Using Warm-Ups to Support and Develop Mathematical Ideas

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In the course of a school year teachers are faced with a full calendar of mathematics to cover. Every minute matters. Engaging students in mathematical activity as soon as they enter the door increases the number of minutes they spend studying mathematics in a year. Warm-ups can be used to review and practice skills. They can be used to build new ideas and forms of mathematical thinking. This paper shares three different sets of warm-ups that can be used with middle school students to encourage different types of mathematical thinking and reasoning. *This article was supported by the* Middle Grades Mathematics Initiative *project funded by* The Ohio Board of Regents Improving Teacher Quality Grant *No. 08-39.*

Introduction

In our work supporting middle grades mathematics teachers, the first two authors are often asked for suggestions on how to cover everything that needs to be covered in a school year and how to incorporate new ideas into an already tightly packed timeline. As part of professional development work with middle grades mathematics teachers, the idea of using warm-ups emerged. In this article we begin by sharing two types of warm-up tasks the project teachers developed to support reasoning related to fractions and to relational reasoning. In addition, the third author, one of the seventh-grade teachers participating in the professional development project, will share a particular warm-up structure she developed and implemented to build up her students' capacity to reason about functional relationships.

The warm-up concept involves using a single problem or a small collection of problems that students can quickly complete upon entering the classroom. Students can complete warm-ups during those few minutes when the teacher returns papers, takes attendance, and so forth. Since the project teachers had a relatively short class period, roughly 45 minutes for most, the warm-up activities were structured to encourage mathematical reasoning yet be quick for students to complete, and to have students share the way they reasoned about the problem (or set of problems) before moving into the daily lesson. Warm-ups don't have to be used everyday, but if incorporated a few days each week across the school year, the potential to support students' mathematical skill and reasoning can be greatly enhanced.

Furthering Students' Understanding of Skills and Concepts

The mathematics curriculum that project teachers were implementing formally taught fractions in sixth-grade. While fractions were revisited in subsequent units across sixth-, seventh-, and eighth-grades they were not formally re-taught in seventh- or eighth-grade. The seventh- and eighth-grade teachers were especially interested in being able to revisit fractions regularly throughout seventh and eighth grades, but did not want to have to teach a formal unit on fractions in these grades.

Fraction operations were one area of concern. Once algorithms were established for operating with fractions, providing practice aimed at developing procedural fluency was something that a quick three- to four-question warm-up could support. For example, giving students one addition, one subtraction, one multiplication, and one division to solve as a warm-up, and then reviewing how to get the answer, can be completed in short order.

While procedures are important, conceptual development and ways of thinking are also important areas to support. During several project workshops the teacher participants explored various types of reasoning associated with fractions. Because flexible reasoning with rational numbers develops across a period of time, not in a single lesson nor even an instructional unit (Lamon, 2005), providing students with ongoing opportunities to build up ways of reasoning can be served through the use of warm-ups. Drawing from the work of Lamon (2005) and Greenwood (1986), the teachers created a collection of fraction warm-ups to use throughout the year.

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A sample of the warm-ups, illustrated in Figure 1, highlight some of the different forms of reasoning the project teachers wanted to encourage. For example, there are problems where students are given a part-of-a-whole and asked to determine what the whole will be or look like. Others ask students to use fractions as an operator (i.e.: What is 2/3 of 12?). With each warm-up, it is assumed that students will explain their reasoning as part of the follow-up discussion.

Developing Relational Reasoning

A second focus of the professional development project was relational reasoning. Molina, Castro, and Ambrose (2005) describe relational reasoning in the following way:

> We say that a person thinks relationally or uses relational thinking when he/she examines two or more mathematical ideas or objects alternatively looking for connections between them and, analyzes or uses those relationships in order to solve a problem, make a decision, or learn more about the situation or concepts involved. (p. 3)

Carpenter, Franke, and Levi (2003) argue that relational reasoning can strengthen number sense, understanding of variable, meaning of the equal sign, and basic arithmetic operations. Three types of warm-ups are shown in Figure 2. In the discussion of these warm-ups we

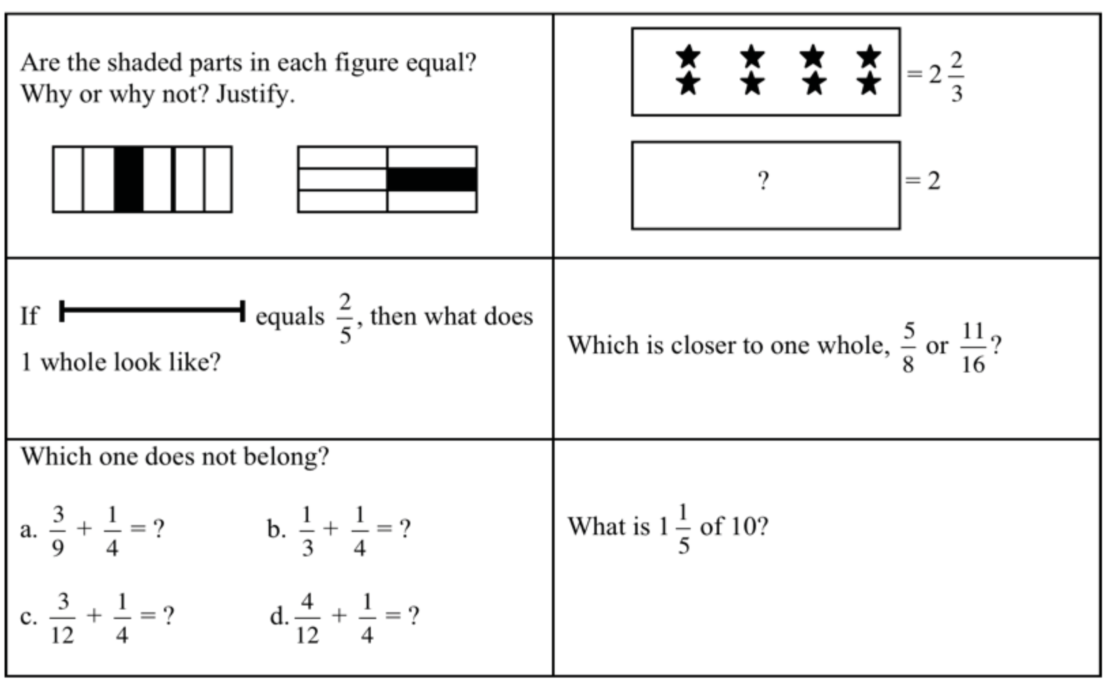

Fig 1 Sample fraction warm-up

True/False Number Sentences	Open Number Sentences	Identities
1. 26 + 35 = 28 + 33	**6.** 11 × 8 = ___ × 8 + 8	**11.** ___ × 1 = ___
2. 43 – 26 = 44 – 25	**7.** 3 + 3 – 345 = 3 × 2 – N	**12.** ___ + 0 = ___
3. 6 × 7 = 5 × 7 + 7	**8.** 19 × 4 = 20 × 4 – ____	**13.** N ÷ 1 = N × 1
4. 5 × 68 = 10 × 34	**9.** 4.6 × 2.3 ÷ ___ = 4.6	**14.** 2 × ___ = ___ + ___
5. 86 ÷ 12 = 43 ÷ 24	**10.** 4.6 + 2.3 – ___ = 4.6	**15.** 3 × N = N + N + N

Fig 2 Sample warm-ups for building relational reasoning

illustrate how relational thinking involves using fundamental properties of number and operations in place of direct calculation.

When using relational reasoning with Problem 1 (26+35=28+33) in Figure 2, one could argue the number sentence is true by comparing 26 and 28. As one goes from 26 to 28 there is an increase of two. However, when looking at the relationship moving from 35 to 33, there is a decrease in two, making the sum on each side of the equal sign equivalent. A second perspective involves rewriting the original problem as 26+2+33=28+33 to highlight the two being shifted. Similarly, rewriting Problem 4 (5×68=10×34) as 5×2×34=10×34 will provide justification that the number sentence is true. With Problem 4 one could also reason that if you double 5 (how many times or how many groups you are making) to 10, then you need to cut the size of the groups in half, from 68 to 34.

Problem 5 can be justified using relationships among quantities instead of direct calculation. By taking into account that division can be interpreted as how many times a group can go into a quantity, a lot more groups of 12 can "go into" 86 compared to how many groups of 24 can go into 43. Another justification draws on the interpretation of a fraction as an indicated division and then applying fraction/ratio equivalence. When 86 ÷ 12 and 43 ÷ 24 is rewritten as $\frac{86}{12} = \frac{43}{24}$, it is easier to see that the two fractions (or ratios) are not equivalent making the number sentence false.

Warm-up Problems 6 through 8 support multiplicative thinking and the meaning of multiplication as repeated addition. Presenting Problem 9 and Problem 10 together in the same warm-up can encourage understanding of inverse relationships as well as the identity properties. Warm-up Problems 11 through 16 are a special group of open-number sentences called identities. Unlike Problems 6 through 10 where there is only one solution for the unknown, identities have infinite solutions. Problem 11 is true regardless of the value is used to represent the unknown.

Once students have had experience with the different problem structures highlighted in Figure 2, asking students to write open-number and true-false sentences to pose to classmates also supports the development of their relational reasoning. If students are discouraged from using direct computation, and asked to talk about how the quantities in the problem are related, warm-ups such as these can support the ability to reason relationally. Relational reasoning develops across time through multiple experiences. Teaching a lesson on relational reasoning will not be effective in the long run.

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Developing Functional Thinking with Visual Patterns

Like work with fractions and relational reasoning, functional thinking can also be built up across experiences using pictorial growth or geometric patterns. When exploring geometric patterns, students analyze and describe how the figures in the pattern grow visually. The visual patterns can be translated into a numeric sequence typically represented in a t-chart. By examining the pattern structure and the numeric sequence, a rule can be generated to represent the functional relationship symbolically. By using pictorial growth patterns as warm-ups throughout the school year students learned to visually analyze how patterns grow and connect this to the numeric sequence represented in the t-chart. Connections between the rule, the numeric pattern in the t-chart, and the visual dot patterns were emphasized. The warm-ups were designed to strengthen students' pattern generalization using something concrete or visual before they shifted to a more number or symbol-driven approach when studying functions in eighth grade.

There were two related warm-ups given during most weeks of the school year—often on Tuesday and Thursday (see Figure 3). The structure of the Tuesday warm-ups used in the first half of the year focused on drawing a provided sequence of figures drawn with dots and then drawing, and describing, what the fifth dot figure would look like. The use of terms like figure number, number of dots, rows, columns, horizontal, and vertical were emphasized when trying to describe the figures. The provided scenario was that students were describing the structure of the dot patterns to someone on the phone and they had to be specific enough for the person listening to visualize the figures in the pattern. On Thursdays students were shown the same four figures and asked to complete a t-chart for various values and a rule for generating the number of dots for any figure number in the sequence. Here the focus was on trying to understand how the functional rule for finding any number in the pattern was related to how students saw the dot patterns grow

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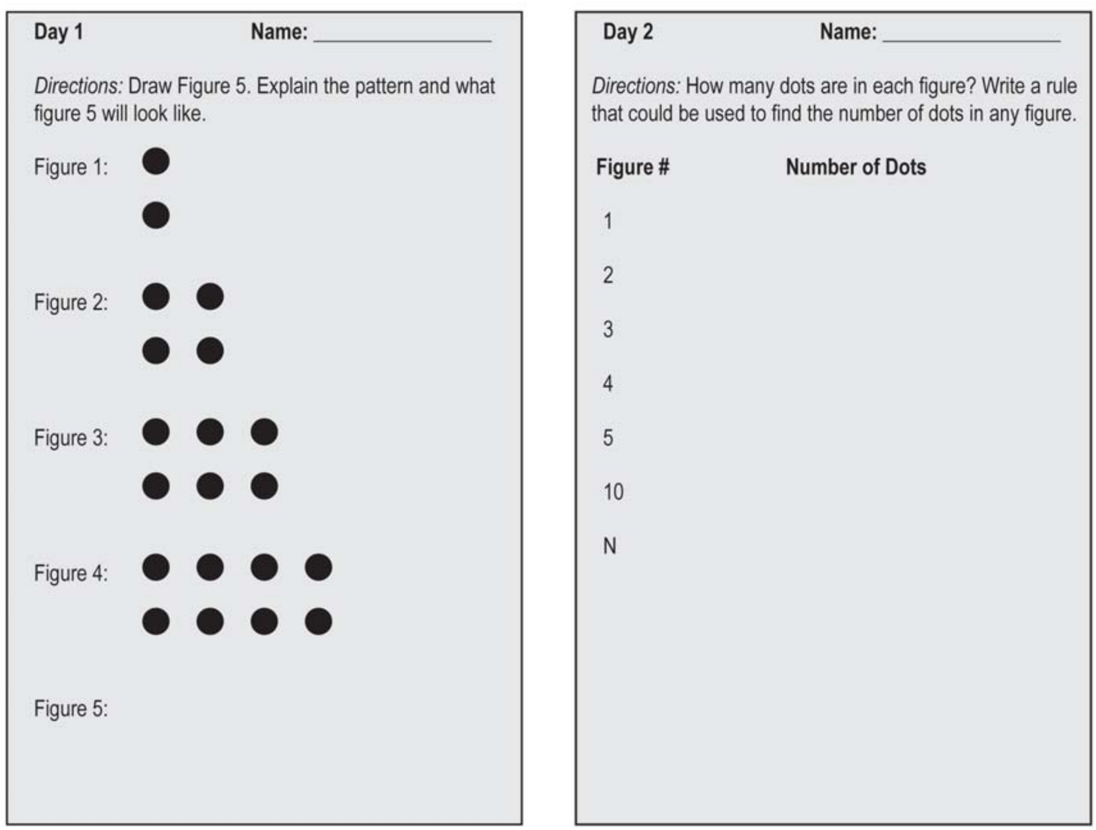

Fig 3 Example warm-up to support functional thinking

and change. An important question asked throughout the year was "What does the figure number have to do with the number of dots in the figure?"

The warm-up in Figure 3 was given early in the year. During the Tuesday warm-up discussion, students described how they visualized each dot pattern in the sequence. For example, the fifth dot pattern has 5 sets of two dots. Each set is arranged in sets of two. Each set of two is stacked horizontally, one on top of the other. There are 5 of these horizontal stacks of two in the fifth dot pattern. One seventh-grader described the dot patterns in Figure 3 as increasing by two each time.

On Thursday, discussions focused on completing the t-chart and describing a rule for finding the number of dots for any figure in the sequence. Early in the year students would offer how many dots to add when moving from one dot pattern in the sequence to the next. For the problem in Figure 3, Mrs. LaCourse pointed out that adding two was not wrong, but would be a lot of work if you wanted to find the 100th figure. Finding a connection between the figure number and the number of dots was encouraged and emphasized. When asked if there was a way to find the number of dots for any figure

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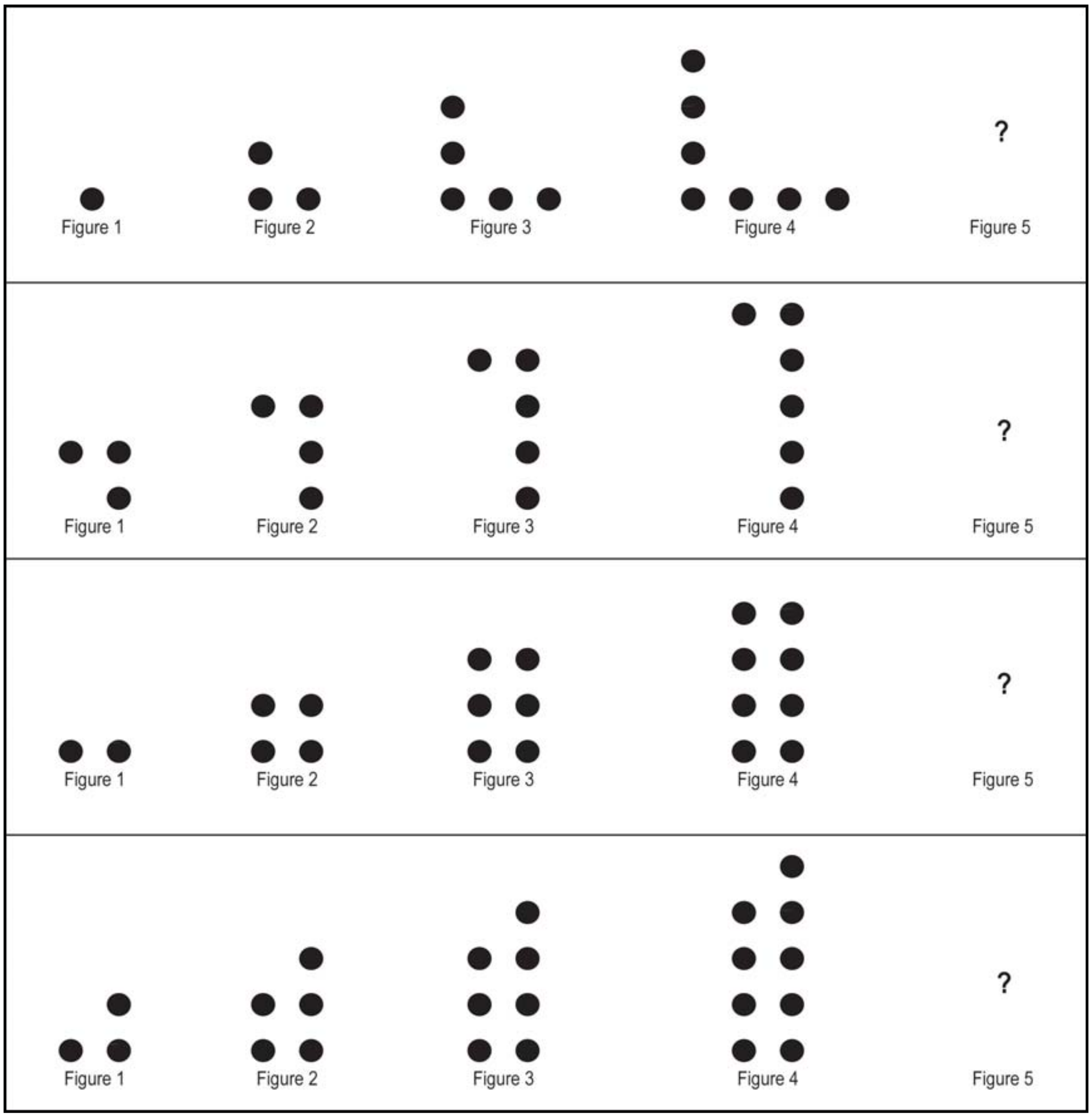

Fig 4 Other figurate patterns

without counting each dot or continually adding two each time, one seventh-grade student offered a "times 2" pattern. In the first dot pattern there is one group of two. In the second dot pattern there are two sets of two or a 2-by-2. In the third dot pattern there is a 2-by-3, or three sets of two. The fourth dot pattern in the sequence is a 2-by-4, or four sets of two (etc.). The dot patterns in Figure 3 uses an array structure which helps to support the "times 2" rule. For any figure, the first, second, third…tenth…if you take the figure number and multiply by two you will be able to determine the number of dots for any figure in the sequence.

Figure 4 includes examples of some of the other figurate pattern structures that were used. Sometimes the pattern structures were related. The two patterns in the bottom row of Figure 4 are similar. The pattern on the left is a "$2N$" pattern with the dots in a different position than those in Figure 3. The pattern on the bottom right is $2N + 1$. It is the same pattern as the one on the left but with one extra dot added to the top of pairs of two that grow or stack up vertically.

Later in the year, different warm-up formats were used to develop and/or strengthen the connection between the visual patterns, the rule, and the graph. For example, in Figure 5a, the dots were arranged in long horizontal rows. On Tuesday the students determined that the fifth dot pattern would have 20 dots and that the rule is "4 times the figure number." On Thursday, students were asked how they could rearrange the dots to better see the rule "4 times N" in the dot patterns. In Figure 5b, students were given this t-chart and asked to determine the rule ($4N – 1$) during the Tuesday warm-up. On Thursday students were asked to create a dot pattern sequence that would go with this rule. During Thursday's discussion they were asked questions such as (1) How can you make any figure? (2) What does the 4 tell you? The N tell you? The "– 1" tell you? (3) If the rule was $4N$ what would that tell you? and (4) How would the dot pattern for $4N$ and for $4N – 1$ be similar and different?

As students became more skilled at extending dot patterns and completing the table with a rule, these were combined and

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5a. *How could the dots be arranged to better show the pattern described by the rule?*

5b. *Draw figures 1 through 3 for this chart.*

Figure #	Number of Dots
1	3
2	7
3	11
4	15
10	39
N	?

How does the rule 4N – 1 help you make Figure 1? Figure 2? Figure 3? Figure 4? Figure 10? Any figure?

Fig 5 Additional warm-up formats for figurate patterns presented as the Tuesday warm-up. On Thursday, students were asked to make a graph for the dot pattern. Initially, all of the patterns were linear. Later students were given a several non-linear patterns, including X2 and X3, and to draw dot patterns for, complete a t-chart, write a rule, and graph. These discussions focused on why the patterns did not result in a straight line.

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Conclusion

Using warm-ups such as these throughout the school year can support learning in numerous ways. Warm-ups provide a way to engage students when they first walk into your classroom. They can be used to review and practice skills. Warm-ups can serve a vehicle for building new ideas. Ways of thinking can be developed across time and not take time away from an already full pacing guide. An important feature assumed in the presentation of each of these three warm-ups is that the students were asked to share their reasoning and communicate their ideas. When skills are revisited and ways of thinking are shared across time, students benefit from hearing the ideas of others. For many students, revisiting ideas across time is more beneficial and in the long run students may take more away than when only one or two lessons are taught.

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