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Report No. 183

CONVERGENCE OF MOLODENSKY'S SERIES

by
Helmut Moritz

Prepared for
Air Force Cambridge Research Laboratories
Air Force Systems Command
United States Air Force
Bedford, Massachusetts 01730

Contract No. F19628-72-C-0120
Project No. 8607
Task No. 8607-01
Work Unit No. 86070101

Scientific Report No. 4

Contract Monitor: Bela Szabo
Terrestrial Sciences Laboratory

The Ohio State University
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ABSTRACT

Molodensky's solution of the geodetic boundary-value problem consists in an asymptotic series expansion with respect to a parameter k. It is shown that this series converges for sufficiently small values of k. The method starts from an integral equation given by Brovar and uses a Neumann series solution of this equation; the norms of the occurring singular integral operators are suitably estimated.

FOREWORD

This report was prepared by Helmut Moritz, Professor, Technische Hochschule Graz, and Adjunct Professor, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. F19628-72-C-0120, OSURF Project No. 3368 A1, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Air Force Systems Command, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Bela Szabo, Project Scientist.

Convergence of Molodensky's Series

by

Helmut Moritz

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CONVERGENCE OF MOLODENSKY'S SERIES

1. Introduction

The best-known and most important solution of the problem of gravimetric determination of the physical surface of the earth has been given by Molodensky in 1960 (Molodenskii et al., 1962, chapter V, sec. 15). It consists in a formal series expansion with respect to a "small parameter" k. To my knowledge, the problem of convergence of this series has not yet been solved. It is the purpose of this report to show that Molodensky's series is indeed convergent for sufficiently small values of the parameter k.

Let us briefly recall how Molodensky's series is obtained. The anomalous potential T is represented as the potential of a single layer, of density φ, on the physical surface S of the earth:

$$T = \iint_S \frac{\varphi}{\ell} \, dS , \qquad (1\text{-}1)$$

where ℓ is the distance between the computation point P and the surface element dS. The function φ is related to the gravity anomaly Δg by the integral equation

$$2\pi \varphi \cos \beta - \iint_\sigma \left(\frac{3R}{2\ell} + \frac{R^2 (h - h_P)}{\ell^3} \right) \sec \beta \cdot \varphi \, d\sigma = \Delta g . \qquad (1\text{-}2)$$

The notations follow (Heiskanen and Moritz, 1967, p. 304): h is the topographic elevation, β is the terrain inclination, R is a mean radius of the earth, σ is the unit sphere (the full solid angle), and $d\sigma$ is the element of solid angle. To a sufficient approximation we have

$$\ell^2 = \ell_0^2 + (h - h_P)^2, \quad \ell_0 = 2R \sin \frac{\psi}{2} , \qquad (1\text{-}3)$$

where ψ is the angular distance between the computation point P and the variable point to which dS and $d\sigma$ correspond.

In order to solve this equation, Molodensky replaces all elevations h by kh, where k is a parameter with $0 \le k \le 1$. The geometric interpretation of this parameter k is as follows. The value $k = 1$ corresponds to the physical earth's surface, $k = 0$ corresponds to the reference surface which is formally a sphere (because of the use of the spherical approximation), and for $0 < k < 1$ we have a set of intermediary surfaces.

Then ℓ is to be replaced by ℓ_k with

$$\ell_k^2 = \ell_0^2 + k^2(h - h_P)^2 , \qquad (1\text{-}4)$$

so that

$$\frac{1}{\ell_k} = \frac{1}{\ell_0}\left[1 + k^2\left(\frac{h - h_P}{\ell_0}\right)^2\right]^{-\frac{1}{2}} , \qquad (1\text{-}5a)$$

$$\frac{1}{\ell_k^3} = \frac{1}{\ell_0^3}\left[1 + k^2\left(\frac{h - h_P}{\ell_0}\right)^2\right]^{-\frac{3}{2}} , \qquad (1\text{-}5b)$$

and the terrain inclination β is replaced by β_k with

$$\tan\beta_k = k\tan\beta , \qquad (1\text{-}6)$$

so that

$$\cos\beta_k = (1 + \tan^2\beta_k)^{-\frac{1}{2}} = (1 + k^2\tan^2\beta)^{-\frac{1}{2}} . \qquad (1\text{-}7)$$

The quantities (1-5a,b) and (1-7) are developed into binomial series which converge as long as

$$k\tan\beta_{max} < 1 \qquad (1\text{-}8)$$

(since also $(h - h_P)/\ell_0 < \tan\beta_{max}$), and substituted in (1-2). In the same way we expand the unknown function

$$\varphi = \sum_{n=0}^{\infty} k^n \varphi_n \qquad (1\text{-}9)$$

and substitute in (1-2). Now all these series in (1-2) are multiplied together, and the result is arranged with respect to equal powers of k. Finally the coefficient of every k^n is set equal to zero. This gives infinitely many equations from which the φ_n can be determined.

The series (1-9) obtained in this way and the series (1-5a) are substituted into (1-1), multiplied together and arranged with respect to equal powers of k. This gives the final result in the form

$$T = \sum_{n=0}^{\infty} k^n T_n \quad . \tag{1-10}$$

Expressions for T_n are found in (Molodenskii et al. 1962, chapter V, sec. 15) or also in (Moritz, 1969, sec. 5).

From the fact that the basic series (1-5a, b) and (1-7) converge under the condition (1-8) one has sometimes concluded that also the resulting series (1-10) will converge under this condition, that is (for k = 1), as long as the terrain inclination $\beta < 45°$. This conclusion is in no way valid, as pointed out in (Moritz, 1969, Appendix); this will also be confirmed by the present investigation.

Another series solution of Molodensky's problem of form (1-10) may be derived in a completely different way, using analytical continuation by means of Taylor series; this was done independently by Marych (1969a, b) and Moritz (1969). In (Moritz, 1970) and, more completely and rigorously, in (Moritz, 1971), it was shown that both the Molodensky series and the new series solution are asymptotic developments of the same function. The fact that two asymptotic series for the same function must be identical, proves the term-by-term identity of the two, apparently different, series (Moritz, 1969, sec. 5). A direct computational verification was given by Pellinen (1972).

Whereas the asymptotic nature of Molodensky's series was rigorously established, it could not be determined along these lines whether it was convergent or divergent. An attempt to link convergence of Molodensky's series to analytical continuation (Moritz, 1969, sec. 6) contained a logical error and was given up. In (Moritz, 1971, sec. 12) we have still considered it "most likely" that Molodensky's series diverges even for small k if analytical continuation is not regular; this view is no longer tenable in view of the results of the present report.

Convergence problems in physical geodesy seem to be a constant challenge to geodesists: interesting results and fallacies that are sometimes also interesting are produced in this way. It is hoped that the present investigations belong to the first category.

2. Convergence Criteria

Instead of Molodensky's original solution as outlined in the preceding section we consider a modification by Brovar (1963), which leads to a simpler integral equation; cf. (Moritz, 1966, sec. 6).

Here T is expressed in the form

$$T = \frac{1}{4\pi} \iint_S \lambda \left[S(r_P, \psi, r) - \frac{1}{r_P} \right] dS \, , \tag{2-1}$$

where $S(r_P, \psi, r)$ is a spatial generalization of Stokes' function; r_P and r are the radius vectors of the computation point P and of the surface element dS, respectively. The function λ, a kind of generalized surface density, is related to Δg by the integral equation

$$\lambda \cos\beta - \frac{R^2}{2\pi} \iint_\sigma \lambda \, \frac{h - h_P}{\ell^3} \sec\beta \, d\sigma = \Delta g; \tag{2-2}$$

the notations are the same as in the preceding section. This integral equation may again be solved by expansion into a series of powers of Molodensky's parameter k and the result substituted in (2-1). This gives an expansion for T of the form (1-10); the result must, of course, be identical to that obtained by solving Molodensky's original integral equation, outlined in the preceding section.

On introducing, in the place of λ, a new function μ by

$$\mu = \lambda \cos\beta \quad ,$$

equation (2-2) becomes

$$\mu - \frac{R^2}{2\pi} \iint_\sigma \mu \, \frac{h - h_P}{\ell^3} \, \frac{1}{\cos^2\beta} \, d\sigma = \Delta g \, . \tag{2-3}$$

This equation may be written symbolically as

$$(I - \Phi)\,\mu = \Delta g \, , \tag{2-4}$$

where I is the unit operator, defined by

$$If = f \, , \tag{2-5}$$

and Φ is the singular integral operator given by

$$\Phi f = \frac{R^2}{2\pi} \iint_\sigma f \, \frac{h - h_P}{\ell^3} \, \frac{1}{\cos^2\beta} \, d\sigma \tag{2-6}$$

for an arbitrary function f for which the singular integral exists.[1)]

The integral (2-6) is strongly singular since the kernel becomes infinite as $1/\ell^2$ if $\ell \to 0$. The integral must be understood in the sense of its Cauchy principal value, that is, a small circle $\psi \leq \epsilon$ around the computation point P is first excluded from the unit sphere σ, and the integral is defined as

$$\lim_{\epsilon \to 0} \iint_{\psi > \epsilon} \; ; \tag{2-7}$$

it would not exist as an ordinary improper integral.

This behavior is significantly different from that of weakly singular integrals in which the kernel becomes infinite as $1/\ell$, such as Stokes

[1)] To avoid unnecessary mathematical complications, we shall assume both Δg and h to be analytic functions on the sphere (cf. Moritz, 1971, p. 52).

integral; cf. (Heiskanen and Moritz, 1967, p. 121). Weakly singular integral operators are "completely continuous", that is, they are not only continuous but also transform a bounded set of functions into a compact set of functions; see any standard text in functional analysis such as (Kantorovich and Akilov, 1964, chapter XIII).

The classical boundary-value problems of Dirichlet and Neumann lead to weakly singular integral equations for which Fredholm's theorems are valid; cf. (Kellogg, 1929, p. 307). Molodensky's problem, however, is a so-called oblique-derivative problem (the derivative is taken along the vertical which is not normal to the earth's surface S), which leads to strongly singular integral equations. For such equations, which involve operators that are not completely continuous, Fredholm's theorems are not immediately valid. This necessitates deeper investigations which, however, lie outside the scope of the present report.

The standard source on (strongly) singular integral equations is (Mikhlin, 1965), to which we shall frequently refer.

The basic fact is that the integral operator (2-6), although not completely continuous, is at least <u>continuous</u>. What does this mean?

In analogy to the norm of a vector $x = (x_1, x_2, \ldots, x_n)$ defined as

$$\| x \| = [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}} ,$$

we may define the norm of a function f on the unit sphere as

$$\| f \| = \left[\iint_\sigma f^2 \, d\sigma \right]^{\frac{1}{2}} \tag{2-8}$$

(this is the so-called L_2-norm; many other definitions of a norm are possible).

The norm of a linear operator Φ is defined as the number

$$\| \Phi \| = \sup_{\|f\|=1} \| \Phi f \| ; \tag{2-9}$$

note that f and Φf are functions for which the norm is defined by (2-8); the symbol sup stands for supremum, or least upper bound; loosely speaking it is the greatest value that the norm of the function Φf can attain for all possible choices of functions f with norm 1. For other functions f, obviously

$$\|\Phi f\| \leq N\|f\| , \tag{2-10}$$

where the number N denotes the norm of Φ,

$$N = \|\Phi\| . \tag{2-11}$$

If the norm (2-9) is finite, then the operator Φ is said to be bounded or continuous.

A formal solution of the equation (2-4) is as follows:

$$\mu = (I - \Phi)^{-1} \Delta g \tag{2-12}$$

with the series

$$(I - \Phi)^{-1} = I + \Phi + \Phi^2 + \Phi^3 + \ldots , \tag{2-13}$$

called Neumann series in the theory of integral equations and analogous to the elementary binomial series

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots . \tag{2-14}$$

Here Φ^2 means, of course,

$$\Phi^2 f = \Phi(\Phi f), \tag{2-15}$$

and similarly for higher powers.

Just as (2-14) converges for $|x| < 1$, the series (2-13) will converge for

$$\|\Phi\| = N < 1 \tag{2-16}$$

and will then represent the operator $(I - \Phi)^{-1}$; cf. (Kantorovich and Akilov, 1964, sec. V. 2. 5).

Now, as we have mentioned, the operator (2-6) is continuous, that is, its norm N is finite, but we do not yet know whether it is smaller than one. Hence we cannot at this moment make any assertion as to its convergence.

The Molodensky Shrinking. -To further investigate this convergence problem, we again use the "Molodensky shrinking" described in the preceding section, that is, we replace all elevations h by kh with $0 \leq k \leq 1$.

Then the operator Φ as given by (2-6) is replaced by $k\Phi_k$ where Φ_k is defined by

$$\Phi_k f = \frac{R^2}{2\pi} \iint_\sigma f \, \frac{h - h_P}{\ell_k^3} \, \frac{1}{\cos^2 \beta_k} \, d\sigma \; ; \tag{2-17}$$

ℓ_k and β_k are given by (1-4) and (1-6).

In the place of (2-4) we now have the operator equation

$$(I - k\Phi_k) \mu = \Delta g \; , \tag{2-18}$$

which, for $k = 1$, reduces to (2-4). A formal solution is again given by the Neumann series

$$\mu = (I + k\Phi_k + k^2 \Phi_k{}^2 + k^3 \Phi_k{}^3 + \ldots) \Delta g \; , \tag{2-19}$$

in analogy to (2-13). This series will converge for

$$k \| \Phi_k \| = N_k < 1 \; . \tag{2-20}$$

Let now all Φ_k be uniformly bounded for $0 \leq k \leq 1$, that is,

$$\sup_{0 \leq k \leq 1} \| \Phi_k \| = C < \infty \; . \tag{2-21}$$

Then the following theorem holds

THEOREM 2-1. The series

$$\mu = (I + \sum_{n=1}^{\infty} k^n \; \Phi_k{}^n) \Delta g$$

converges in the interval $0 \le k < k_0$ where

$$k_0 = \frac{1}{C} . \tag{2-22}$$

Proof. For $k < k_0$ we have

$$k \, \| \Phi_k \| < k_0 C = 1 ,$$

so that (2-20) is satisfied. That the Φ_k are uniformly bounded, will be shown in the following section.

Thus we have obtained the basic result that the series (2-19) will be convergent for all sufficiently small k.

This does not yet mean that the series (2-13) is also convergent. This will only be the case of (2-16) is satisfied.

Convergence of Molodensky's Series. - The convergence of (2-19) does not yet mean that Molodensky's series is also necessarily convergent for $k < k_0$. In fact, Molodensky's series (in the form given by Brovar) may be obtained from (2-19) by expanding the operators Φ_k, Φ_k^2, Φ_k^3, etc., into series of powers of k, substituting these series into (2-19), and arranging with respect to equal powers of k.

Let us study the series expansion of the operator Φ_k. By (2-17) we have

$$\Phi_k f = \frac{R^2}{2\pi} \iint_\sigma f \, \frac{h - h_P}{\ell_k^3} \, \frac{1}{\cos^2 \beta_k} \, d\sigma .$$

We expand $1/\ell_k^3$, as given by (1-5b), into a binomial series:

$$\frac{1}{\ell_k^3} = \frac{1}{\ell_0^3} (1 + k^2 \tan^2 \gamma)^{-\frac{3}{2}}$$

$$= \frac{1}{\ell_0^3} \left(1 + \sum_{n=1}^{\infty} a_{2n} k^{2n} \tan^{2n} \gamma\right) , \tag{2-23}$$

where we have put

$$\tan \gamma = \frac{h - h_P}{\ell_0} . \tag{2-24}$$

The coefficients a_{2n} are given by

$$a_{2n} = \binom{-3/2}{n}$$

$$= \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) \cdots \left(-\frac{2n+1}{2}\right)}{1 \cdot 2 \cdot 3 \cdots n}$$

$$= (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n} ,$$

or, using a customary abbreviation,

$$a_{2n} = (-1)^n \frac{(2n+1)!!}{2n!!} . \tag{2-25}$$

By (1-7) we have

$$\frac{1}{\cos^2 \beta_k} = 1 + k^2 \tan^2 \beta , \tag{2-26}$$

so that, by (2-23),

$$\frac{1}{\ell_k^3 \cos^2 \beta_k} = \frac{1}{\ell_0^3} \left[1 + \sum_{n=1}^{\infty} (a_{2n} \tan^{2n} \gamma + a_{2n-2} \tan^2 \beta \tan^{2n-2} \gamma) k^{2n} \right] \tag{2-27}$$

(with $a_0 = 1$).

This series is substituted into (2-17), which leads to the expansion

$$\Phi_k = \sum_{n=0}^{\infty} k^{2n} B_{2n} , \tag{2-28}$$

where the operators B_{2n} are given by

$$B_0 f = \frac{R^2}{2\pi} \iint_\sigma f \frac{1}{\ell_0^2} \tan\gamma \, d\sigma , \tag{2-29}$$

$$B_{2n} f = a_{2n} \frac{R^2}{2\pi} \iint_\sigma f \frac{1}{\ell_0^2} \tan^{2n+1} \gamma \, d\sigma +$$

$$+ a_{2n-2} \frac{R^2}{2\pi} \iint_\sigma f \frac{1}{\ell_0^2} \tan^2 \beta \tan^{2n-1} \gamma \, d\sigma \tag{2-30}$$

for $n = 1, 2, 3, \ldots,$; here we have put, by (2-24),

$$\frac{h - h_P}{\ell_0^3} = \frac{1}{\ell_0^2} \tan \gamma . \tag{2-31}$$

Let us now assume that the norms of all operators B_{2n} are uniformly bounded, say by a number M, so that

$$M = \sup_n \| B_{2n} \| < \infty . \tag{2-32}$$

For the convergence of an operator series it is sufficient that the series of the corresponding norms converges; hence (2-28) will converge if

$$\sum_{n=0}^{\infty} k^{2n} \| B_{2n} \| \tag{2-33}$$

converges. A majorant of this series is

$$\sum_{n=0}^{\infty} k^{2n} M = \frac{M}{1 - k^2} . \tag{2-34}$$

Thus our operator series will converge together with (2-34), that is, for $k < 1$.

When is the condition (2-32) satisfied? Let us assume that the maximum terrain inclination

$$\beta_{max} < 45^\circ \, . \tag{2-35}$$

Then both β and γ will always be smaller or at most equal to β_{max}:

$$\beta \le \beta_{max}, \quad \gamma \le \beta_{max}; \tag{2-36}$$

this follows easily from the definition (2-24). Hence the integrands in (2-30) will all tend to zero as $n \to 0$, and in sec. 7 it will be shown that this implies

$$\lim_{n \to \infty} \| B_{2n} \| = 0, \tag{2-37}$$

of which (2-32) is a simple consequence. Thus (2-35) is a condition for convergence of the operator series (2-28), for $k < 1$.

It is, however, readily seen that under this condition convergence will still hold for $k = 1$. In fact, since $\tan\beta_{max} < 1$, one can find an angle, denoted by $\bar{\beta}_{max}$, such that

$$\tan \beta_{max} < \tan \bar{\beta}_{max} < 1 \, ,$$

and put

$$k_0 = \frac{\tan \beta_{max}}{\tan \bar{\beta}_{max}} < 1 \, . \tag{2-38}$$

Hence,

$$\tan \beta_{max} = k_0 \tan \bar{\beta}_{max} \, ,$$

and define $\bar{\beta}$ and $\bar{\gamma}$ by

$$\tan \beta = k_0 \tan \bar{\beta}, \quad \tan \gamma = k_0 \tan \bar{\gamma} \, .$$

Then we have

$$\sum_{n=0}^{\infty} k^{2n} B_{2n} = k_0 \sum_{n=0}^{\infty} (k_0 k)^{2n} \overline{B}_{2n} , \tag{2-39}$$

where $\overline{B}_{2n}$ is given by (2-30) with β and γ replaced by $\overline{\beta}$ and $\overline{\gamma}$. Now the operators $\overline{B}_{2n}$ are also uniformuly bounded by a number $\overline{M}$, since obviously

$$\overline{\beta} \le \overline{\beta}_{max} , \ \overline{\gamma} \le \overline{\beta}_{max} < 45^\circ ,$$

so that (2-37) holds also for $\overline{B}_{2n}$. Hence (2-39) will converge for $k_0 k < 1$, which condition is satisfied even if $k = 1$, which was to be shown.

Let us now consider similar series expansions of the operators Φ_k^n entering in (2-19). We have

$$\Phi_k^n = \sum_{r=0}^{\infty} k^{2r} B_{2r}^{(n)} , \tag{2-40}$$

where the operators $B_{2r}^{(n)}$ are found by formally raising the series (2-28) to the n-th power and arranging with respect to equal powers of k; $B_{2r}^{(n)}$ is then the sum of all terms multiplied by k^{2r}.

In the same way as the series (2-34) is a majorant of the operator series (2-28), a majorant of the operator series (2-40) will be given by

$$M^n \sum_{r=0}^{\infty} (-1)^r \binom{-n}{r} k^{2r} = \frac{M^n}{(1-k^2)^n} , \tag{2-41}$$

which likewise converges for $k < 1$. The same then holds for (2-40) provided the basic condition (2-35) is satisfied, and in the same way as before we may show that convergence then holds even for $k = 1$. Thus we have established

THEOREM 2-2. The operator series

$$\Phi_k^n = \sum_{r=0}^{\infty} k^{2r} B_{2r}^{(n)}$$

converge uniformly for $0 \leq k \leq 1$ provided

$$\beta_{max} < 45° .$$

All there remains now to be done is to substitute the series (2-40), for $n = 1, 2, 3, \ldots$, into (2-19) and again arrange with respect to equal powers of k. The result is evidently Molodensky's series.

The convergence of this latter series follows from Theorem 2-1 and Theorem 2-2 by applying Weierstrass' theorem on double series (Knopp, 1947, section 56). This immediately leads to

THEOREM 2-3. Let C be defined by (2-21), and let

$$\beta_{max} < 45° .$$

Then Molodensky's series converges for $k < k_0$ where

$$k_0 = \frac{1}{C} ,$$

if

$$C \geq 1 ,$$

and it converges uniformly for $0 \leq k \leq 1$ if

$$C < 1.$$

Hence, even if $\beta_{max} < 45°$, the convergence of Molodensky's series at the earth's surface (for $k = 1$) is assured only if $C < 1$; otherwise convergence will only hold for $k < k_0$.

Thus the answer to our convergence problem requires finding estimates for the least upper bound C; this will be the subject of sections 5 and 6.

3. Boundedness of the Operators Φ_k

We shall now show that the operators Φ_k are uniformly bounded, that is, that (2-21) holds; this is required for the validity of Theorem 2-1. This section may be omitted in first reading.

The operators Φ_k are defined by (2-17), which is a singular integral containing a parameter k. It is readily shown that this integral is a continuous function of k $(0 \leq k \leq 1)$, by modifying for two dimensions the proof for one-dimensional singular integrals in (Muskhelishvili, 1953, section 18).

Now we shall prove that also the norm $\|\Phi_k\|$ is a continuous function of k. By the definition (2-9),

$$\|\Phi_k\| = \sup_{\|f\|=1} \|\Phi_k f\| \quad . \tag{3-1}$$

If we could assume that this lowest upper bound is actually attained by some function f^*:

$$\|\Phi_k\| = \|\Phi_k f^*\| \ ,$$

and that this function f^* is the same for neighboring values of k, then the continuity of $\|\Phi_k\|$ would follow from the continuity of $\Phi_k f^*$.

This cannot, however, be assumed, and we have to proceed in a somewhat different way. Define a function φ_k by

$$\varphi_k = \Phi_k f \qquad \text{for} \qquad \|f\| = 1 \quad . \tag{3-2}$$

Then (3-1) becomes

$$N_k = \|\Phi_k\| = \sup \|\varphi_k\| \ , \tag{3-3}$$

the supremum being taken with respect to all f with $\|f\| = 1$.

Consider now two neighboring values of k, denoted by k_1 and k_2. In view of the definition of the supremum, we can find a function $\overset{1}{\varphi}$ which "almost attains" the supremum (3-3) for $k = k_1$, more precisely, for which

$$N_1 - \epsilon < \| \overset{1}{\varphi}_{k_1} \| < N_1 , \tag{3-4a}$$

and similarly a function $\overset{2}{\varphi}$ for which

$$N_2 - \epsilon < \| \overset{2}{\varphi}_{k_2} \| < N_2 , \tag{3-4b}$$

for arbitrarily small ϵ. Denote the function $\overset{1}{\varphi}$ taken for the parameter $k = k_2$ by $\overset{1}{\varphi}_{k_2}$, and define $\overset{2}{\varphi}_{k_1}$ analogously. Then we have

$$N_1 - 2\epsilon < \| \overset{1}{\varphi}_{k_1} \| - \epsilon < \| \overset{1}{\varphi}_{k_2} \| < N_2 , \tag{3-5}$$

by (3-4a) and (3-3) and since k_1 and k_2 can be chosen so close to each other that

$$\| \overset{1}{\varphi}_{k_2} \| > \| \overset{1}{\varphi}_{k_1} \| - \epsilon , \tag{3-6}$$

in view of the continuity of the singular integral mentioned above.

Hence we find

$$\begin{aligned} N_1 - 2\epsilon &< N_2 , \\ N_2 - 2\epsilon &< N_1 , \end{aligned} \tag{3-7}$$

the second relation being obtained from the first by interchanging k_1 and k_2.

Equations (3-7) may be written as

$$| N_2 - N_1 | < 2\epsilon , \tag{3-8}$$

which establishes the continuity of the norm since ϵ is arbitrary.

Thus the norm $\| \Phi_k \|$ is a continuous function of k in the closed interval $0 \leq k \leq 1$. Now it is well-known that a function continuous in a closed interval is bounded, which completes the proof of (2-21).

4. Some Mathematical Theorems

Stereographic Projection. - In the present work we are considering singular integrals on a sphere σ, whereas books such as (Mikhlin, 1965), (Morrey, 1966), (Stein, 1970) and (Stein and Weiss, 1971) treat singular integrals in the plane (or more generally, in n-dimensional Euclidean space). In order to be able to use these latter results, we shall map our sphere onto the plane by stereographic projection (Fig. 4-1).

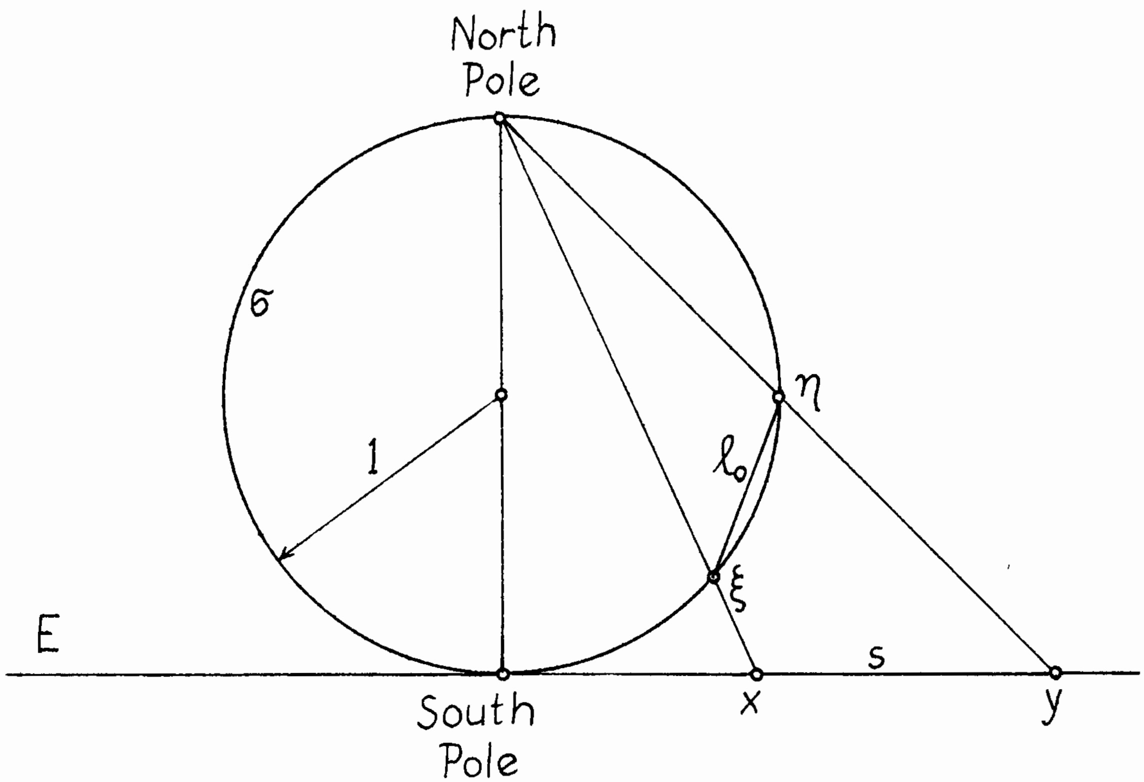

Fig. 4-1

A point ξ on the unit sphere σ is mapped into the point x in the plane E by rectilinear projection from the north pole; the same holds for the image y of another spherical point η .

To the distance $\ell_0 = \xi\eta$ corresponds the plane distance $s = xy$, which is given by

$$s^2 = \ell_0^2 \left(1 + \frac{x^2}{4}\right)\left(1 + \frac{y^2}{4}\right); \tag{4-1}$$

and the spherical surface element $d\sigma$ is mapped into the plane element of area $dy = dy_1\, dy_2$ expressed by

$$dy = d\sigma \left(1 + \frac{y^2}{4}\right)^2 , \tag{4-2}$$

where

$$x^2 = x_1^2 + x_2^2 , \qquad y^2 = y_1^2 + y_2^2 ;$$

x_1 and x_2 denote the plane rectangular coordinates of the point x, and similarly for y_1 and y_2.

Equations (4-1) and (4-2) are taken from (Mikhlin, 1965, section 3) with appropriate modifications as Mikhlin projects the sphere onto the equatorial plane and we are projecting it onto the tangential plane at the south pole.

Consider now a singular integral on the sphere σ of the form

$$g(\xi) = \frac{1}{2\pi} \iint_\sigma \frac{p(\xi, \alpha)}{\ell_0^2} f(\eta)\, d\sigma . \tag{4-3}$$

Here ξ is the computation point, formerly denoted by P, and η is the variable point to which $d\sigma$ refers. As before, ℓ_0 denotes the distance between ξ and η; and α represents the azimuth from ξ to η, taken at ξ. The function $p(\xi, \alpha)$ is a continuous function of the point ξ and the azimuth α satisfying the condition

$$\int_{\alpha=0}^{2\pi} p(\xi, \alpha)\, d\alpha = 0 . \tag{4-4}$$

That under this condition the singular integral exists is shown in (Mikhlin, 1965, section 5.6).

Let us now apply the stereographic projection. By (4-1) and (4-2) we have

$$\frac{1}{\ell_0{}^2} = \frac{1}{s^2} (1 + \frac{x^2}{4}) (1 + \frac{y^2}{4}) ,$$

$$d\sigma = (1 + \frac{y^2}{4})^{-2} dy .$$

On substituting these relations, (4-3) becomes

$$g(\xi) = \frac{1}{2\pi} \iint_E \frac{p(\xi, \alpha)}{s^2} (1 + \frac{x^2}{4}) (1 + \frac{y^2}{4})^{-1} f(\eta) \, dy$$

or

$$g(\xi) (1 + \frac{x^2}{4})^{-1} = \frac{1}{2\pi} \iint_E \frac{p(\xi, \alpha)}{s^2} f(\eta) (1 + \frac{y^2}{4})^{-1} dy .$$

On putting

$$\bar{g}(x) = g(\xi) (1 + \frac{x^2}{4})^{-1} , \quad \bar{f}(y) = f(\eta) (1 + \frac{y^2}{4})^{-1} , \tag{4-5a}$$

$$\bar{p}(x, \alpha) = p(\xi, \alpha) , \tag{4-5b}$$

this finally becomes

$$\bar{g}(x) = \frac{1}{2\pi} \iint_E \frac{\bar{p}(x, \alpha)}{s^2} \bar{f}(y) \, dy . \tag{4-6}$$

This is a plane singular integral equation of standard form (Mikhlin, 1965, section 5), the condition (4-4) becoming

$$\int_{\alpha=0}^{2\pi} \bar{p}(x, \alpha) \, d\alpha = 0 , \tag{4-7}$$

which is also standard.

The fundamental fact in stereographic projection is that if $f(\xi)$ is a function quadratically integrable on the sphere (belonging to $L_2(\sigma)$), then $\bar{f}(x)$ is a function quadratically integrable in the plane (belonging to $L_2(E)$) of the same norm. In fact, by (4-2) and (4-5a) we have

$$\iint_E [\bar{f}(y)]^2 \, dy = \iint_\sigma [f(\eta)]^2 d\sigma , \tag{4-8}$$

which, on taking the square root and using the definition of the norm (2-8), may be abbreviated as

$$\| \bar{f} \|_E = \| f \|_\sigma . \tag{4-9}$$

An Inequality of Calderon-Zygmund Type for the Sphere. - The Calderon-Zygmund inequalities state that singular integral operators of form

$$g(x) = \iint_E \frac{K(\alpha)}{s^2} f(y) \, dy \tag{4-10}$$

and their analogues in n-dimensional space are bounded in L_p-norm (Mikhlin, 1965, section 26; Morrey, 1966, sec. 2-7); the L_2-norms which we are using in the present report--cf. (2-8) and (4-9)--are a special case of L_p-norms.

The following theorem presents an analogous inequality for the sphere.

THEOREM 4-1. Let a singular integral operator on the unit sphere σ be defined by

$$g(\xi) = \iint_\sigma \frac{K(\xi, \alpha)}{\ell_0^2} f(\eta) \, d\sigma , \tag{4-11}$$

where $K(\xi, \alpha)$ is a differentiable function of ξ on the sphere and an absolutely integrable function in α satisfying

$$K(\xi, \alpha + \pi) = -K(\xi, \alpha) \tag{4-12}$$

and hence

$$\int_0^{2\pi} K(\xi, \alpha)\, d\alpha = 0 . \qquad (4\text{-}13)$$

If uniformly in ξ

$$|K(\xi, \alpha)| \leq |F(\alpha)| \qquad (4\text{-}14)$$

for some absolutely integrable function $F(\alpha)$, then

$$\|g\| \leq \|f\| \cdot \frac{1}{2} \int_0^{2\pi} |F(\alpha)|\, d\alpha . \qquad (4\text{-}15)$$

Proof. The stereographic projection transforms (4-11) into the singular integral equation in the plane,

$$\bar{g}(x) = \iint_E \frac{\bar{K}(x, \alpha)}{r^2} \bar{f}(y)\, dy, \qquad (4\text{-}16)$$

where $\bar{f}$ and $\bar{g}$ are given by (4-5a), and r is our former s.

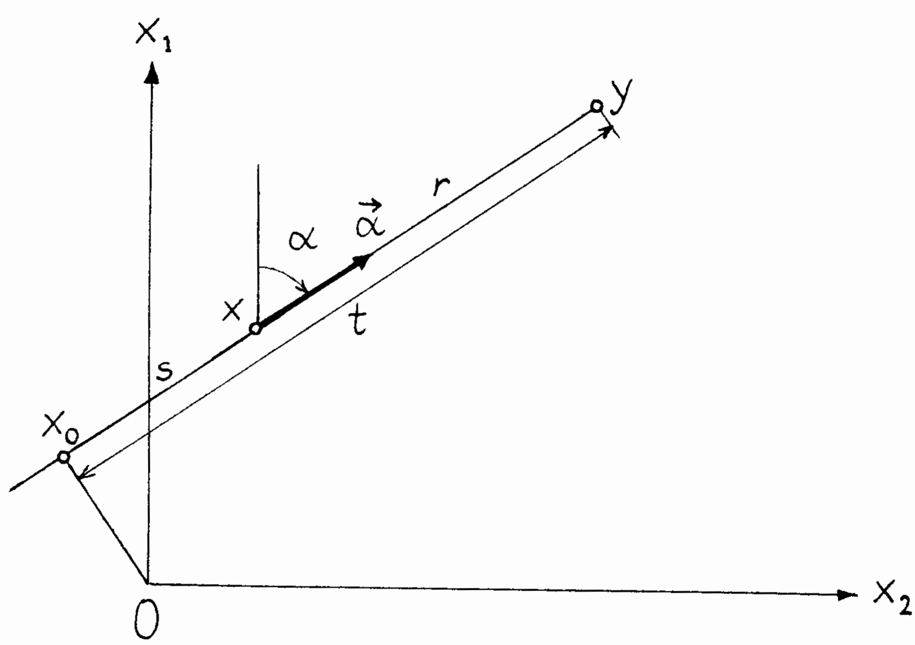

Fig. 4-2

We put (Fig. 4-2)

$$y = x + r\vec{\alpha},$$

where $\vec{\alpha}$ is the unit vector corresponding to the azimuth α. Then, in view of (4-12), eq. (4-16) becomes for polar coordinates r, α

$$\bar{g}(x) = \lim_{\epsilon \to 0} \int_{\alpha=0}^{2\pi} \int_{r=\epsilon}^{\infty} \bar{K}(x, \alpha)\, \bar{f}\, r^{-1}\, dr\, d\alpha$$

$$= \frac{1}{2} \lim_{\epsilon \to 0} \int_{\alpha=0}^{2\pi} \left(\int_{r=-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \bar{K}(x, \alpha)\, \bar{f}\, r^{-1}\, dr\, d\alpha ,$$

where

$$\bar{f} = \bar{f}(x + r\vec{\alpha}) .$$

On setting

$$h(x, \alpha) = \lim_{\epsilon \to 0} \int_{|r| > \epsilon} r^{-1} \bar{f}(x + r\vec{\alpha})\, dr , \tag{4-17}$$

this becomes

$$\bar{g}(x) = \frac{1}{2} \int_{0}^{2\pi} \bar{K}(x, \alpha)\, h(x, \alpha)\, d\alpha . \tag{4-18}$$

Put now (Fig. 4-2)

$$x = x_0 + s\vec{\alpha} , \quad y = x_0 + t\vec{\alpha},$$

where $Ox_0 \perp \vec{\alpha}$, and put further

$$\varphi(t; x_0, \alpha) = \bar{f}(x_0 + t\vec{\alpha}) ,$$
$$\psi(s; x_0, \alpha) = h(x_0 + s\vec{\alpha}, \alpha) . \tag{4-19}$$

Then (4-17) becomes

$$\psi(s;x_0,\alpha) = \lim_{\epsilon\to 0} \int_{|t-s|>\epsilon} \frac{\varphi(t;x_0,\alpha)}{t-s}\,dt \ . \qquad (4\text{-}20)$$

This equation represents a one-dimensional Hilbert transform, which is known to have norm 1 in $L_2(-\infty, \infty)$, that is, we have

$$\int_{-\infty}^{\infty} \psi^2 ds \le \int_{-\infty}^{\infty} \varphi^2 dt \qquad (4\text{-}21)$$

(cf. Stein and Weiss, 1971, p. 187, footnote). Integrating (4-21) over all straight lines parallel to $\vec{\alpha}$ and taking (4-19) into account we get

$$\iint_E h^2\,dx \le \iint_E \bar{f}^2\,dx$$

or

$$\iint_E h^2(x,\alpha)\,dx \le \|\bar{f}\|^2 \ . \qquad (4\text{-}22)$$

Let us now return to (4-18). We have

$$|\bar{g}(x)| \le \int_{\alpha=0}^{2\pi} |h(x,\alpha)| \cdot \frac{1}{2} |\bar{K}(x,\alpha)|\,d\alpha$$

$$= \int_{\omega} |h(x,\alpha)|\,d\omega$$

if we put

$$\frac{1}{2} |\bar{K}(x,\alpha)|\,d\alpha = d\omega \ . \qquad (4\text{-}23)$$

Application of the Hoelder inequality (cf. Kantorovich and Akilov, 1964, pp. 62-63) gives

$$|\bar{g}(x)| \leq \int_{\omega} 1 \cdot |h(x,\alpha)|\, d\omega$$

$$\leq \left[\int_{\omega} d\omega\right]^{\frac{1}{2}} \left[\int_{\omega} h(x,\alpha)^2 d\omega\right]^{\frac{1}{2}} ,$$

or on squaring

$$\bar{g}(x)^2 \leq \int_{\alpha=0}^{2\pi} \frac{1}{2} |\bar{K}(x,\alpha)|\, d\alpha \quad .$$

$$\cdot \int_{\alpha=0}^{2\pi} h(x,\alpha)^2 \frac{1}{2} |\bar{K}(x,\alpha)|\, d\alpha \tag{4-24}$$

or

$$\bar{g}(x)^2 \leq C \int_{\alpha=0}^{2\pi} h(x,\alpha)^2 \frac{1}{2} |F(\alpha)|\, d\alpha \tag{4-25}$$

where

$$C = \int_{\alpha=0}^{2\pi} \frac{1}{2} |F(\alpha)|\, d\alpha \ . \tag{4-26}$$

This follows from the fact that the right-hand side of (4-24) never decreases on replacing $|\bar{K}(x,\alpha)|$ by $|F(\alpha)|$ in view of (4-14);

$$\bar{K}(x,\alpha) = K(\xi,\alpha) \ .$$

Integrating (4-25) over the plane E and interchanging the order of integration on the right-hand side we find

$$\iint_E \bar{g}^2 dx \leq C \int_{\alpha=0}^{2\pi} \iint_E h(x,\alpha)^2 dx \frac{1}{2} |F(\alpha)|\, d\alpha$$

or in view of (4-22) and (4-26),

$$\|\bar{g}\|^2 \leq C^2 \|\bar{f}\|^2 .$$

Taking the square root and considering (4-9) concludes the proof.

In this proof we have followed the structure of the proof of Theorem 2.7.2 of (Morrey, 1966, pp. 56-57), which gives a similar inequality for a singular integral in R_n the kernel of which depends only on $\vec{\alpha}$

A Theorem on Weakly Singular Integral Operators. - The singular integral equations considered in the present report may be split up into a (strongly) singular integral equation of type (4-11) and a weakly singular integral equation, which is dealt with by the following theorem.

THEOREM 4-2. Let a weakly singular integral operator on the unit sphere σ be defined by

$$g(\xi) = \iint_\sigma \frac{A(\xi, \eta)}{\ell_0} f(\eta)\, d\sigma , \tag{4-27}$$

where $A(\xi, \eta)$ is a bounded function, and let

$$A = \sup_\sigma |A(\xi, \eta)| . \tag{4-28}$$

Then this integral operator is bounded in $L_2(\sigma)$ such that

$$\|g\| \leq 4\pi A \|f\| . \tag{4-29}$$

Proof. We have

$$|g(\xi)| \leq A \iint_\sigma \frac{1}{\ell_0} |f(\eta)|\, d\sigma$$

$$= A \iint_\sigma \frac{1}{\sqrt{\ell_0}} \frac{|f(\eta)|}{\sqrt{\ell_0}}\, d\sigma$$

$$\leq A \left[\iint_\sigma \frac{d\sigma}{\ell_0} \right]^{\frac{1}{2}} \left[\iint_\sigma \frac{f(\eta)^2}{\ell_0}\, d\sigma \right]^{\frac{1}{2}} \tag{4-30}$$

by Hoelder's inequality (Kantorovich and Akilov, 1964, pp. 62-63).

Now

$$\iint_\sigma \frac{d\sigma}{\ell_0} = 4\pi \tag{4-31}$$

because this integral may be considered as a surface layer potential of unit density and mass 4π (density 1 times surface 4π). The value of this potential on the sphere is mass 4π divided by radius 1, or 4π.

Hence we have

$$g(\xi)^2 \leq 4\pi A^2 \iint_\sigma \frac{f(\eta)^2}{\ell_0} d\sigma \,.$$

Integration over the unit sphere gives

$$\iint_\sigma |g(\xi)|^2 d\xi \leq 4\pi A^2 \iint_\sigma \iint_\sigma \frac{f(\eta)^2}{\ell_0} d\xi \, d\eta \tag{4-32}$$

$$= 4\pi A^2 \iint_\sigma \left[\iint_\sigma \frac{d\xi}{\ell_0} \right] f(\eta)^2 d\eta$$

$$= 4\pi A^2 \iint_\sigma \frac{d\xi}{\ell_0} \iint_\sigma f(\eta)^2 d\eta \,;$$

here $d\xi$ denotes the surface element $d\sigma$ at ξ, and $d\eta$ denotes $d\sigma$ at η. The expression on the left-hand side is $\|g\|^2$, the square of the norm of g, and the second integral on the right is $\|f\|^2$; the first integral is given by (4-31). Thus we have

$$\|g\|^2 \leq (4\pi)^2 A^2 \|f\|^2 \,,$$

so that (4-29) follows on taking the square root.

Theorem 4-2 is an adaption, to the case of a sphere, of Lemma 1.4 of (Mikhlin, 1965, section 4); the proof given here is an appropriate modification of Mikhlin's proof of his lemma.

Finally, we shall need a theorem, analogous to Theorem 4-2, for the case that the integral is extended over a spherical cap only, instead of being extended over the whole sphere.

THEOREM 4-3. Let a weakly singular integral operator on the unit sphere σ be defined by

$$g(\xi) = \iint_{\varkappa} \frac{A(\xi, \eta)}{\ell_0} f(\eta)\, d\sigma \ , \tag{4-33}$$

where $\varkappa$ is a spherical cap of radius ψ_0 with its center at the point ξ, and where $A(\xi, \eta)$ is a bounded function. Let

$$A = \sup_{\sigma} | A(\xi, \eta) | \ . \tag{4-34}$$

Then this integral operator is bounded in $L_2(\sigma)$ such that

$$\| g \| \leq 4\pi \sqrt{\sin\frac{\psi_0}{2}}\ A \| f \| \ . \tag{4-35}$$

Proof. The proof is very similar to that of Theorem 4-2. Instead of (4-30) we have

$$|g(\xi)| \leq A \left[\iint_{\varkappa} \frac{d\sigma}{\ell_0} \right]^{\frac{1}{2}} \left[\iint_{\varkappa} \frac{f(\eta)^2}{\ell_0} d\sigma \right]^{\frac{1}{2}} \ .$$

We find

$$\iint_{\varkappa} \frac{d\sigma}{\ell_0} = \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_0} \frac{1}{2 \sin\frac{\psi}{2}} \sin\psi \, d\psi =$$

$$= 2\pi \int_0^{\psi_0} \cos\frac{\psi}{2}\, d\psi = 4\pi \sin\frac{\psi_0}{2} \ ,$$

and further

$$\iint_\sigma g(\xi)^2 d\xi \le 4\pi \sin\frac{\psi_0}{2} \cdot A^2 \iint_\sigma \iint_\varkappa \frac{f(\eta)^2}{\ell_0} \, d\xi \, d\eta$$

$$\le 4\pi \sin\frac{\psi_0}{2} \cdot A^2 \iint_\sigma \iint_\sigma \frac{f(\eta)^2}{\ell_0} \, d\xi \, d\eta \, .$$

On comparing this with (4-32), our theorem follows immediately from Theorem 4-2.

5. Decomposition. The Strongly Singular Part

We shall now try to estimate the norm of the operator Φ_k so as to find the least upper bound (2-21).

It will simplify our considerations if we first put $k = 1$, investigating the operator Φ given by (2-6):

$$\Phi f = \frac{R^2}{2\pi} \iint_\sigma f \, \frac{h - h_P}{\ell^3} \, \frac{1}{\cos^2\beta} \, d\sigma \, . \tag{5-1}$$

On introducing the angle γ by (2-24) and letting $R = 1$ this becomes

$$\Phi f = \frac{1}{2\pi} \iint_\sigma \frac{\ell_0}{\ell^3} \, \tan\gamma \, \frac{f}{\cos^2\beta} \, d\sigma \, . \tag{5-2}$$

First, we introduce a new function f^* by

$$f^* = \frac{f}{\cos^2\beta} \tag{5-3}$$

(β is the terrain inclination at the variable point η to which f refers); the norms are obviously related by

$$\| f^* \| \le \frac{1}{\cos^2\beta_{max}} \| f \| \, ; \tag{5-4}$$

this follows from the norm definition (2-8).

Secondly, we approximate the terrain in the neighborhood of the computation point P, by the tangential plane at P(Fig. 5-1). Thus

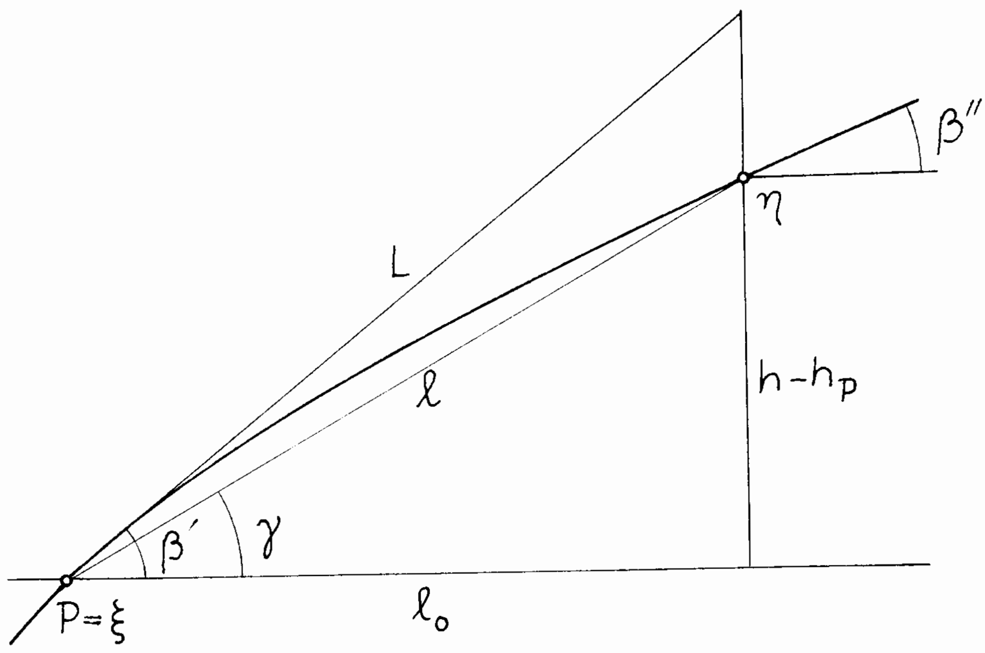

Fig. 5-1

the distance ℓ is approximated by

$$L = \frac{\ell_0}{\cos\beta'} \tag{5-5}$$

and γ is approximated by β', which is the angle of terrain inclination at P along the azimuth α. One easily verifies that

$$\tan\beta' = \tan\beta_0 \cos(\alpha - \alpha_0) ,$$

where β_0 is the maximum inclination at P, and α_0 is tha azimuth of maximum inclination. Without loss of generality we may put $\alpha_0 = 0$, so that

$$\tan\beta' = \tan\beta_0 \cos\alpha \; . \tag{5-6}$$

Thus the integrand in (5-2) is approximated by

$$\frac{1}{2\pi}\frac{\ell_0}{L^3}\tan\beta' = \frac{1}{2\pi\,\ell_0^2}\cos^3\beta'\tan\beta'$$

$$= \frac{1}{2\pi\,\ell_0^2}\frac{\tan\beta'}{(1+\tan^2\beta')^{3/2}}$$

$$= \frac{1}{2\pi\,\ell_0^2}\frac{\tan\beta_0\cos\alpha}{(1+\tan^2\beta_0\cos^2\alpha)^{3/2}} \,. \qquad (5\text{-}7)$$

Now we may write (5-1) in the form

$$\Phi f = \Psi_1 f + \Psi_2 f \qquad (5\text{-}8)$$

where

$$\Psi_1 f = \frac{1}{2\pi}\iint_\sigma \frac{1}{\ell_0^2}\frac{\tan\beta_0\cos\alpha}{(1+\tan^2\beta_0\cos^2\alpha)^{3/2}} f^*\,d\sigma \,, \qquad (5\text{-}9)$$

$$\Psi_2 f = \frac{1}{2\pi}\iint_\sigma \left(\frac{h-hp}{\ell^3} - \frac{1}{\ell_0^2}\frac{\tan\beta_0\cos\alpha}{(1+\tan^2\beta_0\cos^2\alpha)^{3/2}}\right) f^*\,d\sigma \,; \qquad (5\text{-}10)$$

note that we have put R = 1.

We must now find estimates of the norms of the operators Ψ_1 and Ψ_2. The singular integral (5-9) is of the form (4-11) with

$$K(\xi,\alpha) = \frac{1}{2\pi}\frac{\tan\beta_0\cos\alpha}{(1+\tan^2\beta_0\cos^2\alpha)^{3/2}} \,; \qquad (5\text{-}11)$$

here $\tan\beta_0$ is a function of ξ since β_0 is the maximum terrain inclination at the point $P = \xi$. This function is obviously nowhere greater absolutely than

$$F(\alpha) = \frac{1}{2\pi} \tan\beta_{max} \cos\alpha \ , \tag{5-12}$$

where β_{max} is the maximum terrain inclination for the whole earth. Thus Theorem 4-1 can be applied since (4-14) is satisfied, and (4-15) gives

$$\|\Psi_1 f\| \leq \|f^*\| \cdot \frac{1}{2} \int_0^{2\pi} \frac{1}{2\pi} \tan\beta_{max} |\cos\alpha| \, d\alpha$$

$$= \|f^*\| \frac{1}{\pi} \tan\beta_{max} \int_0^{\pi/2} \cos\alpha \, d\alpha$$

$$= \|f^*\| \frac{1}{\pi} \tan\beta_{max} \ . \tag{5-13}$$

Using (5-4) we get

$$\|\Psi_1 f\| \leq \|f\| \frac{1}{\pi} \frac{\tan\beta_{max}}{\cos^2\beta_{max}} \ , \tag{5-14}$$

so that we have found an inequality for the norm of the operator Ψ_1:

$$\|\Psi_1\| \leq \frac{1}{\pi} \tan\beta_{max}(1 + \tan^2\beta_{max}) \ . \tag{5-15}$$

6. The Weakly Singular Part

We shall now try to find an estimate for the norm of the operator Ψ_2 as given by (5-10). Using (1-3), (2-24) and (5-6), we may write (5-10) in the form

$$\Psi_2 f = \frac{1}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} (\cos^3\gamma \tan\gamma - \cos^3\beta' \tan\beta') f^* \, d\sigma \tag{6-1}$$

or

$$\Psi_2 f = \frac{1}{2\pi} \iint_\sigma \frac{A(\xi, \eta)}{\ell_0} f^* \, d\sigma \quad , \tag{6-2}$$

where

$$A(\xi, \eta) = \frac{1}{2\pi \ell_0} \left(\cos^3 \gamma \tan \gamma - \cos^3 \beta' \tan \beta' \right) \tag{6-3}$$

remains bounded for $\ell_0 \to 0$, as we shall see.

Let us put

$$\ell_0 = s \tag{6-4}$$

and consider the function

$$f(s) = s(\cos^3 \gamma \tan \gamma - \cos^3 \beta' \tan \beta') \tag{6-5}$$

for constant azimuth, α = const. By (2-24) we have

$$f(s) = \frac{\Delta h}{(1+t^2)^{3/2}} - s \cos^3 \beta' \tan \beta' \tag{6-6}$$

with

$$\Delta h = h - h_P , \quad t = \tan \gamma = \frac{h - h_P}{\ell_0} \quad . \tag{6-7}$$

Using

$$\frac{\partial t}{\partial s} = \frac{1}{s} \frac{\partial h}{\partial s} - \frac{\Delta h}{s^2} = \frac{\partial h / \partial s - t}{s} \tag{6-8}$$

we may differentiate (6-6) with respect to s. We find

$$\frac{\partial f}{\partial s} = \frac{1 - 2t^2}{(1+t^2)^{5/2}} \frac{\partial h}{\partial s} + \frac{3t^3}{(1+t^2)^{5/2}} -$$

$$- \cos^3 \beta' \tan \beta' , \tag{6-9}$$

$$\frac{\partial^2 f}{\partial s^2} = \frac{1 - 2t^2}{(1+t^2)^{5/2}} \frac{\partial^2 h}{\partial s^2} - \frac{9t - 6t^3}{(1+t^2)^{7/2}} \frac{(\partial h/\partial s - t)^2}{s} .$$

By (6-7), this may be written in the form

$$\frac{\partial^2 f}{\partial s^2} = \cos^3\gamma\,(1 - 3\sin^2\gamma)\frac{\partial^2 h}{\partial s^2} -$$

$$- 9\sin\gamma\cos^4\gamma\,(1 - \frac{5}{3}\sin^2\gamma)\,\frac{(\partial h/\partial s - t)^2}{s} . \tag{6-10}$$

Now,

$$\frac{\partial h}{\partial s} = \tan\beta'' \tag{6-11}$$

is the terrain inclination along the profile at the point η (Fig. 5-1). By Taylor's theorem we have

$$h_P = h - \frac{\partial h}{\partial s}s + \frac{1}{2}\left(\frac{\partial^2 h}{\partial s^2}\right)_1 s^2 ,$$

where $\partial h/\partial s$ refers to the point η and $(\partial^2 h/\partial s^2)_1$ to some point P_1 between ξ and η, so that

$$t = \frac{h - h_P}{s} = \frac{\partial h}{\partial s} - \frac{1}{2}\left(\frac{\partial^2 h}{\partial s^2}\right)_1 s$$

and

$$\frac{\partial h/\partial s - t}{s} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial s^2}\right)_1 . \tag{6-12}$$

By (6-6) and (6-9) we have, for $s = 0$,

$$f(0) = 0 , \quad \frac{\partial f}{\partial s} = 0 ,$$

since for $s = 0$ there is $\beta' = \beta'' = \gamma$. Hence, again by Taylor's theorem,

$$f(s) = f(0) + sf'(0) + \frac{1}{2} s^2 f''(s_2)$$

$$= \frac{1}{2} s^2 f''(s_2) = \frac{1}{2} s^2 \left(\frac{\partial^2 f}{\partial s^2}\right)_2 ,$$

where $0 \leq s_2 \leq s$. By (6-10), (6-11) and (6-12) we thus have

$$f(s) = \frac{1}{2} s^2 \left[\cos^3 \gamma_2 (1 - 3 \sin^2 \gamma_2) \left(\frac{\partial^2 h}{\partial s^2}\right)_2 - \right.$$

$$- \frac{9}{2} \sin \gamma_2 \cos^4 \gamma_2 (1 - \frac{5}{3} \sin^2 \gamma_2) \cdot$$

$$\left. \cdot (\tan \beta_2'' - \tan \gamma_2) \left(\frac{\partial^2 h}{\partial s^2}\right)_1 \right] . \tag{6-13}$$

The two points P_1 and P_2 towhich the subscripts refer, are situated on the line connecting ξ and η, such that P_2 lies between ξ and η, and P_1 lies between ξ and P_2 .

Thus the function (6-3),

$$A(\xi, \eta) = \frac{1}{2\pi s^2} f(s) , \tag{6-14}$$

remains, in fact, bounded as $s \to 0$. Therefore, (6-1) is only a weakly singular integral, so that Theorem 4-2 can be applied.

Since $\gamma \leq \beta_{max}$ and $\beta_2'' \leq \beta_{max}$, where β_{max} is the largest terrain inclination occurring at any point of the earth's surface, (4-28) becomes

$$A = \max | A(\xi, \eta) |$$

$$\leq \frac{1}{4\pi} \left[1 + 9 \tan \beta_{max}\right] \left(\frac{\partial^2 h}{\partial s^2}\right)_{max} . \tag{6-15}$$

Here we have used elementary inequalities such as

$$\cos\gamma_2 \le 1, \quad |\tan\beta_2'' - \tan\gamma_2| \le 2\tan\beta_{max} \quad .$$

The term $(\partial^2 h/\partial s^2)_{max}$ denotes the maximum second derivative of the elevation h at any point of the earth's surface along any direction. It is related to the curvature $\varkappa$ along the same direction in an elementary way: we have

$$\varkappa = \frac{\partial^2 h/\partial s^2}{[1+(\partial h/\partial s)^2]^{3/2}} = \cos^3\beta'' \frac{\partial^2 h}{\partial s^2} \tag{6-16}$$

by (6-11), so that (6-15) becomes

$$A \le \frac{1}{4\pi}(1 + 9\tan^2\beta_{max}) \frac{\varkappa_{max}}{\cos^3\beta_{max}} \quad . \tag{6-17}$$

Now Theorem 4-2 may be applied to give by (4-29) and (5-4)

$$\|\Psi_2 f\| \le (1 + 9\tan^2\beta_{max}) \frac{\varkappa_{max}}{\cos^5\beta_{max}} \|f\| \quad . \tag{6-18}$$

It should be noted that in (6-1) we have put R = 1, that is, we have used the earth's mean radius as our unit of length. If we wish to use our ordinary metric units, we must multiply $\varkappa_{max}$ by R, obtaining for the norm of the operator Ψ_2 :

$$\|\Psi_2\| \le \frac{1 + 9\tan^2\beta_{max}}{\cos^5\beta_{max}} \frac{R}{\rho_{min}} , \tag{6-19}$$

where $\rho_{min} = 1/\varkappa_{max}$ denotes the minimum radius of curvature of a vertical profile of the topographic earth's surface at any point and along any direction.

Discussion. - For the norm of the total operator (5-8) we have the inequality ("triangle inequality")

$$\|\Phi\| \le \|\Psi_1\| + \|\Psi_2\| , \tag{6-20}$$

where $\|\Psi_1\|$ is given by (5-15) and $\|\Psi_2\|$ by (6-19). We remember from sec. 2, eq. (2-16) that we shall have convergence at the earth's surface if $\|\Phi\| < 1$. As far as the first operator, Ψ_1, is concerned, we have for $\beta_{max} < 45°$ by (5-15)

$$\|\Psi_1\| < \frac{2}{\pi} \doteq 0.64 \ . \tag{6-21}$$

However, even for a small maximum inclination β_{max}, (6-19) gives approximately

$$\|\Psi_2\| \le \frac{R}{\rho_{min}} \ , \tag{6-22}$$

in which ρ_{min} will be much smaller than the earth's mean radius R (the topographic surface has much stronger curvature than the earth's ellipsoid), so that (6-22) will be much greater than unity.

This would indicate that Molodensky's series diverges on the earth's surface. It would, however, be rash to draw this conclusion without looking into the matter more closely. In fact, the estimate (6-19) is rigorous, as all our estimates up to now are, but rigorous at the cost of overestimating the right-hand sides: In order to have the ≤ sign apply rigorously, we have been careful always to replace the right-hand sides by some greater value wherever appropriate, never by some average value, say. This is the usual procedure in mathematics, as opposed to the average estimates customary in statistics and geodesy. Thus our present estimates are exact but biased in the sense that they may be overly "pessimistic."

A Less Pessimistic Estimate. - Let us, therefore, try to find a "closer" estimate for $\|\Psi_2\|$.

We split up the operator Ψ_2, given by (6-1), in the following way:

$$\Psi_2 = \Psi_{21} + \Psi_{22} + \Psi_{23} \ , \tag{6-23}$$

where

$$\Psi_{21} f = \iint_{\varkappa} \frac{A(\xi, \eta)}{\ell_0} f^* \, d\sigma , \tag{6-24}$$

$$\Psi_{22} f = \frac{1}{2\pi} \iint_{\sigma - \varkappa} \frac{h - h_P}{\ell^3} f^* \, d\sigma , \tag{6-25}$$

$$-\Psi_{23} f = \frac{1}{2\pi} \iint_{\sigma - \varkappa} \frac{1}{\ell_0^2} \cos^3\beta' \tan\beta' f^* \, d\sigma . \tag{6-26}$$

By Theorem 4-3 we have

$$\|\Psi_{21}\| \leq M \sqrt{\sin \frac{\psi_0}{2}} , \tag{6-27}$$

where M is the expression on the right-hand side of (6-19); cf. (4-29) and (4-35).

Let us now consider (6-25). In view of (5-3) we have

$$\Psi_{22} f = \frac{1}{2\pi} \iint_{\sigma - \varkappa} \frac{1}{\cos^2\beta} \frac{h - h_P}{\ell^3} f \, d\sigma , \tag{6-28}$$

so that for the norm $\|\Psi_{22}\|$ we obtain by (Kantorovich and Akilov, 1964, p. 111)

$$\|\Psi_{22}\|^2 \leq \frac{1}{4\pi^2} \iint_{\sigma - \varkappa} \iint_{\sigma - \varkappa} \frac{1}{\cos^4\beta} \frac{(h - h_P)^2}{\ell^6} \, d\xi \, d\eta , \tag{6-29}$$

where $d\xi$ means $d\sigma$ at the point $P = \xi$, and $d\eta$ denotes $d\sigma$ at our usual integration point η. Since $1/\ell^6 \leq 1/\ell_0^6$, we obtain in the usual way

$$\|\Psi_{22}\|^2 \leq \frac{1}{4\pi^2} \frac{\Delta h_{max}^2}{\cos^4\beta_{max}} \iint_{\sigma - \varkappa} \iint_{\sigma - \varkappa} \frac{d\xi \, d\eta}{\ell_0^6} . \tag{6-30}$$

Integration with respect to η gives

$$\iint\limits_{\sigma-\varkappa} \frac{d\eta}{\ell_0^6} = \int\limits_{\alpha=0}^{2\pi} \int\limits_{\psi=\psi_0}^{\pi} \frac{\sin\psi}{\ell_0^6} \, d\psi \, d\alpha$$

$$= 2\pi \int\limits_{\psi_0}^{\pi} \frac{2 \sin\frac{\psi}{2} \cos\frac{\psi}{2}}{64 \sin^6 \frac{\psi}{2}} \, d\psi$$

$$= \frac{\pi}{16} \int\limits_{\psi_0}^{\pi} \frac{\cos\frac{\psi}{2}}{\sin^5\frac{\psi}{2}} \, d\psi = \frac{\pi}{32} \left(\frac{1}{\sin^4 \frac{\psi_0}{2}} - 1 \right) \leq \frac{\pi}{32 \sin^4 \frac{\psi_0}{2}} ,$$

which is a constant, so that further integration with respect to ξ amounts to multiplication by

$$\iint\limits_{\sigma-\varkappa} d\xi \leq \iint\limits_{\sigma} d\xi = 4\pi \ .$$

Thus (6-30) becomes

$$\| \Psi_{22} \|^2 \leq \frac{1}{32} \frac{\Delta h^2_{max}}{\cos^4 \beta_{max}} \frac{1}{\sin^4 \frac{\psi_0}{2}} ,$$

so that, on admitting a $R \neq 1$,

$$\| \Psi_{22} \| \leq \frac{\sqrt{2}}{8 \cos^2 \beta_{max}} \frac{\Delta h_{max}}{R} \frac{1}{\sin^2 \frac{\psi_0}{2}} . \qquad (6\text{-}31)$$

Finally the operator Ψ_{23}, eq. (6-26), differs from the operator Ψ_1 , eq. (5-9),only in the fact that now the integration is not extended over the whole sphere σ but only over the area $\sigma - \varkappa$. Since this cuts out the strong singularity at $\Psi = 0$, it may be expected that the norm $\|\Psi_{23}\|$ will be smaller than the norm $\|\Psi_1\|$. Thus we may confidently expect, for small ψ_0 ,

$$\|\Psi_{23}\| \leq \|\Psi_1\| \leq \frac{1}{\pi} \tan\beta_{max}(1 + \tan^2\beta_{max}) ; \qquad (6\text{-}32)$$

this equation may also be rigorously established.

Now, again by the triangle inequality, we have

$$\|\Phi\| \leq \|\Psi_1\| + \|\Psi_{21}\| + \|\Psi_{22}\| + \|\Psi_{23}\| , \qquad (6\text{-}33)$$

where the norms on the right-hand side are estimated by (5-15), (6-27), (6-31), and (6-32).

These estimates will be discussed in sec. 8, where we shall also consider the case of Φ_k with $k \neq 1$.

7. Boundedness of the Operators B_{2n}

We shall now prove (2-32) by showing that (2-37) holds, that is,

$$\lim_{n \to \infty} \|B_{2n}\| = 0 . \qquad (7\text{-}1)$$

The operator (2-30) may be split up into a strongly singular part and a weakly singular part, in the same way as we did in sec. 5. We have

$$B_{2n} = B'_{2n} + B''_{2n} \qquad (7\text{-}2)$$

where (we put R = 1)

$$B'_{2n} f = \frac{a_{2n}}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} \tan^{2n+1}\beta' \cdot f \, d\sigma +$$

$$+ \frac{a_{2n-2}}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} \tan^{2n-1}\beta' (f \tan^2\beta) \, d\sigma , \qquad (7\text{-}3)$$

$$B''_{2n} f = \frac{a_{2n}}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} (\tan^{2n+1}\gamma - \tan^{2n+1}\beta') \cdot f\, d\sigma +$$

$$+ \frac{a_{2n-2}}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} (\tan^{2n-1}\gamma - \tan^{2n-1}\beta')(f \tan^2\beta)\, d\sigma . \quad (7\text{-}4)$$

Here γ is defined by (2-24), for β' we have (5-6), and β is the (maximum) terrain inclination at the point η.

The first integral on the right-hand side of (7-3),

$$J_1 f = \frac{a_{2n}}{2\pi} \iint_\sigma \frac{1}{\ell_0^2} \tan^{2n+1}\beta' \cdot f\, d\sigma =$$

$$= \frac{a_{2n}}{2\pi} \tan^{2n+1}\beta_0 \iint_\sigma \frac{\cos^{2n+1}\alpha}{\ell_0^2} \cdot f\, d\sigma ,$$

may be estimated by Theorem 4-1:

$$\|J_1 f\| \le \|f\| \cdot \frac{|a_{2n}|}{4\pi} \tan^{2n+1}\beta_{max} \cdot \int_0^{2\pi} |\cos^{2n+1}\alpha|\, d\alpha ,$$

so that

$$\|J_1\| \le \frac{|a_{2n}|}{\pi} \tan^{2n+1}\beta_{max} \int_0^{\frac{\pi}{2}} \cos^{2n+1}\alpha\, d\alpha$$

$$= \frac{1}{\pi} \frac{(2n+1)!!}{2n!!} \tan^{2n+1}\beta_{max} \frac{(2n)!!}{(2n+1)!!}$$

$$= \frac{1}{\pi} \tan^{2n+1}\beta_{max} . \quad (7\text{-}5)$$

Here we have used (2-25); the value of the definite integral may be found in any integral table.

Similarly we find for the second integral on the right-hand side of (7-3)

$$\|J_2 f\| \leq \|f \tan^2\beta\| \cdot \frac{1}{\pi} \tan^{2n-1}\beta_{max}$$

$$\leq \|f\| \cdot \frac{1}{\pi} \tan^{2n+1}\beta_{max} ,$$

so that, likewise,

$$\|J_2\| \leq \frac{1}{\pi} \tan^{2n+1}\beta_{max} . \tag{7-6}$$

Obviously, for $\beta_{max} < 45^\circ$, the expressions (7-5) and (7-6) tend to zero, so that

$$\lim_{n\to\infty} \|B'_{2n}\| = 0 . \tag{7-7}$$

Let us now consider the weakly singular part (7-4). The first integral may be written in the form

$$J_3 f = \iint_\sigma \frac{A_n(\xi,\eta)}{\ell_0} f \, d\sigma , \tag{7-8}$$

where

$$A_n(\xi,\eta) = \frac{a_{2n}}{2\pi\,\ell_0} (\tan^{2n+1}\gamma - \tan^{2n+1}\beta'). \tag{7-9}$$

for $\ell_0 \neq 0$. For small values of ℓ_0 we find by the method used in sec. 6 to obtain (6-13) and (6-14):

$$A_n(\xi,\eta) = a_{2n} \frac{2n+1}{4\pi} \tan^{2n}\gamma_2 \left(\frac{\partial^2 h}{\partial s^2}\right)_2 +$$

$$+ a_{2n} \frac{n(2n+1)}{4\pi} \tan^{2n-1}\gamma_2 (\tan\beta''_2 - \tan\gamma_2) \left(\frac{\partial^2 h}{\partial s^2}\right)_1 . \tag{7-10}$$

The notations are the same as in (6-13).

It is not difficult to see that the functions $A_n(\xi, \eta)$ converge uniformly in ξ and η to zero as $n \to \infty$. Assume $\ell_0 \geq \delta$ where δ is a small fixed number. Then, by (7-9), we have

$$|A_n(\xi, \eta)| \leq \frac{|a_{2n}|}{\pi \delta} \tan^{2n+1} \beta_{max} \tag{7-11}$$

uniformly as long as $\xi\eta = \ell_0 \geq \delta$. For $\xi\eta < \delta$ we use (7-10); then

$$|A_n(\xi, \eta)| \leq \text{const.} |a_{2n}| (2n+1)^2 \tan^{2n} \beta_{max} \left(\frac{\partial^2 h}{\partial s^2}\right)_{max} . \tag{7-12}$$

The right-hand sides do not depend on ξ, η and tend to zero as $n \to \infty$ (since $|a_{2n}|$ and $(2n+1)^2$ grow more slowly than $\tan^{2n} \beta_{max}$ decreases). Therefore, one may find a n_0 such that for $n > n_0$ the right-hand sides of both (7-11) and (7-12) are smaller than a given ϵ, and therefore

$$|A_n(\xi, \eta)| < \epsilon \quad \text{if} \quad n > n_0 \tag{7-13}$$

uniformly in ξ and η.

Hence,

$$A_n = \max_{\sigma} |A_n(\xi, \eta)| \to 0 \tag{7-14}$$

as $n \to \infty$, and consequently, by Theorem 4-2, also

$$\|J_3\| \to 0 .$$

In the same way one shows that the second integral in (7-4) also tends to zero as $n \to \infty$, so that

$$\lim_{n \to \infty} \|B''_{2n}\| = 0 . \tag{7-15}$$

From

$$\|B_{2n}\| \leq \|B'_{2n}\| + \|B''_{2n}\| \tag{7-16}$$

it finally follows that also

$$\lim_{n\to\infty} \|B_{2n}\| = 0,$$

which was to be shown.

8. Review and Conclusions

It may be helpful to the reader who has worked his way through the mathematical details of the present convergence proof, to review the main lines of the argument.

Our starting point was Brovar's integral equation for Molodensky's problem, eq. (2-3). It may be written symbolically as the operator equation (2-4),

$$(I - \Phi)\mu = \Delta g, \tag{8-1}$$

which is to be solved for μ. On introducing Molodensky's parameter k, this equation becomes

$$(I - k\Phi_k)\mu = \Delta g, \tag{8-2}$$

where the operator Φ_k is given by (2-17). A formal solution of the latter equation is the "Neumann series"

$$\mu = (I + k\Phi_k + k^2\Phi_k^2 + k^3\Phi_k^3 + \ldots)\Delta g. \tag{8-3}$$

The operators Φ_k and their n-th powers Φ_k^n still depend on k in view of (2-17). They are also expanded in series with respect to k, of the form

$$\Phi_k^{\ n} = \sum_{r=0}^{\infty} k^{2r} B_{2r}^{(n)} , \tag{8-4}$$

obtained by raising the series (2-28) to the n-th power. On substituting the series (8-4) into (8-3) and arranging with respect to equal powers of k, we obtain Molodensky's series.

Using the method of majorants, it is not difficult to see that all series (8-4) converge for $0 \le k \le 1$ provided the maximum terrain inclination β_{max} is smaller than 45° (Theorem 2-2); the required boundedness of the operators, expressed by (2-32), was proved in sec. 7.

Now Weierstrass' theorem on double series may be applied to (8-3) and (8-4) to show that Molodensky's series is convergent whenever (8-3) is (always provided $\beta_{max} < 45^\circ$). Theorem 2-1 asserts that (8-3) converges in the interval $0 \le k < k_0$ where

$$k_0 = \frac{1}{C} \tag{8-5}$$

with

$$C = \sup_{0 \le k \le 1} \| \Phi_k \| , \tag{8-6}$$

and, therefore, we have Theorem 2-3 stating the same assertion for the Molodensky series.

In this way, our problem is reduced to the determination of C. First we have put $k = 1$, investigating the norm $\| \Phi \|$. For this purpose, we have, in sec. 5, split up the operator Φ into a "strongly singular" part (5-9) and a "weakly singular" part (5-10). The first part is of such a form that Theorem 4-1 can be applied to it to give the estimate (5-15). To the second part we have applied Theorem 4-2 to obtain the estimate (6-19), and an alternative estimate, given by (6-33) with (6-27), (6-31) and (6-32), was derived using Theorem 4-3; this was the subject of sec. 6.

The comparison between (2-6) and (2-17) shows that Φ and Φ_k differ only by

$$\frac{1}{\ell^3} = \frac{1}{\ell_0^{\,3}} (1 + \tan^2\gamma)^{-\frac{3}{2}}$$

and

$$\frac{1}{\cos^2\beta} = 1 + \tan^2\beta$$

being replaced by

$$\frac{1}{\ell_k^{\,3}} = \frac{1}{\ell_0^{\,3}} (1 + k^2\tan^2\gamma)^{-\frac{3}{2}}$$

and

$$\frac{1}{\cos^2\beta_k} = 1 + k^2\tan^2\beta \quad ,$$

respectively; $\tan\gamma$ is defined by (2-24). If we re-examine the derivatives of our basic estimates (5-15), (6-19), (6-27), (6-31), and (6-32), we see that they continue to hold if Φ is replaced by Φ_k. For instance, the function (5-12) remains a majorant if (5-11) is now replaced by

$$\frac{1}{2\pi} \frac{\tan\beta_0 \cos\alpha}{(1 + k^2\tan^2\beta_0\cos^2\alpha)^{3/2}} \quad .$$

Instead of (6-1) we now have the expression

$$\frac{1}{2\pi k} \iint_\sigma \frac{1}{\ell_0^{\,2}} (\cos^3\gamma_k \tan\gamma_k - \cos^3\beta'_k \tan\beta'_k) f^* d\sigma$$

with

$$\tan\beta'_k = k\tan\beta' , \qquad \tan\gamma_k = k\tan\gamma .$$

Thus, (6-15) is now replaced by

$$A_k \le \frac{1}{4\pi}\left[1+9k^2\tan^2\beta_{max}\right]\left(\frac{\partial^2 h}{\partial s^2}\right)_{max},$$

the right-hand side of which is not greater than the right-hand side of (6-15), so that (6-15) is also an estimate for A_k. In (6-30) we have used $1/\ell_0^{\,6}$ as a majorant for $1/\ell^3$, and it is also a majorant for $1/\ell_k^{\,6}$.

Thus the estimates given in sections 5 and 6 are indeed valid for Φ_k as well as for Φ.

Let us now consider these estimates more closely, always presupposing $\beta_{max} < 45^\circ$. The estimate (5-15)

$$N_1 = \frac{1}{\pi}\tan\beta_{max}(1+\tan^2\beta_{max}), \tag{8-7}$$

is then always smaller than

$$\frac{2}{\pi} \doteq 0.64 ,$$

as we have already mentioned in sec. 6. The estimate (6-19),

$$N_2 = \frac{1+9\tan^2\beta_{max}}{\cos^5\beta_{max}}\frac{R}{\rho_{min}}, \tag{8-8}$$

will, for small β_{max}, be approximately equal to

$$N_2 \doteq \frac{R}{\rho_{min}}, \tag{8-9}$$

which can be expected to be much greater than unity since the maximum radius of curvature of a terrain profile will be much smaller than the mean earth's radius R.

Thus,

$$\|\Phi\| \le \|\Psi_1\| + \|\Psi_2\| \le N_1 + N_2 . \tag{8-10}$$

We have just seen that N_1 and N_2 are also estimates for the case $k \neq 1$, so that also

$$C = \sup \| \Phi_k \| \leq N_1 + N_2 . \qquad (8-11)$$

If we were justified in assuming that, at least approximately,

$$C = N_1 + N_2 ,$$

then k_0 as given by (8-5) would be very small since

$$N_1 + N_2 >> 1,$$

and we could assert convergence only for very small values of k and certainly not for $k = 1$, that is, for the actual earth's surface.

It must, however, be carefully noted that the $\leq$ sign, which holds exactly, cannot be replaced by an equality sign, not even approximately. As already pointed out in sec. 6, in order to get mathematically "safe" $\leq$ estimates, we have, statistically speaking, admitted biases, which may have accumulated in such a way that our estimates are not even approximately unbiased, and they may be far too great.

It would probably be much more realistic if in estimates such as (8-7) and (8-8) the extremum values β_{max} and ρ_{min} were replaced by some average values of the terrain inclination and the radius of curvature. For instance, take the proof of Theorem 4-2: in (4-30), it is mathematically "safe" to consider A as the maximum value (4-28), but it would probably result in an estimate closer to reality if A were considered as some kind of average value. A rigorous formulation and appropriate use of such averages would, however, greatly increase the mathematical difficulties.

If β_{max} and ρ_{min} were replaced by suitable averages, then the estimates (8-7) and (8-8) would probably decrease very considerably. It might even be possible in this way to get a value $C < 1$, so that by (8-5) convergence would result for $k = 1$, i.e. for the actual earth's surface.

Still, the estimate (8-8) is too large also for another reason. As (6-14) shows, it amounts to replacing $f(s)$ by a parabola const. s^2 all over the earth, which results in far too great values for larger s. Therefore, we used the decomposition (6-23), resulting in the estimate

$$C \leq N_1 + N_{21} + N_{22} + N_{23}, \tag{8-12}$$

where, by (6-27), (6-31), and (6-32),

$$N_{21} = N_2 \sqrt{\sin \frac{\psi_0}{2}} \quad , \tag{8-13}$$

$$N_{22} = \frac{\sqrt{2}}{8 \cos^2 \beta_{max}} \; \frac{\Delta h_{max}}{R} \; \frac{1}{\sin^2 \frac{\psi_0}{2}} \quad , \tag{8-14}$$

$$N_{23} = N_1 \quad , \tag{8-15}$$

N_1 and N_2 being given by (8-7) and (8-8). Here ψ_0 may be any angle such that $0 < \psi_0 < 180°$.

The main advantage in using these new estimates consists in the possibility in reducing the effect of the large term N_2 by selecting a small ψ_0. As an example, take $\psi_0 = 30'$ corresponding to a linear distance of about 55.5 km. Then

$$\sin \frac{\psi_0}{2} = 0.00436 \ ,$$

so that

$$N_{21} \doteq 0.066 \, N_2 \ ,$$

$$N_{22} \doteq 0.35 \cdot 0.0014 \cdot 229^2 \doteq 26 ,$$

if we take Δh_{max} equal to the elevation of Mount Everest, 8882 m. and $\beta_{max} \doteq 45°$

Still, these values are not close to unity, but they are probably much too high, especially N_{22}. By replacing Δh_{max} by some average Δh, we might well be able to reduce it by a factor of 100, which would be sufficient.

Another reason for our estimates being too high is the frequent replacement of quantities that change their sign, by their absolute amount which is always positive. Thus positive and negative terms instead of tending to cancel each other, add up by their absolute amounts.

As the result of these considerations we can assert that Molodensky's series converges for all $k < k_0$, where k_0 is some fixed value, but we do not know what this value is. Our present estimates give a value k_0 that is very small, but our estimates are almost certainly far too high, so that k_0 is almost certainly considerably underestimated. In particular, we do not know if not, after all, $k_0 > 1$, in which case Molodensky's series would converge at the earth's surface.

We see, however, clearly from which parameters the convergence proof depends:

1. the maximum terrain inclination, by (8-7);
2. the maximum curvature of vertical profiles, by (8-8) and (8-13);
3. the maximum elevation difference, by (8-14).

These maximum values are global; it is not sufficient to consider only a neighborhood of the computation point. We could probably replace these maximum values by suitable average values, on which, however, nothing definite can be said at this point.

Since a small k_0 corresponds to a smoothed topography (all elevations h being replaced by $k_0 h$), we may say that Molodensky's series will converge on a sufficiently smoothed topography. Some degree of smoothing is, anyway, always necessary. To avoid mathematical difficulties, it is convenient to assume the earth's topographic surface to be an analytical surface; this is always justified, cf. (Moritz, 1971, p. 52). It is also necessary to smooth out vertical slopes ($\tan \beta = \infty$), sharp ridges (infinite curvature), etc.

Thus one may even say that the question of convergence of Molodensky's series on the earth's surface becomes meaningful only after this surface has been replaced by a smooth analytical surface. By the precise but overly pessimistic estimates given in the present report, convergence is warranted only for a topography smoothed beyond a reasonable extent, but it is not ruled out even for a more "realistic" topography.

Convergence, however, is not a prerequisite for a successful practical use of Molodensky's series, as we have repeatedly pointed out; cf. (Moritz, 1969, Appendix; Moritz, 1971, sec. 12).

Since the series solution found by analytical continuation (Marych, 1969 a,b; Moritz, 1969, 1971) is equivalent to Molodensky's series, its convergence behavior is also described by the results of the present report.

The same holds for the iterative solution of Brovar's integral equation given in (Moritz, 1966, equations (143a,b)), which is equivalent to the Neumann series (8-3).

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13. ABSTRACT

Molodensky's solution of the geodetic boundary-value problem consists in an asymptotic series expansion with respect to a parameter k. It is shown that this series converges for sufficiently small values of k. The method starts from an integral equation given by Brovar and uses a Neumann series solution of this equation; the norms of the occuring singular integral operators are suitably estimated.

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