface must be applied rigorously), the reference surface may be approximated by another figure such as an ellipsoid, or even a sphere in the linearized integral equations, depending upon the desired accuracy. In this section the equations for an ellipsoidal approximation will simply be presented. Details of the derivation may be found in Koch (1968). In the following section, the simple derivation of the spherical approximation will be presented.

According to Koch, approximating the reference surface by an ellipsoid whose surface potential is U_0 and whose second eccentricity is e', and tolerating errors on the order of e'^4 we obtain the following two integral equations from [19] and [20]:

$$[21]\quad \zeta = \frac{1}{\gamma_Q}\iint_E \frac{1}{\ell_Q} X\, dE + \frac{U_0 - W_0}{\gamma_Q}$$

$$[22]\quad \Delta g - G = 2\pi X_Q \cos^2\alpha_Q - \iint_E \frac{1}{\ell_Q}\left[\frac{2}{a+H_Q} + \frac{e'^2(1-2\sin^2\Phi'_Q)}{a} + \frac{2\omega^2}{\gamma_Q}\right] X dE$$

$$+ \iint \frac{1}{\ell_Q{}^3}\Big\{(a+H_Q) - (a+H)\cos\psi - ae'^2[\sin\Phi'_Q(\tfrac{1}{2}\sin\Phi'_Q + \sin\Phi')$$

$$- \cos\psi(\sin^2\Phi'_Q + \tfrac{1}{2}\sin^2\Phi'))]\Big\} XdE$$

where

a = semimajor axis of approximating ellipsoid
Φ' = geocentric latitude
ω = angular velocity of the earth

and

$$dE = (a+H)^2(1-e'^2\sin^2\Phi')\cos\Phi'\, d\Phi' d\lambda$$

$$\cos\psi = \sin\Phi'_Q \sin\Phi + \cos\Phi'_Q \cos\Phi' \cos\Delta\lambda$$

$$\ell^2_Q = (a + H_Q)^2 + (a+H)^2 - 2(a+H_Q)(a+H) - ae'^2(\sin^2\Phi'_Q + \sin^2\Phi')(1-\cos\psi)$$

$$G = -\left[\frac{2}{a+H_Q} + \frac{e'^2}{a}(1-2\sin^2\Phi'_Q) + \frac{2\omega^2}{\gamma_Q}\right](U_0 - W_0)$$

8. Spherical Approximation of the Reference Surface

Instead of an ellipsoidal approximation, the reference surface may be replaced by a sphere of radius R. This will result in errors on the order of the square of the second eccentricity. The following substitutions may be made in the basic integral equations ([19,20]), their derivativation being obvious from Figure 4:

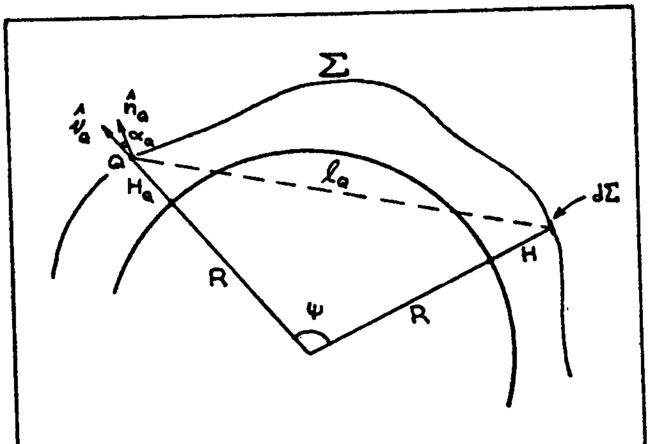


FIGURE 4

[23]

$$\ell_Q^2 = (R+H_Q)^2 + (R+H)^2 - 2(R+H_Q)(R+H)\cos\psi$$

$$\frac{\partial}{\partial\nu_Q}\left(\frac{1}{\ell_Q}\right) = \frac{\partial}{\partial H_Q}\left(\frac{1}{\ell_Q}\right) = -\frac{(R+H) - (R+H)\cos\psi}{\ell_Q^3}$$

$$\gamma = \frac{k^2 M}{(R+H)^2}$$

$$\frac{\partial\gamma}{\partial\nu} = \frac{\partial\gamma}{\partial H} = -\frac{2\gamma}{(R+H)}$$

$$dS = (R+H)^2 d\omega$$

so that [20] becomes

$$\Delta g - G = 2\pi X_Q \cos^2\alpha_Q + \iint_\omega \frac{(R+H_Q) - (R+H)\cos\psi}{\ell_Q^3} X(R+H)^2 d\omega - \frac{2}{R+H_Q} \iint_\omega \frac{X}{\ell_Q}(R+H)^2 d\omega$$

Eliminating $\cos\psi$ by means of [23] and setting $r = R + H$ we obtain, after algebraic manipulation, the following two integral equations analogous to [21] and [22]:

$$[24] \quad \zeta = \frac{1}{\gamma_Q} \iint_\omega \frac{X}{\ell_Q} r^2 d\omega + \frac{U_Q - W_Q}{\gamma_Q}$$

$$[25] \quad \Delta g - G = 2\pi X_Q \cos^2\alpha_Q - \frac{1}{2r_Q} \iint_\omega \frac{r^2(r^2 - r_Q^2)}{\ell_Q^3} X d\omega - \frac{3}{2r_Q} \iint_\omega \frac{X}{\ell_Q} r^2 d\omega$$

These equations agree with what Molodensky gives (Molodensky et al, p. 118).

9. Application of the Basic Integral Equations To The Geoid

By neglecting the topography

$$H = H_Q = \alpha_Q = O$$

in [24] and [25], we may apply the basic integral equations to the geoid:

$$[26] \quad \zeta = \frac{R^2}{\bar{\gamma}} \iint_\omega \frac{X}{\ell_Q} d\omega + \frac{U_0 - W_0}{\bar{\gamma}}$$

$$[27] \quad \Delta g - G = 2\pi X_Q - \frac{3R}{2} \iint_\omega \frac{X}{\ell_Q} d\omega$$

where $\bar{\gamma}$ represents a mean value of gravity over the earth, for example 979.8 gals. Again, these equations are valid to a spherical approximation of the reference surface. They will be useful later as approximations to the integral equations of the modern theory, particularly in the study of truncation error where theory required the property of isotropy (see Appendix "A").

10. Mathematical Model

The mathematical model for the global density solution will be based on the two integral equations [21] and [22] (or [26] and [27] if less accuracy is required).

Much attention has been given to the analytical solution of integral equations in the literature of physical geodesy. Such solutions do exist and are well documented. Molodensky solved his "Basic Integral Equation" (our [20]) by the method of "Successive Approximations;" the result takes the form of the infinite sum of integral terms, the first of which is Stokes' integral and the succeeding terms of which form a convergent series of "correction terms" (Molodensky et al, p. 118). The method has been implemented for electronic computer by Koch (1967) in such a way as to minimize the effort involved in computing the successive approximations. Convergence problems do arise, especially in mountainous areas.

An analytical-type solution is convenient when there is only one observed quantity (in the case of Molodensky's Basic Integral Equation it is the gravity anomaly) having global coverage. The present problem is, however, a "mixed boundary value problem" having two observed quantities -- the gravity anomaly on the continents and the height anomaly on the oceans -- neither of which alone has global coverage. Koch (1970) suggests a method for solving this problem which is based on the simple analogy between a linear integral equation and a system of linear algebraic equations (see also Heiskanen and Moritz, p. 294). Koch's idea will be pursued here with certain simplifications.

In order to facilitate a discussion of Koch's proposal, it will be convenient to introduce the following notation:

let ξ denote the point of computation
let η denote the variable point
let δW denote the quantity $U_0 - W_0$

then we may write each of the equation sets [21,22] or [24,25] or [26,27] symbolically as

$$[28] \quad \zeta(\xi) = A(\xi)\delta W + \iint_S K_\zeta(\xi, \eta)\, X(\eta)\, dS(\eta)$$

$$[29]\quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X(\xi) + \iint_S K_g(\xi,\eta)X(\eta)dS(\eta)$$

where the meaning of the various terms can be found be referring to the appropriate equation set.

Let the earth of the first approximation be divided into N "squares" of equal size and let these be projected onto the local horizon ($dS = d\Sigma \cos\alpha$). Assume for each "square" a constant density X_J. Then we may replace continuous integration over S by summation over a finite number (N) of discrete elements

$$[30]\quad \zeta(\xi) = A(\xi)\delta W + \sum_{J=1}^{N} X_J \iint_J K_\zeta(\xi,\eta)dS(\eta)$$

$$[31]\quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X_I + \sum_{J=1}^{N} X_J \iint_J K_g(\xi,\eta)\,dS(\eta)$$

which may be further simplified in notation:

$$[32]\quad \zeta(\xi) = A(\xi)\delta W + \sum_{J=1}^{N} C_\zeta(\xi,\eta,J)\,X_J$$

$$[33]\quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X_I + \sum_{J=1}^{N} C_g(\xi,\eta,J)\,X_J$$

where the "discretization error" depends on the size of the element chosen. Quite appropriately [32] will be referred to as the "satellite altimetry equation" and [33] will be referred to as the "gravimetry equation".

Theoretically speaking, an equation of type [32] or [33] could be written at each point observation of ζ or Δg. Point observations are, however, widely scattered; furthermore, a point observation is generally not representative of an area. In the practical implementation of these equations we will therefore interpret the "observations" $\zeta(\xi)$ and $\Delta g(\xi)$ as mean values over a finite area centered at ξ. The size of these "observational units" may depend on the number of actual point observations therein. Thus for a remote land area with only a few gravimetry observations scattered over an area of several thousand square miles, we might use a mean 5° anomaly; for an ocean area with plentiful altimetry data we might drop down to 1° as the basic unit.

The observational unit should not be confused with the system of discrete elements $J = 1, 2, \ldots N$. These are elements of fixed size location chosen according to some scheme. In the Zhonogolovich scheme, for example, there are 1640 elements each equal in area to a $5^{\circ} \times 5^{\circ}$ block at the equator. To each block corresponds a single density X_J whose value we seek to determine.

We will develop [32] and [33] as observation equations. Since the number of possible "observational units" is in great excess of the number of unknowns, our system has a redundancy which can be exploited in a least squares solution. Weights can be assigned to the observations on the basis of area, a larger weight for a smaller area.

Equations 32 and 33 are impractical in their present form since they do not take into consideration the mathematical nature of the kernal functions K_{ζ} and K_g. Both kernals are nearly proportional to $1/\ell$ so that the integrands are singular at the computation point and decrease to a minimum at the antipodal point.

In the areas remote from the computation point ξ ("remote zones"), the kernal suppresses the effect of the density function so that only its lower harmonics contribute significantly to the integral. We may take advantage of the fact that the lower harmonics are well known to compute this remote zone contribution explicitly. The remote zone contribution (RZC) is then treated as an "observation" and is transferred to the left-hand side of the equation where it combines with the actual observation to yield a modified observation. It only remains to determine the extent of the remote zones for each type of equation. We will speak of a "Cap C of half-opening angle ψ_0 centered at ξ". The remote zones are then defined as the region exterior to this cap, namely the region $\{S-C\}$. The size of this cap will in general be different for each type of equation and will depend on the accuracy of the RZC.

The reason for modifying the model in remote zones is simple: it greatly reduces the number of unknowns X_J which must be carried in each equation and hence the number of coefficients which must be computed. This results in a sparse coefficient matrix which may possibly be handled by special techniques.

Two remote zones modification schemes currently in use are (Meissl, 1971b):

1. Representing the function x by an analytical function with a finite number of parameters -- for example, a spherical harmonic expansion truncated at degree p -- and using this representation to estimate the effect of the remote zones;

2. Using bigger "squares" as the distance from the computation point increases.

Both schemes are employed in this paper. The first scheme is described in Appendix "A", the second in Appendix "B". In the following two sections, the first scheme will be employed in modifying [32] and [33].

11. Application of Truncation Theory (Appendix "A") To The Gravimetry Equation

In order to determine the extent of "remote zones" in the case of the gravimetry equation ([33]), the accuracy of the RZC must be considered. The statistical background for this problem has been given in Appendix "A", and requires that we return to the original integral equation on which [33] is based, namely [22] or [25]. Attention much specifically be focused on the integral terms since these contain the RZC.

The theory in Appendix "A" applies to an integral of the form:

$$[34] \quad g(\xi) = \iint_{\omega} K(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta)$$

where the kernal (K) depends only on the cosine of the angle between ξ and η; this is denoted by writing K as a function of the inner product $\xi \cdot \eta$. Equation A.19 expresses the error in $g(\xi)$ which results when the exact function f is approximated by the function f' over that portion of the sphere outside the circular cap C of half-opening angle ψ_0.

Clearly neither kernal in [22] or [25] is isotropic. For the purposes of error analysis, however, [27] (which neglects the topography) is a valid approximation; furthermore its kernal is isotropic. Denoting the integral term of [27] as δg and rewriting it in the notation of Section 10 we obtain:

$$[35] \quad \delta g(\xi) = \frac{3R}{2} \iint_{\omega} \frac{1}{\ell(\xi \cdot \eta)} X(\eta)\, d\omega(\eta)$$

Then by [A.21] the spherical harmonic expansion of the error resulting from using a simplified version of X in remote zones is given by:

$$[36] \quad e_{nm} = \Delta\lambda_n (1-\beta_n) X_{nm}$$

11.1 Eigenvalues of the Kernal

By the results of Appendix "A", the eigenvalues are given by the Funk-Hecke formula:

$$[37] \quad \Delta\lambda_n = 2\pi \int_{t=-1}^{+1} \Delta K(t)\, P_n(t)dt = 2\pi \int_{t=-1}^{t_o=\cos\psi_o} \frac{3R}{2\ell(t)} P_n(t)dt = 3\pi R \int_{t=-1}^{t_o} \frac{P_n(t)}{\ell(t)}\, dt$$

The distance ℓ depends only on t (= cosψ). From Figure 5

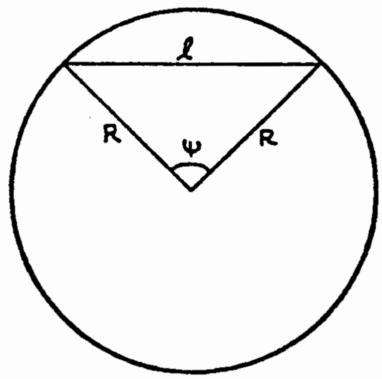


FIGURE 5

$$[38] \quad \ell(t) = \sqrt{2R^2 - 2R^2\cos\psi} = \sqrt{2}\,\sqrt{1-t}\; R$$

so that [37] becomes:

$$[39] \quad \Delta\lambda_n = \frac{3\pi}{\sqrt{2}} \int_{t=-1}^{t_o} \frac{P_n(t)}{\sqrt{1-t}} dt \equiv \frac{3\pi}{\sqrt{2}}\, b_n(t_o)$$

Following Meissl (1971b) a recursive formula for $b_n(t_o)$ is developed:

$$[40] \quad b_n(t_o) = 2\left(\frac{n-\frac{1}{2}}{n+\frac{1}{2}}\right)b_{n-1}(t_o) - \left(\frac{\sqrt{1-t_o}}{n+\frac{1}{2}}\right)[P_n(t_o) - P_{n-2}(t_o)] - \left(\frac{n-\frac{3}{2}}{n+\frac{1}{2}}\right)b_{n-2}(t_o)$$

The first two terms in the sequence are:

$$[41] \quad b_o(t_o) = -2(1-t_o)^{\frac{1}{2}} + 2\sqrt{2}$$

$$[42] \quad b_1(t_o) = \frac{2}{3}(1-t_o)^{\frac{3}{2}} - 2(1-t_o) + \frac{2}{3}\sqrt{2}$$

11.2 Eigenvalues Of The Smoothing Operator

In remote zones, X will be represented by a truncated harmonic expansion through degree p. From the results of Appendix "A", the eigenvalues of such a smoothing operator are:

$$[43] \quad \beta_n = \begin{cases} 1 & n \le p \\ 0 & n > p \end{cases}$$

The smoothed version of X, namely X^1, has the following spherical harmonic representation:

$$[44] \quad X' = \sum_{n=0}^{\infty} X'_n = \sum_{n=0}^{p} X_n$$

11.3 Expansion Of The Surface Density In Spherical Harmonics

Neglecting the topography, the surface density X may be expressed in terms of Δg and T (Heiskanen and Moritz, p. 303):

$$[45] \quad X = \frac{1}{2\pi}\Delta g + \frac{3}{4\pi R}T$$

The spherical harmonic expansion of X can thus be obtained by expanding the right-hand side of the above equation:

$$[46] \quad X_n = \frac{1}{2\pi}\Delta g + \frac{3}{4\pi R}T_n$$

Applying the well-known relationship between the spherical harmonic expansion of the gravity anomaly and that of the disturbing potential (Heiskanen and Moritz, p. 89):

$$[47] \quad T_n = \frac{R}{n-1}\Delta g_n \qquad n \geq 2$$

and substituting into [46] we obtain

$$[48] \quad X_n = \frac{1}{4\pi}\left(\frac{2n+1}{n-1}\right)\Delta g_n \qquad n \geq 2$$

Suppose that the gravity anomaly is expanded into fully normalized spherical harmonics:

$$[49] \quad \Delta g_n = \sum_{m=0}^{n}(\bar{a}_{nm}\bar{R}_{nm} + \bar{b}_{nm}\bar{S}_{nm})$$

Then [48] becomes

$$[50] \quad X = \frac{1}{4\pi}\sum_{n=2}^{\infty}\left(\frac{2n+1}{n-1}\right)\sum_{m=0}^{n}(\bar{a}_{nm}\bar{R}_{nm} + \bar{b}_{nm}\bar{S}_{nm})$$

A final expression for the error of the integral δg is obtained by substituting the results of Sections 11.1 to 11.3 into [36]:

$$[51] \quad e(\xi) = \frac{3}{4\sqrt{2}}\sum_{n=p+1}^{\infty}\left(\frac{2n+1}{n-1}\right) b_n(t_o)\Delta g_n$$

This is the error due to using an approximate representation of X in remote zones. It is also equal to the error of the RZC since the inner zones contribute no error whatsoever. It is the error at a particular point. The mean square error (variance) over the entire sphere is of more value in the analysis of truncation error:

$$[52] \quad \sigma^2_{RZC_{33}} = \frac{9}{32} \sum_{n=p+1}^{\infty} \left(\frac{2n+1}{n-1}\right)^2 b_n(t_o)^2 c_n$$

where the c_n are "degree variances," i.e. the average square of the Laplace harmonic Δg_n of degree n over the entire sphere (Heiskanen and Moritz, p. 259), and are given by

$$[53] \quad c_n = \sum_{m=0}^{n} (\bar{a}^2_{nm} + \bar{b}^2_{nm})$$

The subscript RZC_{33} has been chosen to signify that the error is associated with the remote zone contribution of [33].

12. Application Of Truncation Theory (Appendix "A") To the Satellite Altimetry Equation

For purposes of truncation error analysis of the satellite altimetry equation ([32]) we again must return to the original integral equation ([21] or [24]). For the sake of obtaining an isotropic kernal, topography is again neglected resulting in [26]. Again our interest centers on the integral term in [26] which we denote by $\delta\zeta$ and rewrite in the notation of Section 10:

$$[54] \quad \delta\zeta(\xi) = \frac{R^2}{\bar{\gamma}} \iint_{\omega} \frac{1}{\ell(\xi \cdot \eta)} X(\eta) d\omega(\eta)$$

By the results of Appendix "A", the spherical harmonic expansion of the error resulting from using a simplified version of X in remote zones is:

$$[55] \quad e_{nm} = \Delta\lambda_n (1-\beta_n) X_n$$

The kernal of [54] is the same as the kernal of [35], within a constant. The error in $\delta\zeta$ is therefore obtained by direct comparison with [51]:

$$[56] \quad e(\xi) = \frac{\sqrt{2}R}{4\bar{\gamma}} \sum_{n=p+1}^{\infty} \left(\frac{2n+1}{n-1}\right) b_n(t_o) \Delta g_n$$

Once again the mean square error (variance) over the entire sphere is of more value:

$$[57] \quad \sigma^2_{RZC_{32}} = \frac{R^2}{8\bar{\gamma}^2} \sum_{n=p+1}^{\infty} \left(\frac{2n+1}{n-1}\right)^2 b_n(t_o)^2 c_n$$

The subscript RZC_{32} signifies that the error is associated with the remote zones contribution of [32].

13. Computational Test of [57] And [52]

In order to determine the extent of remote zones in [32] and [33], [57] and [52] were plotted as functions of truncation angle (ψ_o) and the degree (p) of the harmonic expansion used to represent X in remote zones (equivalently, the degree of the reference surface). We seek to determine those truncation angles ψ_{32} and ψ_{33} at which $\sigma^2_{RZC_{32}}$ and $\sigma^2_{RZC_{33}}$ are compatible with the observational accuracies of ζ and Δg, respectively. By "compatible" accuracy we mean that the magnitude of the RZC error should be at most one-half as great as the observational accuracy with which it combines.

Figure 6 shows $\sigma^2_{RZC_{32}}$ as a function of truncation angle (ψ_o) and the degree (p) of the harmonic expansion used to represent X in remote zones. Figure 7 shows $\sigma^2_{RZC_{33}}$ as a function of ψ_o and p.

It will be noted that the curves do not decrease monotonically as would logically be expected. In order to obtain conservative estimates from these curves, it would therefore be wise to replace the original curves by a set of smoothed, monotonically decreasing curves which bound the original set from above. This is illustrated in Figures 6 and 7 by the set of dashed lines.

No single set of values can be quoted as a standard for the observational accuracies. Recall that the observations $\zeta(\xi)$ and $\Delta g(\zeta)$ are really block averages over a finite area surrounding the computation point. The accuracy assessment must be made separately for each observational unit based on the number of point observations within the unit, their distribution and accuracy. Statistical methods will certainly play a part in the evaluation of observational accuracy.

To illustrate the use of Figures 6 and 7, suppose that a mean height anomaly for a 1° block is observed. Suppose further that the projected accuracy of 1 m^2 (see Section 1) is achieved for this observation. A compatible RZC accuracy would then be on the order of 0.25 m^2. Figure 6 indicates that to obtain this accuracy we may truncate [32] at 170° (p = 4), 165° (p = 6), 140° (p = 8),... 35° (p = 16).

Next suppose that a mean 1° gravity anomaly with an observational accuracy of 9 $mgal^2$ is "observed". Then a compatible RZC accuracy would be on the order of 2.25 $mgal^2$. Figure 7 indicates that to obtain this accuracy we may truncate [33] at 10° (p = 4), 2° (p = 6), 1° (p = 8), ... 0° (p = 16).

It must be realized that as the degree (p) of the reference surface increases, the accuracy with which its coefficients are known decreases. Our derivations assume that the coefficients through degree p are known perfectly. We should therefore choose a conservative value for p, for example p = 6 or p = 8 at most. If p is taken much larger, an error component will enter the RZC through the coefficients themselves; this is highly undesirable.

An interesting conclusion can be drawn from Figures 6 and 7. For a given value of p, typical observational accuracies require much larger truncation angle for [32] than for [33]. This emphasizes the fact that satellite altimetry is much more sensitive to the shape of the earth than gravimetry (given present levels of observational accuracy); a 1° mean height anomaly with a variance of 1 m^2 tells us more (about the shape of the earth) than the same square with a 9 $mgal^2$ gravity anomaly.

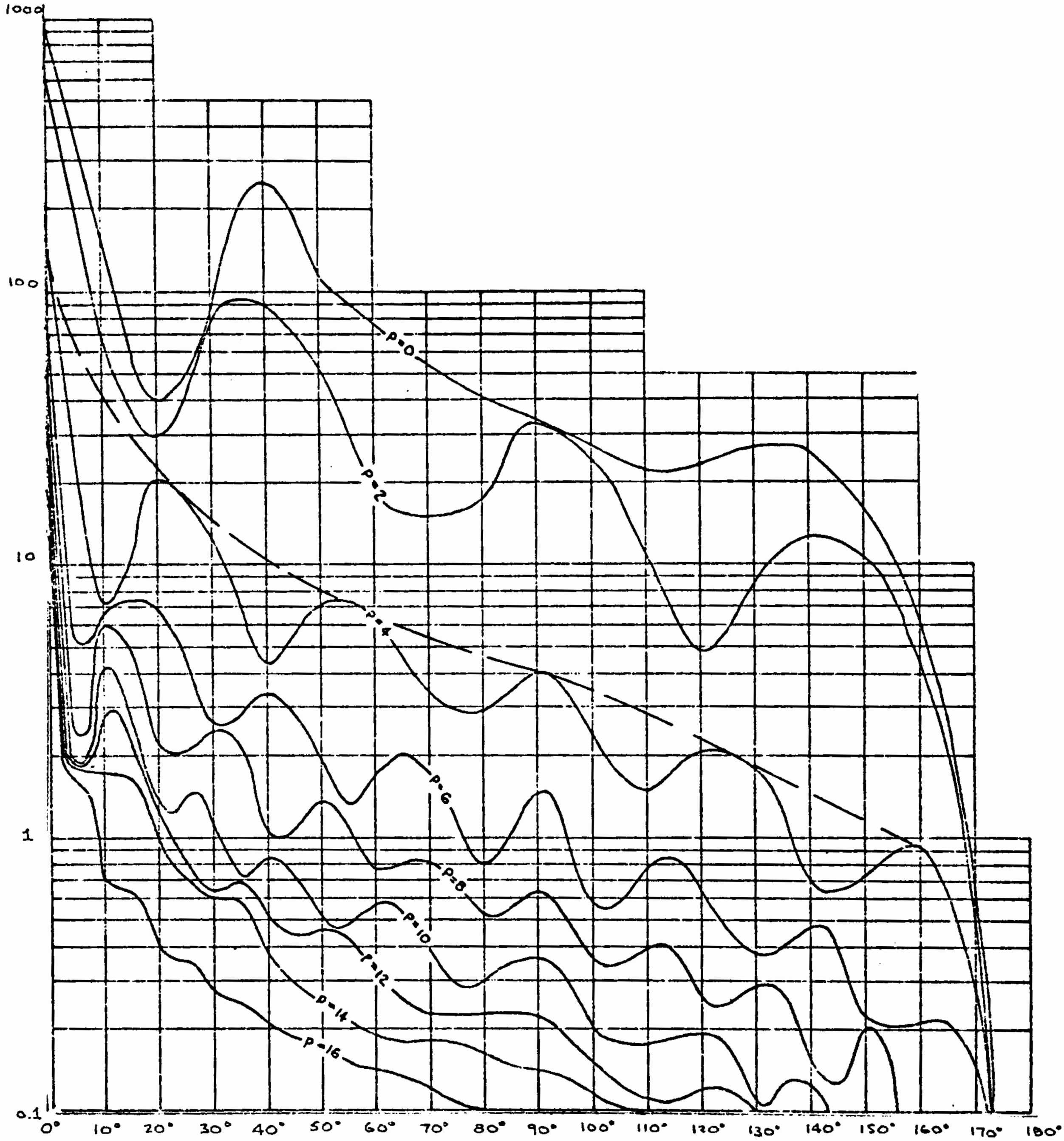


FIGURE 6

$\sigma^2 RZC_{32}$ AS A FUNCTION OF TRUNCATION ANGLE (ψ_0) AND THE DEGREE (p) OF THE HARMONIC EXPANSION USED TO REPRESENT X IN REMOTE ZONES

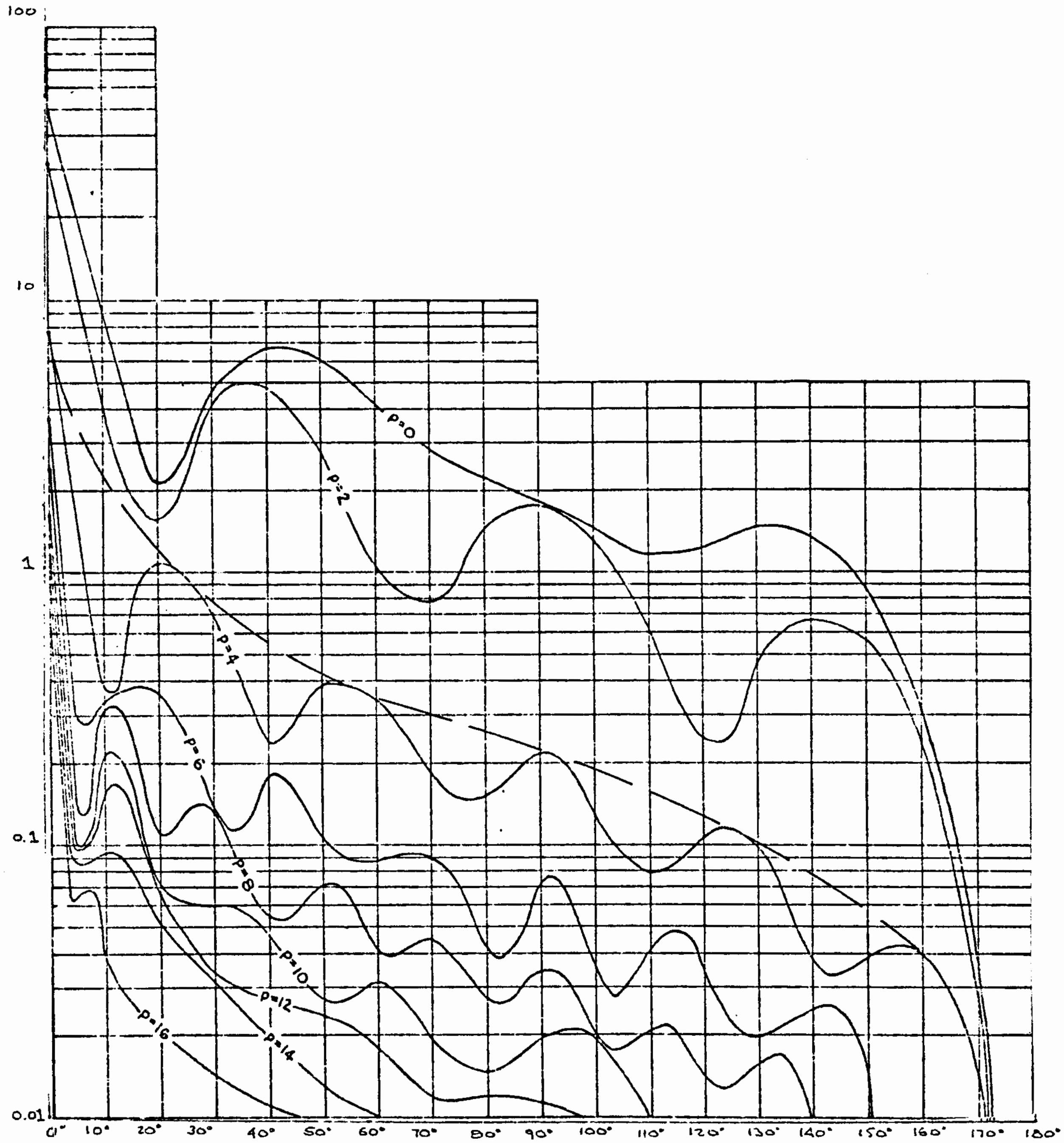


FIGURE 7

$\sigma^2_{RZC_{33}}$ AS A FUNCTION OF TRUNCATION ANGLE (ψ_o) AND THE DEGREE (p) OF THE HARMONIC EXPANSION USED TO REPRESENT X IN REMOTE ZONES

14. Modification Of The Model In Remote Zones

Equations 32 and 33 may now be rewritten to show the effect of remote zones explicitly:

$$[58] \quad \zeta(\xi) = A(\xi)\delta W + \sum_{J\in C32} C_{\zeta}(\xi,\eta,J)X_J + \iint_{S-C32} K_{\zeta}(\xi,\eta)\, X'(\eta)\, dS(\eta)$$

$$[59] \quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X_1 + \sum_{J\in C33} C_g(\xi,\eta,J)\, X_J + \iint_{S-C33} K_g(\xi,\eta)\, X(\eta) dS(\eta)$$

Due to the spherical harmonic representation X' ([44]), the right-hand integrals in the above two equations have continuously defined integrands and may therefore be evaluated explicitly by numerical integration. Simplification of the kernal functions K_{ζ} and K_g will facilitate this evaluation, and Meissl's theory can be extended to predict the effect of such kernal simplification on the accuracy of the integrals (Meissl, 1971b).

Define

$$[60] \quad \zeta'(\xi) = \zeta(\xi) - \iint_{S-C32} K_{\zeta}(\xi,\eta)\, X'(\eta)\, dS(\eta)$$

$$[61] \quad \Delta g'(\xi) = \Delta g(\xi) - \iint_{S-C33} K_g(\xi,\eta)\; X'(\eta)\; dS(\eta)$$

Then the mathematical model, modified for remote zones, may be written:

$$[62] \quad \zeta'(\xi) = A(\xi)\delta W + \sum_{J\in C32} C_{\zeta}(\xi,\eta,J)\, X_J$$

$$[63] \quad \Delta g'(\xi) = B(\xi)\delta W + D(\xi)X_1 + \sum_{J\in C33} C_g(\xi,\eta,J)X_J$$

where the modified "observations $\zeta'(\xi)$ and $\Delta g'(\xi)$ have variances equal to the

$$[64] \quad \sigma^2_{\zeta'} = \sigma^2_{\zeta} + \sigma^2_{RZC_{32}}$$

$$[65] \quad \sigma^2_{\Delta g'} = \sigma^2_{\Delta g} + \sigma^2_{RZC_{33}}$$

15. Normal Equations For 5° Zhonogolovich Blocks

Admittedly, a global-type solution for the tens of thousands of unknown densities in a 1° subdivision of the earth's surface is out of the question. The steps leading to the formation of the normal equations alone would require computation of literally hundreds of millions of fairly complex coefficients. The storage of these coefficients would require several hundred reels of magnetic tape. The most efficient arrangement of unknowns and observations, coupled with the most efficient solution algorithm, would still require many hours on the world's fastest computer.

We may, however, pursue the idea of a global solution for some smaller number of blocks; for example, instead of 1° blocks, we may try 5° blocks. If such a solution is feasible, the resulting 5° densities can be made to serve as a framework of "superior control" into which the 1° densities can be fitted piecemeal using some sort of non-global solution method. In this section, some of the factors in a global-type solution will be examined in order to demonstrate that the 5° solution is feasible. In the following sectiion, we will look at the problem of densification.

15.1 Choice Of A Subdivision Scheme

The problem of choosing a scheme for subdividing the earth's surface is not new. In the past, anomaly information has been stored in the form of block averages, the blocks being bordered by meridians and parallels in the simplest case. When satellite information began flowing in, its combination with existing terrestrial data was hampered by statistical problems which arose because of the unequal area of the terrestrial mean anomaly blocks. The need for an equal area subdivision scheme became apparent, and many suggestions were made. In 1970 the IAG formed a working group to study the various suggestions and arrived at a scheme suitable for international use (Rapp, 1971b). The working group expressed preference for schemes meeting the requirements set forth by Rapp in 1970 (Rapp, 1971a):

1. Blocks should have approximately equal area
2. Blocks should be bordered by meridians and parallels
3. Starting longitude for blocks should be 0° Greenwich
4. Block borders should be easily definable or locatable
5. A consistent system of computing the block coordinates for various size blocks should be available
6. A mean (anomaly) in a larger block should be obtained from the mean (anomalies) of smaller blocks that fall solely within the larger block.

One scheme satisfying all the above requirements was proposed by Zhonogolovich in 1952. Zhonogolovich proposed a subdivision of the earth into 410 blocks approximately equal in area to a 10° x 10° block at the equator. Each of these 10° blocks may be subdivided into 4 blocks, each equal in area to a 5° x 5° block at the equator. The 5° blocks may be subdivided into 25-1° blocks and so on, ad infinitum. The basic Zhonogolovich 10° scheme is given (for the Northern hemisphere) in Table 1, taken from Rapp (1971a):

Zone #	Northern Latitude Limit	#Blocks	Longitude Increment ($\Delta\lambda_o$)	Area (A_o)
1	10.1100	36	10.0000	.03063726
2	19.9700	34	10.5900	.03067479
3	29.8400	32	11.2500	.03064066
4	40.0800	30	12.0000	.03063617
5	49.9800	25	14.4000	.03065276
6	60.2600	21	17.1400	.03065757
7	70.3000	15	24.0000	.03065572
8	80.1800	9	40.0000	.03063254
9	90.0000	3	120.0000	.03068619

TABLE 1

Sucessive subdivisions of the basic Zhonogolovich scheme are obtained by the following set of formulas, derived by Rapp (1971a): Define the kth scheme as the subdivision of each block of the k-1st scheme into q_k^2 blocks of equal area. Then...

$$[66a] \quad A_k = A_{k-1}/q_k^2$$

$$[66b] \quad \Delta\lambda_k = \Delta\lambda_{k-1}/q_k$$

$$[66c] \quad \sin\varphi_{i+1} = \sin\varphi_i + A_{k-1}/q_k \Delta\lambda_{k-1} \qquad (\Phi_o = 0^\circ)$$

Zhonogolovich blocks were adopted by this author. The 5° scheme currently under consideration is given by substituting $k = 1$, $q_1 = 2$ into [66]. The proposed 1° scheme is then obtained from the 5° scheme by taking $k = 2$ and $q_2 = 5$. The 5° scheme is plotted in a polar sterographic projection in Figure 8.

15.2 Ordering Of Unknowns And Observation Equations

The 5° Zhonogolovich scheme of subdivision results in 410 x 2^2 = 1640 blocks of approximately equal area, to each of which corresponds an unknown density X_j. The numbering of these unknowns is completely arbitrary, but there is one numbering scheme which has the virtue of simplicity and which, when the observations are ordered in the same sequence (all observat ions within block 1 followed by all observations within block 2, etc.) results in a diagonally dominant pattern in the normal equations. This numbering scheme is called "pole-to-pole spiral ordering" by D. Brown (Brown, 1968). Brown was faced with the problem of adjusting a closed block of circular-format photographs of a sphere, which, in its geometry, is similar to the problem at hand. Pole-to-pole spiral ordering is illustrated for 5° Zhonogolovich blocks in Figures 9 and 10. A subtelty of this ordering is the manner in which the transition is made from one latitude zone to the next. If spiraling is to the east (as in the present case), the stepdown in latitude must also bear to the right or the bandwidth of the normal equations (to be defined) will actually be increased (Brown, 1968).

A series of circular cap templates of various half-opening angle (ψ_o) were computed for each of the 5° Zhonogolovich latitude zones. The templates

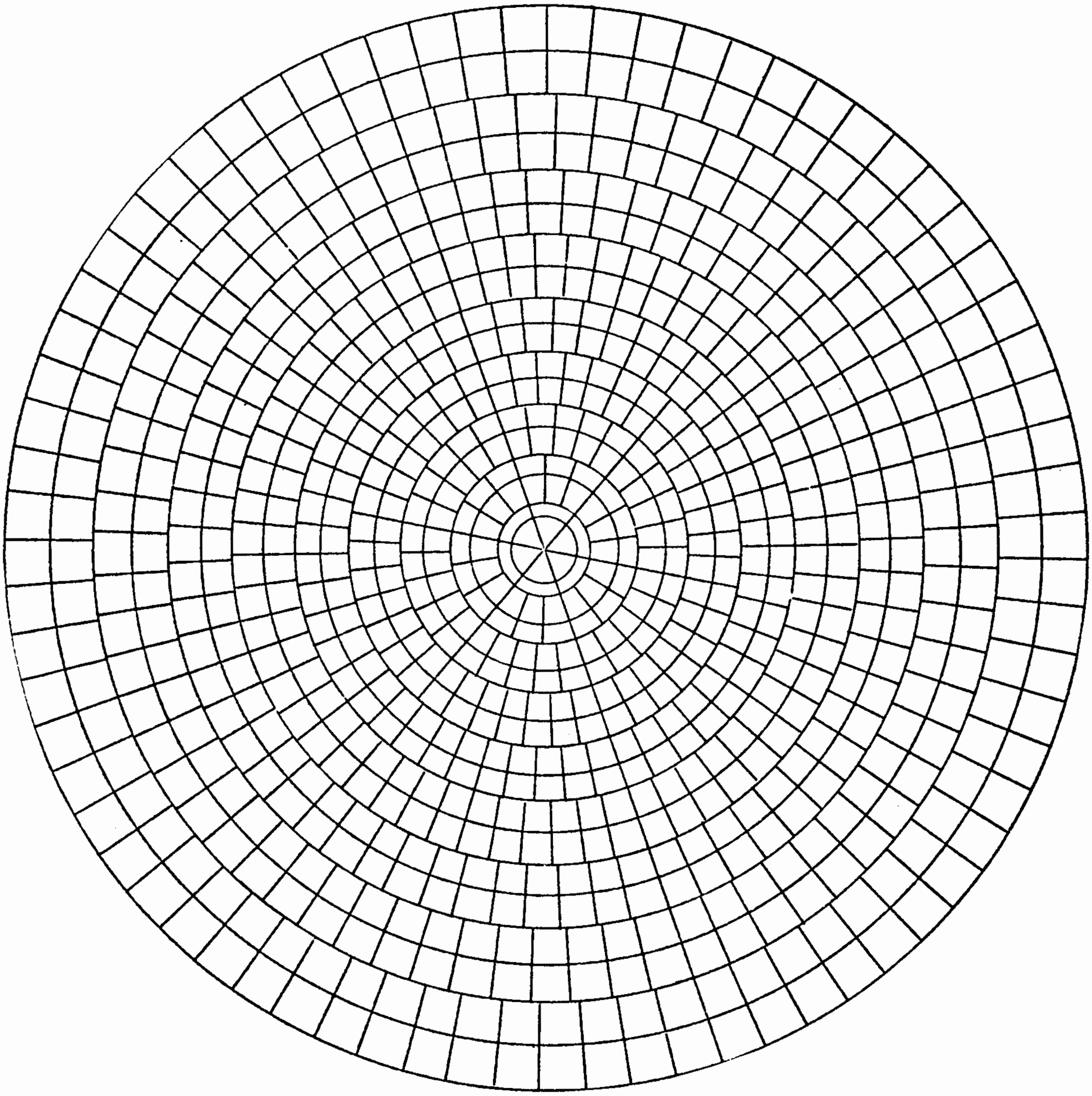


FIGURE 8

ZHONOGOLOVICH 5° SCHEME
POLAR STEREOGRAPHIC PROJECTION

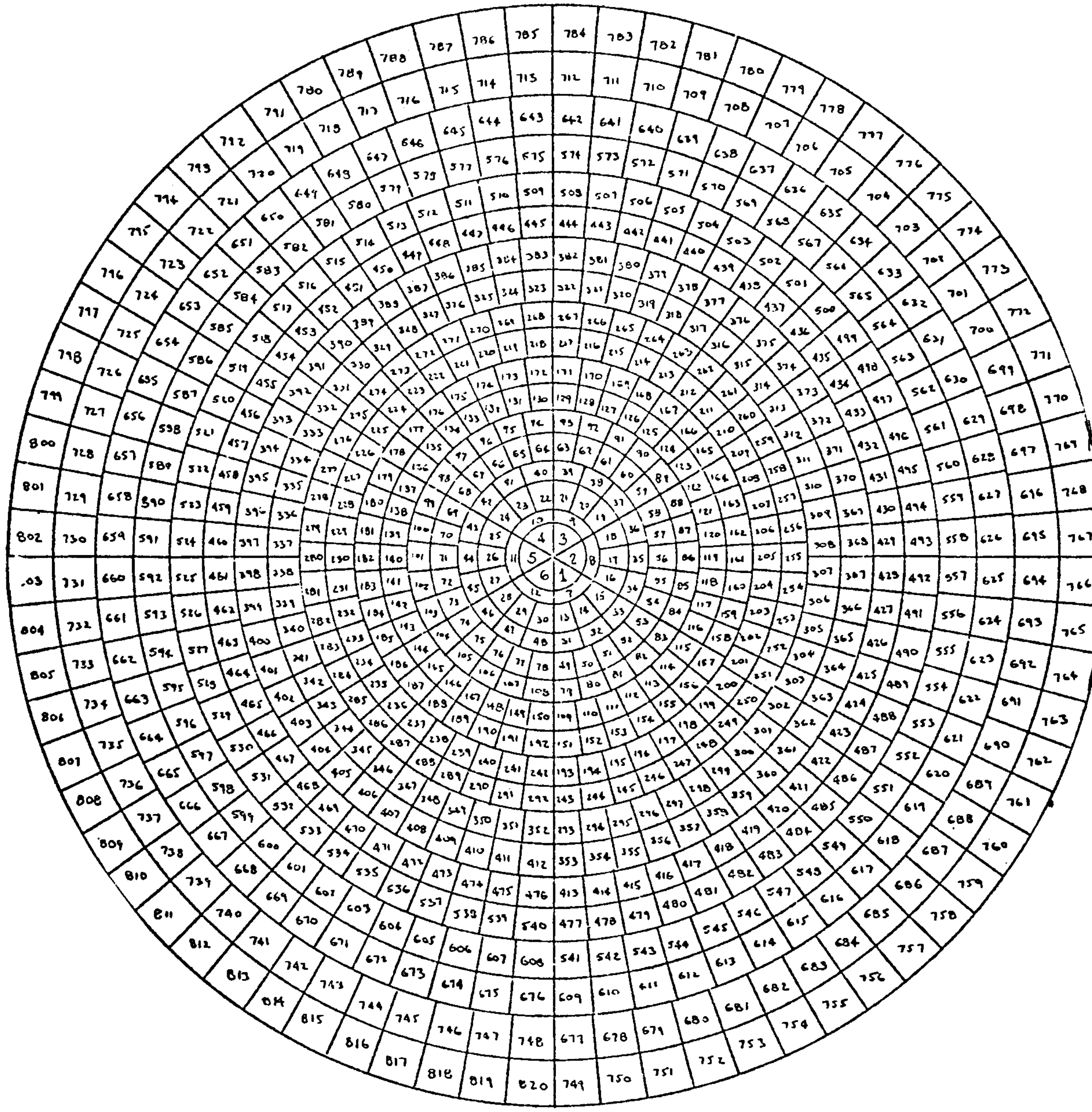


FIGURE 9

ZHONOGOLOVICH 5° SCHEME ILLUSTRATING POLE-TO-POLE SPIRAL ORDERING FOR NORTHERN HEMISPHERE POLAR STEREOGRAPHIC PROJECTION

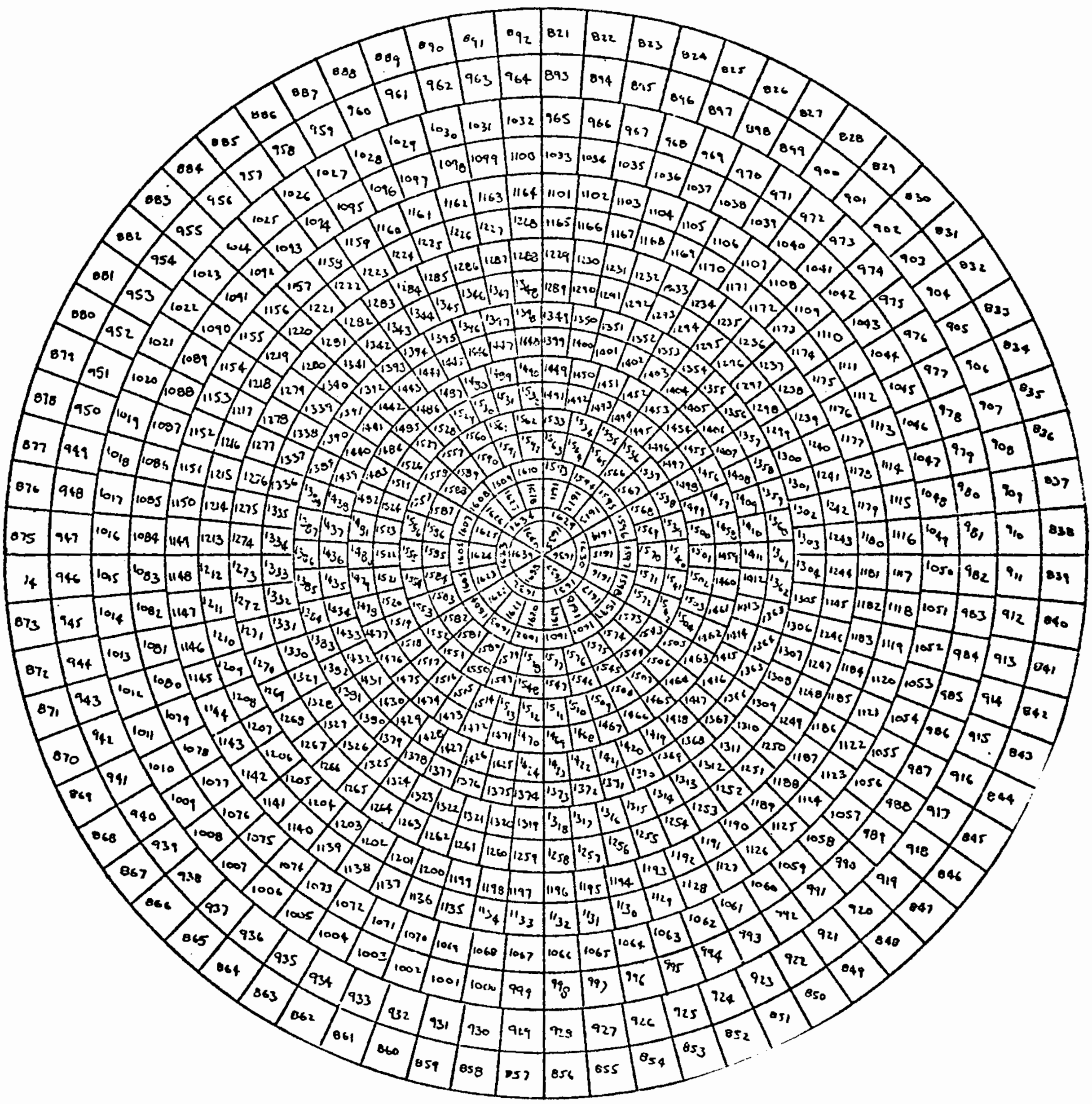


FIGURE 10

ZHONOGOLOVICH 5° SCHEME ILLUSTRATING POLE-TO-POLE SPIRAL ORDERING FOR SOUTHERN HEMISPHERE POLAR STEREOGRAPHIC PROJECTION were plotted in the same polar stereographic projection as the 5° scheme. By centering a cap template on a given Zhonogolovich block, the observation equation pattern for all observations within that block is obtained by noting <u>which</u> blocks are fully or partially contained within the template. This is illustrated in Figure 11 for a $\psi_o = 30°$ template centered on block #292.

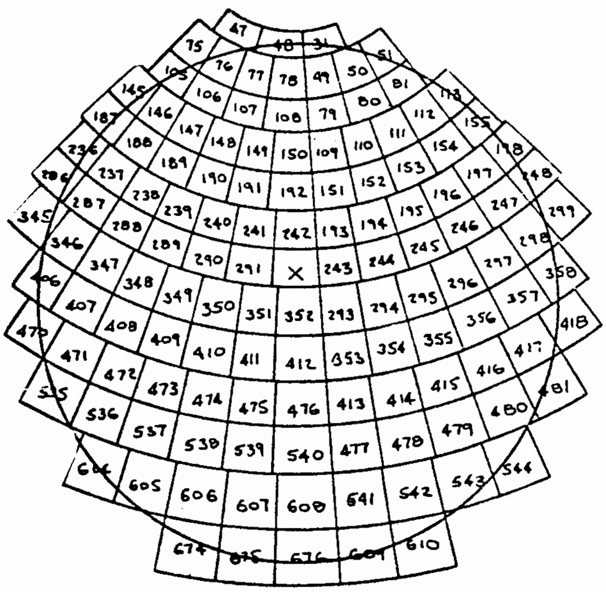


FIGURE 11

CIRCULAR CAP OF HALF-OPENING ANGLE 30° GENERATED BY AN OBSERVATION IN BLOCK NO. 292 OF THE 5° ZHONOGOLOVICH SCHEME

Continuing in this fashion for each observation, the pattern of the observation equations is generated.

As noted in Section 13, there is a marked discrepancy between truncation angles for altimetry equations and those for gravimetry equations. For a reference surface of $p = 4$ the altimetry truncation angles were found to be in the vicinity of 170° while the gravimetry truncation angles were closer to 10° for a typical observation. This suggests a natural partitioning of the mathematical model: group all gravimetry equations and all altimetry equations separately, but maintain pole-to-pole spiral ordering within each group. To facilitate a discussion of the observation and normal equations, let the mathematical model ([62, 63]) be written in the following matrix notation:

$$[67] \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X_Q = \begin{bmatrix} F_{b1} \\ F_{b2} \end{bmatrix}$$

where A_1 is the gravimetry coefficient matrix and A_2 is the altimetry coefficient matrix. Because of truncation A_1 will be extremely sparse while A_2 will be practically full.

A FORTRAN program was written to simulate the A_1 submatrix configuration. For this purpose two assumptions were made:

1. One and only one observation in each and every Zhonogolovich block
2. A constant truncation angle of 10° for all equations.

Although these are not quite realistic assumptions, they give results which are readily generalized. Figure 12 depicts the A_1 submatrix; this figure is only schematic since the actual submatrix is too large to reproduce at such a small scale.

The A_1 submatrix is seen to have a distinctly banded-bordered pattern (the right-hand border is one column wide and arises from the single "nuisance parameter" δW). The most surprising discovery is that the banded portion is symmetric. The number of bands depends only on the truncation angle; thus, for example, $\psi_0 = 10^\circ$ yields a 5-band pattern while $\psi_0 = 30^\circ$ yields a 13-band pattern.

In order to exploit the partitioning, we will adopt a "Sequential Adjustment" scheme (Uotila, 1967). Write the observation equations corresponding to [67] as

$$[68] \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X = \begin{bmatrix} F_{b1} \\ F_{b2} \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The "loss function" is given by

$$[69] \quad \Phi = V_1' P_1 V_1 + V_2' P_2 V_2 - 2K_2'(A_1 X - F_{b1} - V_1) - 2K_2'(A_2 X - F_{b2} - V_2)$$

where K_1 and K_2 are two vectors of Lagrangian multipliers. It has been assumed

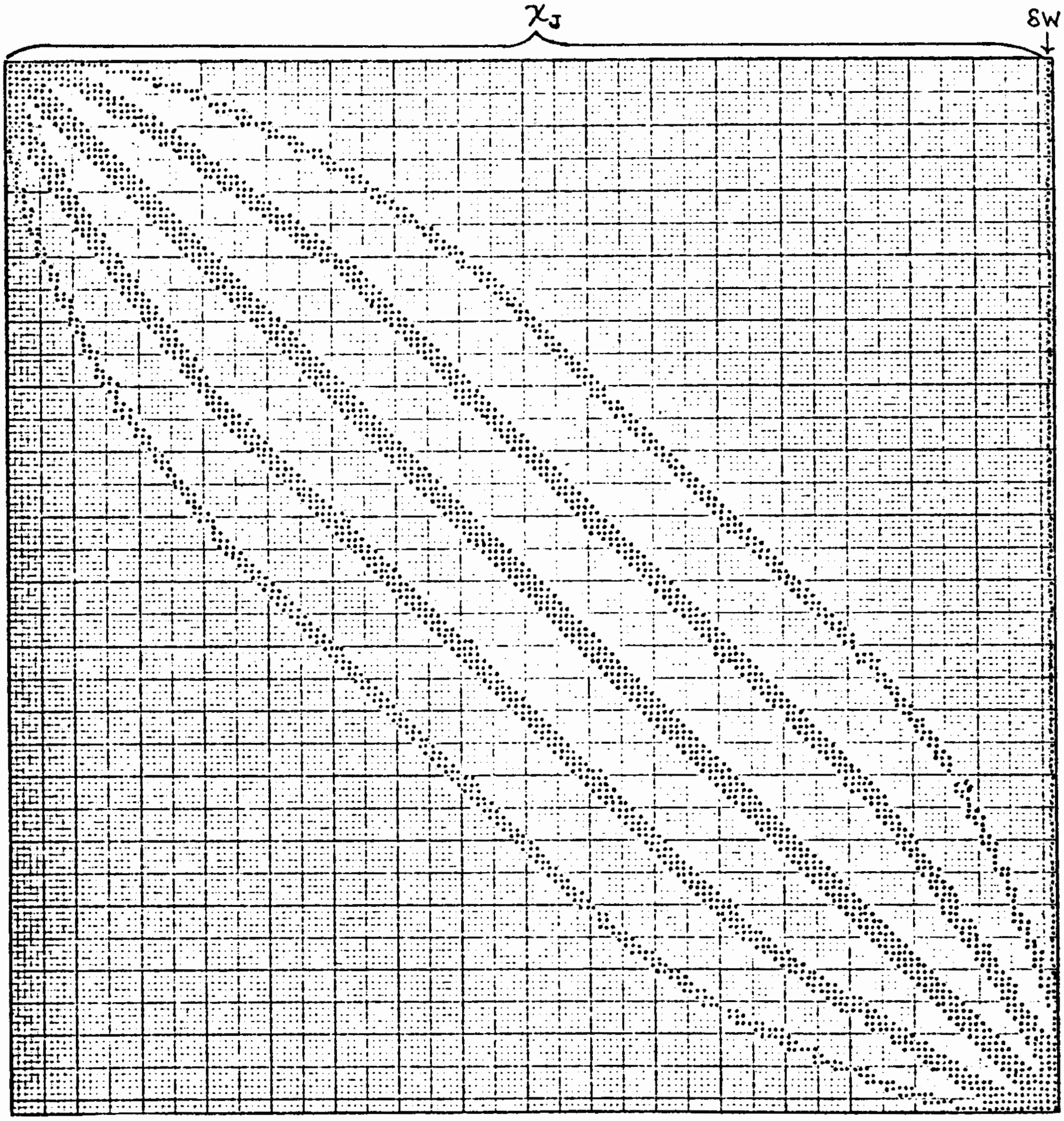


FIGURE 12

COMPUTER-GENERATED CONFIGURATION OF A_1 SUBMATRIX FOR A UNIFORM TRUNCATION ANGLE OF 10° ON 5° ZHONOGOLOVICH BLOCKS (ONE OBSERVATION PER BLOCK) that the altimetry and gravimetry observations are uncorrelated, although they may be correlated among themselves:

$$[70]\quad P = \begin{bmatrix} P_1 & O \\ O & P_2 \end{bmatrix}$$

Minimization of the loss function results in the following system of normal equations:

$$[71a]\quad P_1 V_1 + K_1 = 0$$

$$[71b]\quad P_2 V_2 + K_2 = 0$$

$$[71c]\quad A_1' K_1 + A_2' K_2 = 0$$

$$[71d]\quad A_1 X - V_1 = F_{b1}$$

$$[71e]\quad A_2 X - V_2 = F_{b2}$$

We can eliminate V_1 and V_2 from the system by solving for them in [71a] and [71b]

$$[72a]\quad V_1 = -P_1^{-1} K_1$$

$$[72b]\quad V_2 = -P_2^{-1} K_2$$

so that [71c - 71e] become

$$[73a]\quad A_1' K_1 + A_2' K_2 = 0$$

$$[73b]\quad A_1 X + P_1^{-1} K_1 = F_{b1}$$

$$[73c]\quad A_2 X + P_2^{1} K_2 = F_{b2}$$

Next solve for K_1 in [73b]

$$[74]\quad K_1 = P_1(F_{b1} - A_1 X)$$

Substituting [74] into [73a] and [73c] we obtain the final reduced system of normal equations:

$$[75a] \quad A_1' P_1 A_1 \; X - A_2' K_2 = -A_1' P_1 F_{b1}$$

$$[75b] \quad A_2 X + P_2^{-1} K_2 = F_{b2}$$

In partitioned matrix form [75] may be written

$$[76] \quad \begin{bmatrix} A_1' P_1 A_1 & A_2' \\ A_2 & -P_2^{-1} \end{bmatrix} \begin{bmatrix} X \\ -K_2 \end{bmatrix} = \begin{bmatrix} -A_1 P_1 F_{b1} \\ F_{b2} \end{bmatrix}$$

This may be written symbolically as

$$[77] \quad \begin{bmatrix} N_{11} & N_{12} \\ N_{12}' & N_{22} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

There are two possible forms for the solution, depending on whether N_{11} or N_{22} is inverted. The solution which inverts N_{11} is given by (Uotila, 1967)

$$[78] \quad \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} (N_{11}^{-1} + N_{11}^{-1} N_{12} Q^{-1} N_{12}^{1} N_{11}^{-1}) & (-N_{11}^{-1} N_{12} Q^{-1}) \\ (-Q^{-1} N_{12}' N_{11}^{-1}) & (Q^{-1}) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

where

$$[79] \quad Q = (N_{22} - N_{12}' N_{11}^{-1} N_{12})$$

The nature of Sequential Adjustment is to calculate an initial vector of parameter estimates Z^* based only on the first set of observations (F_{b1}) and then cal ulate a correction vector ΔZ which incorporates the effect of the second set of observations (F_{b2}). Quite obviously from [77]

$$[80] \quad Z_1^* = N_{11}^{-1} U_1$$

so that the correction vector is given by

$$[81] \quad \begin{bmatrix} \Delta Z_1 \\ \Delta Z_2 \end{bmatrix} = \begin{bmatrix} (N_{11}^{-1} N_{12} Q^{-1} N_{12}' N_{11}^{-1}) & (-N_{11}^{-1} N_{12} Q^{-1}) \\ (-Q^{-1} N_{12}' N_{11}^{-1}) & (Q^{-1}) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

which yields

$$[82] \quad \Delta Z_1 = (N_{11}^{-1} N_{12} Q^{-1} N_{12}' N_{11}^{-1}) U_1 - (N_{11}^{-1} N_{12} Q^{-1}) U_2$$

In the original notation

$$[83] \quad \Delta X = (A_1' P_1 A_1)^{-1} A_2' [P_2^{-1} + A_2 (A_1' P_1 A_1)^{-1} A_2']^{-1} A_2 (A_1' P_1 A_1)^{-1} A_1' P_1 F_1 + (A_1' P_1 A_1)^{-1} A_2' [P_2^{-1} + A_2 (A_1' P_1 A_1)^{-1} A_2']^{-1} F_{b2}$$

The total solution is then

$$[84] \quad X = X^* + \Delta X = -(A_1' P_1 A_1)^{-1} A_1' P_1 F_{b1} + \Delta X$$

Examination of [83] and [84] reveals that two matrix inversions are necessary (aside from the usually diagonal weight matrix):

$$[85a] \quad (A_1' P_1 A_1)$$

$$[85b] \quad [P_2^{-1} + A_2 (A_1' P_1 A_1)^{-1} A_2']$$

Matrix [85a] had dimensions equal to the number of unknowns (1641 x 1641) which is quite large even by the standards of modern computers. Figure 13 gives a computer-simulated configuration of $A_1' P_1 A_1$ which shows the banded-bordered nature of the matrix. The bandwidth of $A_1' P_1 A_1$ is exactly twice that of A_1*. In the present case (10° caps on 5° Zhonogolovich blocks) the bandwidth is about 800. Inversion routines for banded-bordered matrices exist and are quite capable of performing this inversion in a few minutes on a large, fast computer

* Bandwidth (p) is defined as that integer for which $a_{ij} = 0$ if $|i - j| > p$. Border-width (q) is simply the width (in columns or rows) of the borders.

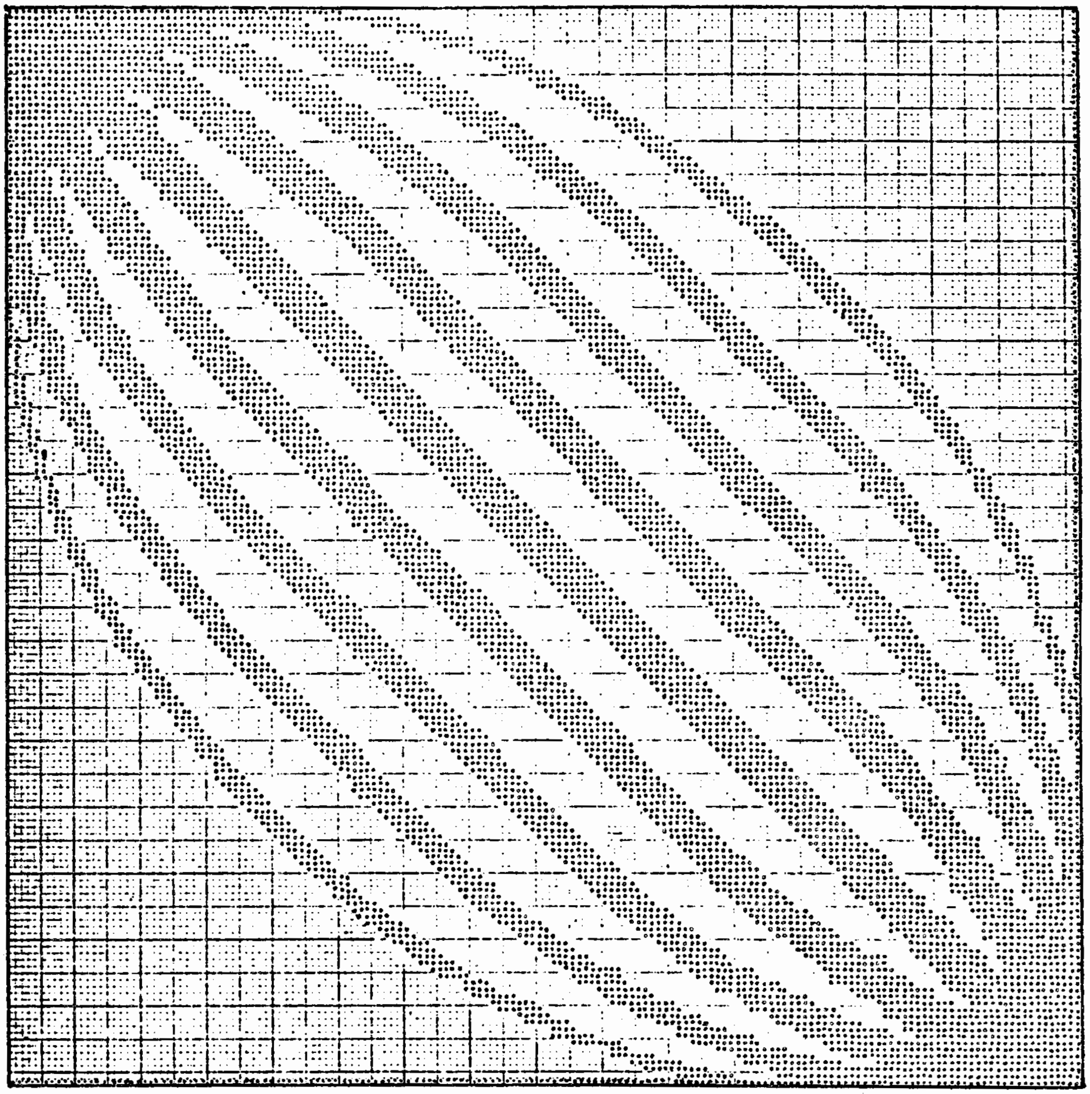


FIGURE 13

Computer-Generated Configuration of $A_1P_1A_1$ Submatrix for a Uniform Truncation Angle of 10° on 5° Zhonogolovich Blocks (One Observation Per Block) such as the CDC 6600.

Matrix [85b] has a dimension equal to the number of altimetry observations in the system. It is nearly full due to the generally large altimetry truncation angles.

The adjustment method described above is limited insofar as the number of altimetry observations it can accomodate; this limit is determined by the largest inversion feasible. Thus if the inversion of a 1000 x 1000 matrix is deemed feasible, 1000 altimetry observations may be incorporated into the global adjustment. Since we are limited as to the total number, these observations should be chosen with care and should be as uniformly distributed as possible for a strong geometry. The number of gravimetry observations which the system can handle is, on the other hand, unlimited; the dimension of $A_1^{\prime}P_1A_1$ remains constant.

16. Non-Global Solution For 1° Densities

The global solution for densities of 5° blocks may be made to serve as a framework of "superior control" into which the 1° densities may be fitted piecemeal using a non-global method of solution. Such "densification" may be performed in some locations, but not in others. The method suggested here is flexible in that an arbitrary area may be densified. The only requirement is that there be sufficient additional information within and in the vincinity of that area.

16.1 Mathematical Model

Equations 32 and 33 are again the basis for our mathematical model:

$$[32]\quad \zeta(\xi) = A(\xi)\delta W + \sum_{J=1}^{N} C_{\zeta}(\xi,\eta,J)\,X_J$$

$$[33]\quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X_I + \sum_{J=1}^{N} C_g(\xi,\eta,J)\,X_J$$

Equations 62 and 63 resulted when [32] and [33] were modified by splitting off the remote zone contribution to the summation and computing this contribution explicitly using spherical harmonics; the contribution was then treated as an "observation" by transferring it to the left-hand side of the equal sign. The global solution was then implemented using 5° mean densities inside the caps.

16.2 Modification Of The Model In Remote Zones

For the 1° solution, we again modify [32] and [33] by splitting off the RZC and treating it as an "observation." The inner zones will be represented by 1° mean densities in a completely analagous manner. However the RZC will now be computed explicitly using moving averages over 5° blocks, the values of which were determined in the global solution. This process is summarized in the following set of equations:

$$[86] \quad \zeta(\xi) = A(\xi)\delta W + \sum_{j \in C_{32}} C_\zeta(\xi,\eta,j)X_j + \sum_{J \in S-C_{32}} C_\zeta(\xi,\eta,J)X_J$$

$$[87] \quad \Delta g(\xi) = B(\xi)\delta W + D(\xi)X_1 + \sum_{j \in C_{33}} C_g(\xi,\eta,j)X_j + \sum_{J \in S-C_{33}} C_g(\xi,\eta,J)X_J$$

where capital J denotes summation over elements of the 5° Zhonogolovich scheme and small j denotes summation over the 1° scheme. Defining the "observations" as:

$$[88] \quad \zeta''(\xi) = \zeta(\xi) - A(\xi)\delta W - \sum_{J \in S-C_{32}} C_\zeta(\xi,\eta,J)X_J$$

$$[89] \quad \Delta g''(\xi) = \Delta g(\xi) - B(\xi)\delta W - \sum_{J \in S-C_{33}} C_g(\xi,\eta,J)X_J$$

then the mathematical model, modified for remote zones, may be written

$$[90] \quad \zeta''(\xi) = \sum_{j \in C_{32}} C_\zeta(\xi,\eta,j)X_j$$

$$[91] \quad \Delta g''(\xi) = D(\xi)X_1 + \sum_{j \in C_{33}} C_g(\xi,\eta,j)X_j$$

The designation "remote zones" is again applied to the area outside of a spherical cap of half-opening angle ψ_0. The half-opening angle of this cap must again be determined by an error study. Since we are working with the same two integral formulas, the derivations on pp. 33 - 38 are also applicable in the present case, except for the eigenvalues β_n of the smoothing operator which are given by [B.3]. Modifying [57] and [52] in accordance with [B.3] yields

$$[92] \quad \sigma^2_{RZC_{32}} = \frac{R^2}{8\bar{\gamma}^2} \sum_{n=5}^{\infty} \left(\frac{2n+1}{n-1}\right)^2 b_n(t_o)^2 [1 - d_n(\cos\alpha_o)]^2 c_n$$

$$[93] \quad \sigma^2_{RZC_{33}} = \frac{9}{32} \sum_{n=5}^{\infty} \left(\frac{2n+1}{n-1}\right)^2 b_n(t_o)^2 [1 - d_n(\cos\alpha_o)]^2 c_n$$

It is important to understand the precise meaning of the variances given in [92] and [93]. These are the RZC variances assuming that the 5° density values in remote zones are true block means. The X_J determined in the global solution were, however, not true means but have variances of their own (which information is contained in the inverse matrix of the normal equations). Hence the RZC variances given in [92] and [93] each need a second term expressing the propagation error. Unfortunately this problem was discovered too late to derive and test the necessary equations. Hence no quantitative results are available to illustrate typical truncation angles for [88] and [89]. Preliminary tests utilizing [92] and [93] indicate that the altimetry equations again requires a much greater truncation angle. In the following discussion, the symbols ψ_{32} and ψ_{33} will be used to denote the truncation angles for [90] and [91], respectively.

16.3 Observation Equations

What follows is merely a suggestion as to how the non-global solution might be handled. Further investigation will doubtless determine better methods, but the basic principle should remain the same; only the densities within a given area are considered unknown in any one solution.

Suppose that we want to determine the 25-1° densities within block I of the 5° Zhonogolovich scheme (heavily outlined in Figure 15). Suppose further that the area within a radius of ($\psi_{32} + 2\frac{1}{2}°$) of the center of block I contains (as a minimum condition) mean observations of either ζ or Δg for every 1° block in the area.

For each observation within the boundary of block I, an observation equation of type [90] or [91] may be written. In the most general case (observations of both types within block I), a circle of radius ($\psi_{32} + 2\frac{1}{2}°$) from the center of I will form a locus of inner zones such that no observation equation for any observation within I will have a cap which extends beyond this boundary.

Consider all 1° densities within this cap as unknowns X_j. Consider all densities outside of the cap as known (from the global solution). There will be approximately

$$[94] \quad \frac{41000}{2} \left[1 - \cos \left(\psi_{32} + 2\tfrac{1}{2}^\circ \right) \right]$$

unknowns inside the cap. If a particular area contains as especially large amount of observational date, the size of the observational unit may be decreased to 30' or even 5' blocks, increasing the degrees of freedom of the system.

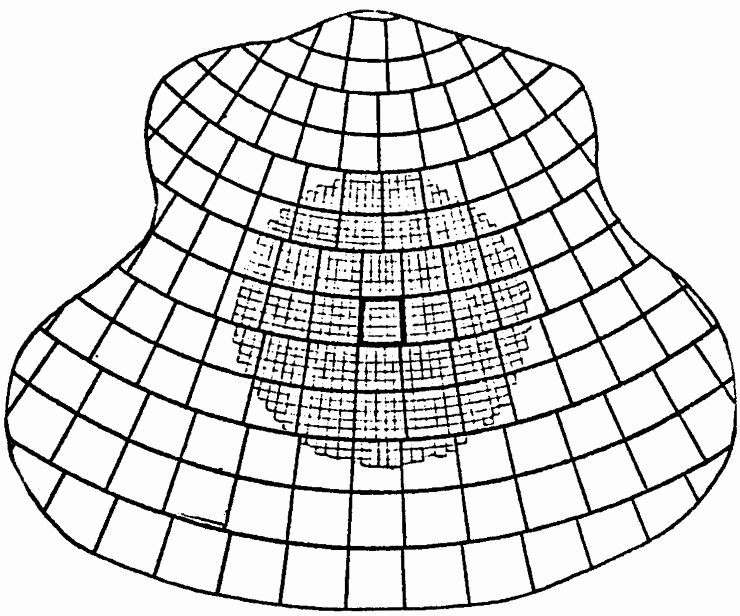


FIGURE 15

Solution of the above system will yield values for all densities within the cap; only the 25 X_j in block I should be retained. The reason for this is as follows: consider a block on the fringe of the cap. An altimetry or gravimetry observation in this block generates its <u>own</u> cap of radius ψ_{32} or ψ_{33}. A large portion of the latter cap will fall outside of the $(\psi_{32} + 2\frac{1}{2}^\circ)$ radius cap, resulting in a fairly large model error in [90] or [91]. The influence of fringe densities on the central blocks is, however, minimal. As we consider blocks closer and closer to block I, their model error decreases as their influence on the central blocks increases. It therefore stands to reason that the quality of the solution is inversely proportional to the distance from block I. More study needs to be done in this area.

17. Recovery Of Useful Information From Surface Densities

The solution of mean surface densities for 1° squares is not an end in itself. In first adopting the surface density parameterization, we traded physical significance for mathematical convenience. Now we are faced with the problem of converting the X_j back into physically meaningful information. Possibilities include: densities to point gravity anomalies; densities to point height anomalies; densities to potential coefficients; densities to mean anomalies. A brief discussion of each of these types of conversion follows.

17.1 Densities To Point Gravity Anomalies

A gravity anomaly could be computed at some point in an ocean area as follows: define a small area 1-2° in radius surrounding the computation point. For example, we could choose the 1° block containing the computation point as well as all contiguous degree blocks. From the discrete altimetry observations in this area, compute an inner zone contribution Δg_I using the explicit zero-order solution of [27] (Molodensky et al, p. 50):

$$[95] \quad \Delta g_I(\xi) = -\frac{\bar{\gamma}\zeta(\xi)}{R} - \frac{\bar{\gamma}R^2}{2\pi}\iint_I \frac{\zeta(\eta) - \zeta(\xi)}{\ell^3(\xi \cdot \eta)}\, d\omega(\eta)$$

The zero-order solution is justified over the oceans since there is no topography and hence the classical formulas hold. The spherical approximation of the reference surface (see [27]) is justified because the inner zone is such a small area.

For the remaining contribution, the Zhonogolovich blocks and their discrete density values may be used. From [33] we obtain

$$[96] \quad \Delta g_{S \cdot I}(\xi) = B(\xi)\delta W + D(\xi)X_i + \sum_{j \in S-I} C_g(\xi, \eta, j)X_j$$

The 1° Zhonogolovich blocks may be used out to a certain radius, $\psi = \psi_1$; 5° blocks may be used from $\psi = \psi_1$ to $\psi = \psi_2$; and 10° blocks may be used beyond $\psi = \psi_2$. Larger block means are obtained from the block means that fall solely within the larger block, as described previously. This whole procedure resembles the procedure used in evaluating Stokes' integral using mean anomaly blocks.

The anomalies computed in this fashion refer to the higher order reference surface (Section 4). In order to convert them to a reference ellipsoid (such as the International Ellipsoid) we must add the quantity

$$[97] \quad |\text{grad } U| - |\text{grad } U_I|$$

which is the inverse of the procedure described in Section 4.

17.2 Densities To Point Height Anomalies

A second problem frequently encountered would be that of computing a height anomaly ζ at some point on the land mass. Again we must define a small inner zone surrounding the computation point. From the discrete gravimetry observations within this area, compute an inner zone contribution ζ_I using the explicit second-order solution of [24] (Molodensky et al, pp. 122 - 123):

$$[98] \quad \zeta_I(\xi) = \frac{R}{4\pi\bar{\gamma}} \iint_I S(\xi\cdot\eta)[\Delta g(\eta) + G_1(\eta) + G_2(\eta)]d\omega(\eta) - \frac{1}{4\pi R\bar{\gamma}} \iint_I \frac{[h(\eta)-h(\xi)]^2}{\ell^3(\xi\cdot\eta)}\Delta g(\eta)d\omega(\eta)$$

where G_1 and G_2 are the first - and second - order correction terms, respectively

$$[99] \quad G_1(\xi) = \frac{1}{2\pi R} \iint_\omega \frac{h(\eta)-h(\xi)}{\ell^3(\xi\cdot\eta)} \Delta g(\eta)d\omega(\eta)$$

$$[100] \quad G_2(\xi) = \frac{1}{2\pi R} \iint_\omega \frac{h(\eta)-h(\xi)}{\ell^3(\xi\cdot\eta)} G_1(\eta)d\omega(\eta) + \frac{1}{R^2}\Delta g(\xi)\,|\nabla h(\xi)|^2$$

For mountainous terrain, the second-order solution is necessary; for average terrain, a first-order solution will suffice.

It should be noted that [95], [98], [99] and [100] can not be applied in the immediate vicinity of the computation point (the "neighborhood") because of numerical instability. They may, however, be transformed into workable forms by a process Meissl calls "regularization." This transformation, as well as the other preparation necessary prior to the actual evaluation of the inner zone contribution, may be found in Meissl (1971b).

For the remain contribution, the Zhonogolovich blocks are used. From[32] we obtain

$$[101] \quad \zeta_{S-I}(\xi) = A(\xi)\delta W + \sum_{j \in S-I} C_{\zeta}(\xi, \eta, j)\, X_j$$

Again 5° and 10° block means may be used as the distance from the computation point increases.

17.3 Densities To Potential Coefficients

The conversion of surface densities to potential coefficients is fully described by Koch (1968).

17.4 Densities To Mean Anomalies

Mean gravity anomalies for 1° blocks in ocean areas would definitely be desired. A simple way of obtaining these would be to compute a regular grid of point gravity anomalies using the method outlined in Section 17.1. Statistical methods could then be used to predict the mean anomalies. The latter procedure is outlined in Heiskanen and Moritz (Ch. 7).

The same method could be used to obtain mean height anomalies for 1° blocks in the continental areas.

18. Areas For Further Study

In this paper I have attempted to deal with a number of factors in the solution of a mixed boundary value problem in physical geodesy. I have suggested certain techniques which might be used. In particular I have followed closely the suggestion of Koch, who proposed the surface density parameterization of the anomalous potential.

This is not to ignore the many valid alternate proposals which have been made (for a brief survey of such proposals, see Chovitz, 1971). Each of these attacks the problem in a unique way and results in a unique mathematical formulation; all such proposals warrant careful study.

Within the scope of this paper, certain questions remain unanswered and therefore provide opportunity for further study. This section is devoted to the unanswered questions.

Probably the biggest criticism of this paper will concern the suggested global solution. What about a strictly non-global approach? One degree densities could be solved for at the outset by a method similar to the one suggested in Section 16; the 5° and 10° solutions would then follow by simple averaging of the smaller block means. The two approaches certainly need to be compared.

As far as the non-global solution is concerned, a deeper study of the "aliasing effect" alluded to at the end of Section 16 is certainly warranted. It should aim to determine the size of the cap radius for the unknowns, a task which was not completed in this paper.

Koch's proposal to represent integral equations by algebraic equations also raises some interesting questions. In the "forward" evaluation of an integral formula (such as Stokes' formula) we invariably use smaller and smaller block averages as we approach the computation point; this is necessitated by the fact that the kernal is approaching singularity. Yet in the "inverse" formulation of Koch, all squares are of constant size. One wonders how well these neighborhood densities are determined in the solution. Would it be better if we could somehow eliminate those few unknowns in the immediate vicinity of each computation point?

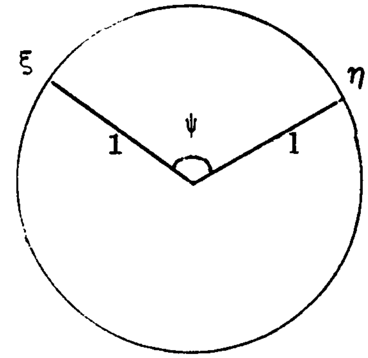
Situations exist where the results of the truncation error studies (Sections 11, 12, and 16.2) are falsified. Suppose that a reference surface of degree p is chosen, and that the coefficients are not errorless as assumed. How much error enters the RZC when we compute it explicitly using the harmonic coefficients? How much should the truncation angles be increased to compensate for this effect?

In the 5° global solution, an inverse matrix is obtained as a byproduct of the sequential adjustment. This inverse will be very helpful in assessing the accuracies of the 5° densities. There will be, however, objections to the calculation of an inverse based on reasons of economy. What equation solution methods can be efficiently applied to the system [62.62]? In the absence of an inverse, how can the accuracy of the densities be assessed?

Finally I mention the conversion of surface densities to useful information discussed in Section 17. Given the accuracy estimates for the surface densities, how do errors in the densities propagate to gravity anomalies? to height anomalies? to potential coefficients? How can we assess the accuracy of these computed quantities?

APPENDIX "A"

Consider the integral operator on the unit sphere ω which maps the function f into the function g:



$$[A.1] \quad g(\xi) = \iint_{\omega} K(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta)$$

The kernal (K) is said to be isotropic since it depends only on the cosine of the angle between ξ and η, denoted by the inner product $\xi \cdot \eta$.

The Funk-Hecke Theorem states that the eigenfunctions of such an operator are the surface spherical harmonics S_{nm} ($P_{nm} \cos m\lambda$ or $P_{nm} \sin m\lambda$). Thus

$$[A.2] \quad \iint_{\omega} K(\xi \circ \eta) S_{nm}(\eta) d\omega(\eta) = \lambda_n S_{nm}(\xi)$$

where the eigenvalues λ_n are given by the Funk-Hecke formula:

$$[A.3] \quad \lambda_n = 2\pi \int_{t=-1}^{+1} K(t) P_n(t) dt$$

Suppose that the functions f and g have the following spherical harmonic expansions:

$$[A.4a] \quad f(\xi) = \Sigma f_n(\xi) = \Sigma\Sigma f_{nm} S_{nm}(\xi)$$

$$[A.4b] \quad g(\xi) = \Sigma g_n(\xi) = \Sigma\Sigma g_{nm} S_{nm}(\xi)$$

Substituting these into [A.1] we obtain

$$[A.5] \quad \Sigma\Sigma g_{nm} S_{nm}(\xi) = \Sigma\Sigma f_{nm} \iint_{\omega} K(\xi \cdot \eta) S_{nm}(\eta) d\omega(\eta)$$

* Based on Meissl (1971b)

which, by the Funk-Hecke theorem, becomes

$$[A.6] \quad \sum\sum g_{nm} S_{nm}(\xi) = \sum\sum f_{nm} \lambda_n S_{nm}(\xi)$$

Equating the series term by term the following important result is obtained

$$[A.7] \quad g_{nm} = \lambda_n f_{nm}$$

Suppose that f is the given function and that f' is the "smoothed"version of f which is to be used instead of f in representing the effect of "remote zones" (the area outside of a circular cap "C" of half-opening angle ψ_0). For example, a finite number of terms in the spherical harmonic expansion of f might be used. Let us restrict f' to be the result of applying an isotropic smoothing operator Z:

$$[A.8] \quad f'(\xi) = \iint_{\omega} Z(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta)$$

By analogy to [A.7], the coefficients of the harmonic expansion of f' are

$$[A.9] \quad f'_{nm} = \beta_n f_{nm}$$

where β_n are the eigenvalues of the smoothing operator. Representation by a truncated harmonic series is evidently such an isotropic smoothing process with eigenvalues

$$[A.10] \quad \beta_n = \begin{cases} 1 & n \le p \\ 0 & n > p \end{cases}$$

where p is the highest order harmonic retained in the representation.

Let the true value of the integral be denoted by g where

$$[A.11] \quad g(\xi) = \iint_{\omega} K(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta)$$

Define the piecewise function $\tilde{K}$ as follows:

$$[A.12] \quad \tilde{K}(\xi \cdot \eta) = \begin{cases} K(\xi \cdot \eta) & \psi \le \psi_0 \\ O & \psi > \psi_0 \end{cases}$$

Denoting the truncated integral by $\tilde{g}$ we have

$$[A.13] \quad \tilde{g}(\xi) = \iint_{\omega} \tilde{K}(\xi \cdot \eta) f(\eta) d\omega(\eta)$$

The error committed by completely neglecting the "remote zones" is given by the difference between [A.11] and [A.13] which we denot by Δg:

$$[A.14] \quad \Delta g(\xi) = \iint_{\omega} K(\xi \cdot \eta) f(\eta) d\omega(\eta) - \iint_{\omega} \tilde{K}(\xi \cdot \eta) f(\eta) d\omega(\eta)$$

The "residual kernal" is defined as

$$[A.15] \quad \Delta K(\xi \cdot \eta) = K(\xi \cdot \eta) - \tilde{K}(\xi \cdot \eta) = \begin{cases} O & \psi \le \psi_0 \\ K(\xi \cdot \eta) & \psi > \psi_0 \end{cases}$$

Substituting this expression into [A.14] we obtain a single integral

$$[A.16] \quad \Delta g(\xi) = \iint_{\omega} \Delta K(\xi \cdot \eta) f(\eta) d\omega(\eta)$$

Now suppose that the smoothed version of f, namely f', is used to compute the error function Δg. The residual error in Δg, denoted as $\Delta g'$, would be given by the following difference:

$$[A.17] \quad \Delta g'(\xi) = \iint_{\omega} \Delta K(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta) - \iint_{\omega} \Delta K(\xi \cdot \eta) f'(\eta) d\omega(\eta)$$

Defining the function Δf as

$$[A.18]\quad \Delta f(\xi) = f(\eta) - f'(\eta)$$

and substituting into [A.17] we obtain

$$[A.19]\quad \Delta g'(\xi) = \iint_\omega \Delta K(\xi \cdot \eta) \Delta f(\eta) d\omega(\eta)$$

Equation A.19 has the same form as [A.1]. Hence by analogy to [A.7]:

$$[A.20]\quad \Delta g'_{nm} = \Delta\lambda_n \Delta f_{nm}$$

Substituting [A.18] and [A.9] into [A.20] we obtain the desired result:

$$[A.21]\quad \Delta g'_{nm} = \Delta\lambda_n (f_{nm} - f'_{nm}) = \Delta\lambda_n (1 - \beta_n) f_{nm}$$

It will now be demonstrated that $\Delta g'$ is equivalent to the error committed by using the exact function f in the "inner zones" and the smoothed function f' in "remote zones". Denoting this error by e...

e = (Exact integral) - (Exact integral over inner zones) - (Approximate integral over remote zones)

$$= \iint_\omega Kf \, d\omega - \iint_c Kf \, d\omega - \iint_{\omega - c} Kf' \, d\omega$$

$$= \iint_\omega (\tilde{K} + \Delta K)(f' + \Delta f) \, d\omega - \iint_\omega \tilde{K} f \, d\omega - \iint_\omega \Delta K f' \, d\omega$$

$$= \iint_\omega (\tilde{K} f + \Delta K f' + \Delta K \Delta f) \, d\omega - \iint_\omega \tilde{K} f \, d\omega - \iint_\omega \Delta K f' d\omega$$

$$= \iint_\omega \Delta K \Delta f \, d\omega = \Delta g'$$

APPENDIX "B"

EIGENVALUES OF THE BLOCK-AVERAGING-TYPE SMOOTHING OPERATOR*

In this section we determine the eigenvalues β_n of the smoothing operator Z which maps the function f into the function f', where f' is the block-average representation of f. Such an operator is not isotropic, but may be approximated (for the purposes of error analysis) by the operator $\bar{Z}$ which maps f into $\bar{f}'$, where $\bar{f}'$ is the average value of f over circular caps of half-opening angle α_o. The eigenvalues β_n of this operator are:

$$[B.1] \quad \beta_n = \frac{1}{1-\cos\alpha_o} \int_{t=\cos\alpha_o}^{1} P_n(t)dt$$

Utilizing the identity

$$[B.2] \quad \int_{t=\cos\alpha_o}^{1} P_n(t)\,dt = \frac{1}{2n+1}\left[P_{n-1}(\cos\alpha_o) - P_{n+1}(\cos\alpha_o)\right]$$

we obtain a final expression for the eigenvalues β_n

$$[B.3] \quad \beta_n = \frac{P_{n-1}(\cos\alpha_o)-P_{n+1}(\cos\alpha_o)}{(2n+1)(1-\cos\alpha_o)} = d_n(\cos\alpha_o)$$

where the shorthand notation $d_n(\cos\alpha_o)$ is introduced.

* Based on Meissl (1971a and 1971b)

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13. ABSTRACT

The utilization of satellite altimetry by itself, and in combination with existing gravity material is considered for the determination of the gravity field of the earth. This is done by developing equations that relate surface density values defined in discrete blocks to geoid undulations and gravity anomalies. The use of a higher order reference field defined by a set of spherical harmonics is considered and truncation errors are computed when the contribution of an area outside a spherical cap is obtained from a spherical harmonic expansion of the anomaly field. A suggested solution to recover 5° equal area blocks is made with specific recommendations made on the ordering of these blocks so that structured sets of normal equations will result. The determination of a more local field (such as 1°) is discussed using the global 5° field as a basis.

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