FIBONACCI AROUND THE CLOCK

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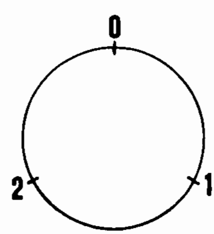
"One o'clock, one o'clock, two o'clock rock. Three o'clock, five o'clock, eight o'clock rock. We're gonna Fibonac around the clock!"

Problem solving is a struggle for most students. By making problems more interesting and by showing students how to use problem solving strategies, teachers can turn confusion into confidence. Fibonacci numbers can help do just that. Activities with the Fibonacci numbers illustrate many problem solving strategies, such as using a formula, looking for patterns, considering various cases, making a list, and developing an algorithm.

When introduced to Fibonacci numbers for the first time, students can apply the problem solving strategy of looking for a pattern. Show the students the first few numbers of the Fibonacci sequence (1 , 1 , 2 , 3 , 5 , 8 , ...), and let them tell you in words how these numbers are formed. A student should say something like "The sequence starts with two ones and each number that follows is the sum of the two previous numbers." Next, ask the students to express the same idea by means of a formula. Here is an opportunity to introduce subscript notation. Let F_n be the number we want to find, where n is the number of the term in the sequence. Then, F_{n-1} and F_{n-2} are the two previous numbers and $F_n = F_{n-1} + F_{n-2}$. Have the students use this formula to generate the next several numbers in the sequence (... 13 , 21 , 34 , 55 , 89 , ...) .

In order to investigate some additional patterns involving the Fibonacci numbers, we introduce modular arithmetic, better known to some as "clock arithmetic". Modular arithmetic involves dividing numbers by an integer, called the modulus, and using the remainders, called residues. For example, we will examine the number 7 (modulo 3). Three divides 7 two times with a remainder of 1. This can be demonstrated on a 3-hour clock or circle. Divide the circle into thirds, and label the divisions 0, 1, and 2.



To count to 7 on this circle, starting at zero, you would go around the circle two times and 1 more tic mark. So we say that 7 is congruent to 1 modulo 3 , written $7 \equiv 1 \pmod 3$. After the students are familiar with modular arithmetic, ask them to generate the Fibonacci sequence using different moduli and look for new patterns. By making a list, they can see the patterns more easily.

Examples:

$F_n \pmod 2$ 1 , 1 , 0 , 1 , 1 , 0 , ...
$F_n \pmod 3$ 1 , 1 , 2 , 0 , 2 , 2 , 1 , 0 , 1 , 1 , 2 , 0 , 2 , 2 , 1 , 0 , ...
$F_n \pmod 4$ 1 , 1 , 2 , 3 , 1 , 0 , 1 , 1 , 2 , 3 , 1 , 0 , ...
$F_n \pmod 5$ 1 , 1 , 2 , 3 , 0 , 3 , 3 , 1 , 4 , 0 , 4 , 4 , 3 , 2 , 0 , 2 , 2 , 4 , 1 , 0 , ...

The students should notice that the sequence starts to repeat whenever the terms 1 , 0 occur in succession. After 3 terms, the mod 2 sequence repeats, after 8 terms mod 3 repeats, 6 terms are needed for mod 4, and after 20 terms for mod 5. The sequence of residues modulo n between repetitions is called the cycle. The number of terms it takes to complete a cycle is called the period. For example, the period for the Fibonacci numbers modulo 5 is 20. What else might be noticed? How many times in each cycle does zero appear? In each cycle of $F_n \pmod 2$ zero occurs once, mod 3 twice, in mod 4 once, and mod 5 four times. After generating the sequence for more moduli, students might hypothesize that, for any modulus, the cycle will contain 1 , 2 , or 4 zeros only. This exercise helps students develop skills of looking for patterns, making lists, and formulating hypotheses.

To further investigate this topic, students can use a computer. Modern technology provides terrific teaching tools for mathematics classrooms. First challenge the students to develop a computer algorithm to generate the Fibonacci sequence, then to modify their algorithm to compute the sequence modulo m for a user-specified value of m. The following program stops once the sequence has come to the end of its cycle.

Sample GW – BASIC PROGRAM

```
1 REM FIBONACCI NUMBERS MOD M
10 FOR M=2 TO 25
20 FB1=1
30 FB2=1
40 PRINT "FIBONACCI NUMBERS MOD" M
50 PRINT FB1;" ";FB2;
```

```
60 FB3=FB2+FB1
70 FM3=FB3 MOD M
80 PRINT " ";FM3;
90 IF FM3=0 AND FB2=1 THEN GOTO 130
100 FB1=FB2
110 FB2=FM3
120 GOTO 60
130 NEXT M
140 END
```

To have the computer print the period (length of each cycle) insert the following lines.

```
35 P=2
55 P=P+1
90 IF FM3=0 AND FB2=1 THEN GOTO 125
125 PRINT "  Period  =  ";P
```

For those using Applesoft Basic, the following changes need to be made to the program.

```
65 GOSUB 1000
(Delete line 70.)
1000 IF FB3 < M THEN GOTO 1040
1010 FB3=FB3-M
1020 GOTO 1000
1040 FM3=FB3
1050 RETURN
```

With the help of these computer programs, students can quickly generate various cases, where they will see patterns appear and be able to predict the period for a specific modulus. For example, if the modulus is a power of a prime, we get the following results:

Powers of 2	Period
mod 2	3
mod 4	6
mod 8	12
mod 16	24

What will be the period of mod 32? Students will likely guess 48, which is 2 times the period for the preceeding power of 2.

Powers of 3	Period
mod 3	8
mod 9	24
mod 27	72

What will be the period of mod 81? The period is 216, which is 3 times 72. Look at the powers of mod 5.

Powers of 5	Period
mod 5	20
mod 25	100
mod 125	500

What will be the period of mod 625? The period is 2500, which is 5 times 500. Have the students formulate conjectures about the periods of Fibonacci numbers (mod m) where $m=p^k$ for a prime number p. A good hypothesis might be that the period equals p times the period for F_n modulo the preceeding power of p. Also note that, equivalently this is p^{k-1} times the period for F_n(mod p). Have them examine the Fibonacci numbers modulo 7 and 11 to test their hypothesis. Remind the students that testing cases may support a conjecture, but only a proof covers all cases.

In summary, students have prepared a table of values by hand, written a computer program to check and extend that table, learned useful algebraic notation, and conjectured a generalization to describe their findings. There are many problem solving skills to be learned by students as they Fibonac Around the Clock!

"Cauchy, there are limits to your knowledge of real numbers," Dedekind said cuttingly.

Mathematical Maxims and Minims
Compiled by Nicholas J. Rose
Rome Press, Inc., 1988, p. 135

Answers to "Hatching Answers" by William H. Kraus, in Issue No. 20, Winter 1991:

7 Wonders of the World
88 Piano Keys
52 Weeks in a Year
40 Days and Nights of the Great Flood
9 Planets in the Solar System
1 Wheel on a Unicycle
50 Ways to Leave Your Lover
2 Scoops of Raisins in Kellogg's Raisin Bran
7 Colors of the Rainbow
18 Wheels on a Semi
9 Digits in a Social Security Number
12 Signs of the Zodiac
24 Hours in a Day
10 Little Indians
8 Sides on a Stop Sign
64 Squares on a Checkerboard
29 Days in February in a Leap Year
13 Doughnuts in a Baker's Dozen
5 Digits in a Zip Code
2 All Beef Patties on a Sesame Seed Bun
4 Beats in a Whole Note
8 Notes in an Octave

"The only Americans who have ever accepted the metric system are the dope dealers. Here are guys who probably couldn't get a D in grade-school math, and they're converting grams to ounces to kilos at the bat of an eye."

What Cops Know
by Connie Fletcher
Villard Books, 1991

MATH SCRAMBLER answers: METER FIRST SQUARE NINETY

"That was SUM TEST !"