CIRCLES IN NEW WORLDS

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Introduction

Circles are defined in middle school geometry by the "compass construction". That is, we mark off all of those points on a piece of paper which are at a fixed distance (the radius) from a given point (the center). The diameter is defined as twice the radius and, by grade 6, the ratio of the circumference (distance around the circle) to the diameter is shown to be about 3. The number pi (or π) is introduced and the equation C = pi × d (the circumference is pi times the diameter) is then explained.

The main mathematical idea of this presentation is that the results of the above paragraph are keyed to the "usual" (Euclidean) distance. A circle, via its compass definition, is defined as all points at a fixed distance from a given point and so "distance" is a crucial concept.

The main pedagogical idea of this paper is that "circles in new worlds" can be used both to excite the imagination of middle and high school children and teachers and deepen their understanding of basic concepts such as circle, diameter, circumference, what pi really is and why it is important. If different kinds of distances are used then the resulting "circles" look quite different.

The author has presented this material to inservice middle school teachers (at the Wright State University Winning in Science Education program), a general math class at the tenth grade level, and to a class of secondary mathematics education majors at Wright State University.

In the sections to follow, we will describe the mathematics of the geometries, explore its consequences for the shape of circles and the "value of pi", and describe further directions and related work. The two geometries which will be discussed use the "taxicab" and "max" distances.

The Taxi and Max Distances

The taxicab distance between two points is the distance that it would take a taxi to drive from one point to another along streets arranged in a grid. In terms of a formula, the taxi distance is the sum of the x change and the y change between the two points. Thus the distance between (1, 2) and (3, 5) is (3–1) + (5–2) = 5. If A = (a, b) and C = (c, d) then the x change is |a–c| and the y change is |b–d| so the algebraic expression for the taxi distance, d_t (A, C) is:

$$d_t(A, C) = |a-c| + |b-d|$$

Graphically, it can be interpreted as the distance a taxicab travels, as can be seen from Figure 1. It is called the Manhattan distance because of the way in which one travels in Manhattan.

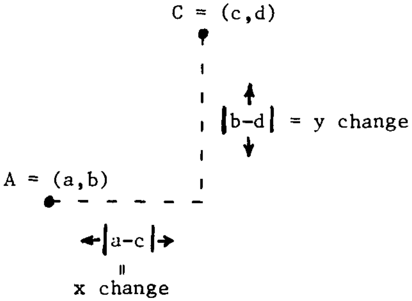

Figure 1: Taxi Distance

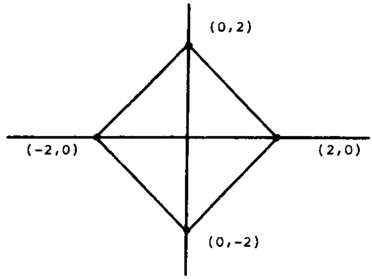

Figure 2: Taxi Circle

In presenting this material, I give some examples to the class and then ask them to do the following in-class exercise.

Exercise 1: Find the taxi distance from (0, 0) to

a) (2, 0)	d) (0, 2)	g) (1/4, 1 3/4)
b) (1, 1)	e) (-1, -1)	h) (0, -2)
c) (-1/2, -1 1/2)	f) (1/2, 1 1/2)	i) (1/5, 1 4/5)

Since the answer is 2 to each of the parts of the question, the students are now convinced that we have something that could be a "circle". The "taxi circle" is graphed in Figure 2. (It is important to call this figure a "taxi circle" as opposed to a "circle" because students just do not have the appreciation for abstraction yet that allows them to visualize a circle as in Figure 2).

The second distance that we explore is called the max distance because it deals with the maximum of two quantities. Geometrically the <u>max distance</u>

between two points is the larger of the x change and the y change between the two points. For example, the max distance between S = (-1, 5) and T = (1, 2) is found as follows. The x change is $|-1 - 1| = 2$. The y change is $|2 - 5| = 3$. The larger of these numbers is 3 and so the max distance between S and T is 3. If A = (a, b) and C = (c, d) then the <u>max</u> <u>distance</u>, d_m, is defined as

$$d_m(A, C) = \text{Maximum of } \{|a - c|, |b - d|\}$$

Other examples can easily be constructed to give the students facility with computing the max distance.

What interpretation can we give to the max distance? Suppose that you were at the point P = (x, y) and want to get to the origin. If there were a fast bus which ran on the x-axis and another one which ran along the y-axis, then you would want to go to one of the axes in order to take the bus. The longer distance to the bus is then the maximum of $|x|$ and $|y|$, i.e. $d_m(P, (0, 0))$. A max circle of radius 2 is the set of points at max distance 2 from a given point.

The following exercise can be given to have the students discover the "max circle".

Exercise 2: Find the max distance from (0, 0) to each of the following points.

a) (2, 0)
b) (-2, 1)
c) (1.2, -2)
d) (0, 2)
e) (2, 1.3)
f) (0.4, -2)
g) (0, -2)
h) (-2, -1)
i) (-1.2, 2)

Since the answer to each question of the exercise is once again 2, the students get the idea that this is a circle in the max distance. The "max circle" sure does look like (and is!) a Euclidean square. See Figure 3.

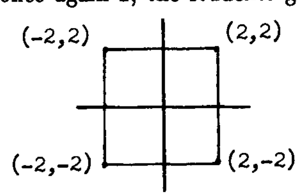

Figure 3: A Max Circle

<u>The Relationship Between Circumference and Diameter of Circles</u>

While the reason for the astounding nature of pi is a matter of taste, as a geometer, I claim that the fact that it is a constant is in itself a miracle. Although our textbooks do not define it as such, really pi is the ratio of the circumference of a circle to its diameter or the ratio of the area of a circle to the square of its radius. There really is no reason to expect in advance that this ratio will not depend on the circle chosen. Why should it be that a circle of radius 1 has the same circumference to diameter ratio as a circle of radius 2? This miracle should be emphasized more in our middle and high schools. Students who have studied regular polygons might be aware of this fact if they think of a circle as a (very) regular polygon. For example, every regular hexagon has a perimeter equal to three times its diameter.

The idea of pi as a ratio can be explored naturally in the setting of the new worlds which was described above. What is the ratio of the circumference of a taxi (or max) circle to its diameter? We shall compute that ratio in a number of cases for both the taxi and max geometry and see that it is independent of the "circle" chosen. (The answer is 4 in both cases). This does not mean the value of pi changes (it is still 3.1415...) but rather that there is a different constant in these other geometries. We shall put pi in quotes ("pi") when referring to the other geometries.

We will first do the computation for the taxi circle of radius 2 about the origin (Figure 2). To find the circumference we need to know the length of each of the four straight line segments of the circle. The length of the one in the first quadrant is the distance from (0, 2) to (2, 0). Now comes the key point: what distance do we use to find the length? Since we are in the taxi geometry, the distance that is used should not be the usual Euclidean distance but rather the taxi distance! Thus:

$$\text{Length of first segment is } d_t\,(\,(0, 2), (2, 0)) = 2 + 2 = 4.$$

It is now easy to see that all four segments have the same length which is 4 and so the circumference is 16. If the "diameter" is defined to be twice the "radius" then

the diameter of the circle of radius 2 is 4.

The ratio of the circumference to the diameter of this taxi circle is therefore 16/4 = 4. Have the students in your class try circles of other radii and you will see that this always happens; that is, "pi" is 4 in the strange new world of taxi geometry. In fact it is not hard to prove this fact using a little high school algebra.

We now repeat the computation of the ratio of the circumference of a "circle" in the max distance to its diameter. The max circle centered around the origin with radius 2 was sketched in Figure 3. The first step is to find the distance from (2, 2) to (2, -2):

$$d_m((2, 2), (2, -2)) = \text{maximum of } 0 \text{ and } 4 = 4.$$

Since each of the segments has the same length, the entire circumference is 4(4) = 16. On the other hand the "diameter" is, as always, twice the "radius" which is 4. The value of "pi" for the max geometry is 16/4 = 4. Once again, other "circles" can be experimented with and the same results will hold.

Further Directions and Related Work

Many of the geometric constructions that we do in our standard geometry classes carry over directly to the new worlds of taxi and max geometry. These concepts can be assigned as an enrichment topic for the most inquisitive students. One of my favorites is to ask for the set of all points equidistant from two points. The students understand that the notion of distance is what is important by this time. In the taxi and max distances, we get back our old Euclidean friend, the perpendicular bisector.

On the other hand, what is the set of all points at distance 2 from two fixed points? In Euclidean geometry, the answer depends on the location of the points. We would draw circles of radius 2 about those points and then see where they intersect. This could be in zero, one or two points. See Figure 4.

Figure 4: Points common to two Euclidean circles of radius 2

When we try to follow the same procedure for the max distance, it is still possible to get zero or one point of intersection, as in Figure 5, but there is also a strange possibility. The circles of radius 2 about both (-2, 0) and (2, 0) are sketched in Figure 5. Note that they meet in infinitely many points! Ask your students to experiment with the taxi cab distance.

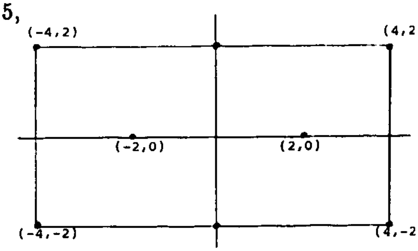

Figure 5: Points common to two max circles of radius 2

Another exercise which is intriguing to better students is the actual algebratization of the geometry. By "algebratization", I mean that we turn the subject into analytic geometry just as Descartes would have meant it. We write out the equations that govern the lines which describe the taxi circle. This gives extra practice in algebra and especially absolute values. It is also a way to provide proofs for various statements made in this article. The key is contained in Exercise 3. (I would not recommend assigning it except to outstanding students unless there were hints given).

Exercise 3: Find the equation for a circle of radius 2 about the origin in the taxi geometry and sketch the resulting equation.

Solution: The taxi distance from the origin (0, 0) to the arbitrary point P = (x, y) is

$$d_t\,(P, (0, 0)) = |x - 0| + |y - 0| = |x| + |y|.$$

Thus, the equation of a taxi circle of radius 2 about the origin is

$$|x| + |y| = 2 \tag{1}$$

In order to graph equation (1), we split it into four cases. If $x \geq 0$ and $y \geq 0$ then equation (1) is $x + y = 2$. In this case, however, the condition that $x \geq 0$ and $y \geq 0$ forces $2 \geq x$ and $2 \geq y$. (If x were bigger than 2 then y would be negative.) If $x \geq 0$ and $y \leq 0$ then equation (1) becomes $x - y = 2$ and $-2 \leq y \leq 0$, all four cases are

2(a)	$x + y = 2$	$0 \leq x \leq 2;$	$0 \leq y \leq 2$
2(b)	$x - y = 2$	$0 \leq x \leq 2;$	$-2 \leq y \leq 0$
2(c)	$-x + y = 2$	$-2 \leq x \leq 0;$	$0 \leq y \leq 2$
2(d)	$-x - y = 2$	$-2 \leq x \leq 0;$	$-2 \leq y \leq 0$

Equation (1) is equivalent to the four equations (2). Each of them is a line segment. By graphing (2), Figure 2 reappears.

Taxicab and max geometries are not new. They appear in topology as a useful counterexample to theorems about metrics. Taxi geometry has been discussed in connection with mathematics education in Byrkit (1). His approach and content are completely different from ours. His emphasis is on the formulation of axioms to present a non-Euclidean, geometric system whereas ours is on distance. His version is discrete (using only those points in the plane with both coordinates integers) whereas ours uses all of the points of the plane. After establishing his axioms he then defines lines (which he calls <u>trips</u>) and discusses parallelism. He then shows the marvelous fact that the geometry has infinitely many lines (by his definition) through a point parallel to a given line. (Compare with Euclid's Fifth Postulate!)

Care must be taken when making assertions about distance related phenomena in the taxi geometry. For example, Byrkit points out (p. 421) that the usual triangle inequality ("the sum of the lengths of any two sides of a triangle is greater than the length of the third") is false in taxi geometry. The statement is correct if "greater than or equal to" is substituted for "greater than". Ask your students to try this on the triangle whose vertices are at A = (0, 0), B = (5, 4) and C = (5, 0).

Clearly there are many other exploration possibilities. Taxis are not only the right way to travel in a big city but also provide travel down an exciting road to geometry.

Bibliography

Byrkit, Donald. "Taxicab Geometry: A Non-Euclidean Geometry of Lattice Points." Mathematics Teacher. 64 (May 1971), 418-422.

Millman, Richard and George Parker. Geometry: A Metric Approach with Models. New York, NY: Springer-Verlag, 1981.

Answers to the circle puzzle on p. 21:

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    0   3              2   0              3   0
8 5  7 4  6 9      7 8  5 6  1 9      9 1  2 8  6 7
    2   1              4   3              5   4


    1   3              4   1              3   5
4 0  6 7  5 8      8 0  2 5  3 9      4 0  8 9  1 2
    9   2              7   6              7   6
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