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Institute of Geodesy, Photogrammetry and Cartography

Report No. 38

On the Accuracy of Geoid Heights and Deflections of the Vertical

by

E. Groten
and
H. Moritz

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
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Contract No. AF 19(628)-2771
Project No. 7600
Task No. 760002

The Ohio State University
Research Foundation
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March 1964

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INSTITUTE OF GEODESY, PHOTOGRAMMETRY AND CARTOGRAPHY

W. A. Heiskanen, Director

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FOREWORD

This report was prepared by Dr. H. Moritz and Mr. E. Groten, Research Associates, of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF 19 (628)-2771, OSURF Project No. 1613, under the supervision of Dr. Weikko A. Heiskanen, Director of the Institute. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen Williams and Mr. Bela Szabo, Project and Task scientists.

ABSTRACT

The accuracy of the geoid undulations and the deflections of the vertical, obtainable by means of different idealized gravity nets, is investigated. The following cases are assumed: one gravity station or one gravity profile in every $1^\circ \times 1^\circ$, $2^\circ \times 2^\circ$, $5^\circ \times 5^\circ$, and $10^\circ \times 10^\circ$ blocks.

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ON THE ACCURACY OF GEOID HEIGHTS AND DEFLECTIONS OF THE VERTICAL

BY E. GROTEN AND H. MORITZ

A. INTRODUCTION

Previous work on this subject was done by Hirvonen [3] and Kaula [5]. They estimated the accuracy obtainable from the present gravity material and from idealized nonuniform distributions of gravity stations respectively.

The purpose of this report, however, is to estimate the accuracy obtainable by uniform coverage of the whole earth by gravity measurements. Point and profile measurements of different density are considered. This is necessary for a rational planning of gravity surveys: given a certain accuracy standard, the observational density necessary to achieve it can be found.

Similar investigations for the accuracy of spherical harmonics were made by Kaula [6] and Moritz [12]; related work is also found in [4] and [10].

The mathematical apparatus for such problems is described in [9], [10], or [12]. The difficulty in the present problems is the singularity of Stokes' and Vening Meinesz' functions at the origin, which makes somewhat delicate manipulations necessary.

The problem is also solved by means of spherical harmonics developements, making extensive use of ideas of Molodensky [8].

B. ACCURACY OF GEOID HEIGHTS

B1. PRINCIPLE.

The principle is straightforward. Take Stokes' formula:

$$N = \frac{R}{4\pi G} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \Delta g(\psi,\alpha)\, S(\psi) \sin\psi \, d\psi \, d\alpha \qquad (1)$$

where

N: geoid height,

Δg= gravity anomalies,

R = 6371 km: mean radius of the earth,

G = 980 gal: mean gravity of the earth,

ψ,α: spherical polar coordinates,

$S(\psi)$: Stokes' function.

The formula for error propagation [10] immediately gives

$$M^2(N) = \frac{R^2}{16\pi^2G^2} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi,\alpha,\psi',\alpha')S(\psi)S(\psi')\sin\psi \sin\psi' \, d\psi d\alpha d\psi' d\alpha' \tag{2}$$

where

$M(N)$: standard error of N

$\sigma(\psi,\alpha,\psi',\alpha')$: error (covariance) function of Δg.

If the error function of Δg is given, M (N) can be computed from (2). The rigorous evaluation of this fourfold integral is unreasonably laborious; it would involve an excessive amount of high speed computation.

However, it is possible to achieve a drastic simplification by considering that the error function σ is practically zero except for points (ψ,α) and (ψ',α') that are rather close together, because the gravity anomalies at two distant points can be considered to be uncorrelated. Thus we can approximately replace $S(\psi')$ by $S(\psi)$. This is, of course, wrong if ψ' is much different from ψ, but then the integrand is almost zero anyway.

Then (2) becomes .

$$M^2(N)= \frac{R^2}{16\pi^2G^2} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi,\alpha,\psi',\alpha')[S(\psi)]^2 \sin\psi \sin\psi' d\psi d\alpha d\psi' d\alpha'$$

Now the integration over ψ' and α' can be performed beforehand; we put

$$R^2 \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi,\alpha,\psi',\alpha') \sin\psi' \, d\psi' \, d\alpha' = L \tag{3}$$

L is the so called error integral. In [11] and [12] we have denoted it by S; here we use L in order to avoid confusion with Stokes' function.

Note that L is a constant if the error function σ only depends on the relative position of the points (ψ,α) and (ψ',α'); this is the case of uniform accuracy all over the earth, which we shall assume in this paper.

So we get

$$M^2(N) = \frac{L}{16\pi^2 G^2} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} S^2(\psi) \sin\psi \, d\psi \, d\alpha$$

Performing the integration over α finally yields

$$M^2(N) = \frac{L}{8\pi G^2} \int_0^{\pi} S^2(\psi) \sin\psi \, d\psi \tag{4}$$

Thus we have reduced our problem to a simple and practicable formula. Unfortunately $S^2(\psi)\sin\psi$ is infinite at $\psi = 0$; the formula (4) yields the value ∞, which is obviously wrong.

B2. ELIMINATION OF THE CENTRAL ZONE: EFFECT OF THE OUTER ZONE.

The reason for this unpleasant behavior is the singularity of Stokes' function at the origin. The approximate substitution of $S(\psi)$ for $S(\psi')$ evidently works only if for small $\psi' - \psi$ also the differences $S(\psi') - S(\psi)$ are small. This is the case everywhere except at the origin where $S(\psi)$ is discontinuous.

We must therefore exclude the singularity by leaving out the central zone $\psi \leqq \psi_0$ for a certain fixed ψ_0. Then (4) becomes

$$m^2(N) = \frac{L}{8\pi G^2} \int_{\psi_0}^{\pi} S^2(\psi) \sin\psi \, d\psi \tag{5}$$

This gives the standard error of N if inside the central circle $\psi<\psi_o$ we have an ideally dense gravity net, so that Δg can be considered errorless there; for $\psi>\psi_o$ we have uniform coverage with error covariance function $\sigma(\psi,\alpha,\psi',\alpha')$.

Formula (5) is accurate except for very small ψ_o, where it gives somewhat too large values, as is seen on closer inspection.

Because the complicated function of Stokes

$$S(\psi) = \frac{1}{\sin\frac{\psi}{2}} + 1 - 6 \sin \frac{\psi}{2} - 5 \cos \psi - 3 \cos \psi \, \ln \left(\sin \frac{\psi}{2} + \sin^2 \frac{\psi}{2}\right)$$

and also $S(\psi) \sin \psi$ have been tabulated [7], it is practical to compute the integral

$$J \equiv \int_{\psi_o}^{\pi} S^2(\psi) \sin\psi \, d\psi \qquad (6)$$

by numerical integration:

$$J = \Sigma \frac{y_i+y_{i+1}}{2} \Delta\psi_i$$

where

$$y_i = S(\psi_i) \cdot S(\psi_i) \sin \psi_i$$

$$\Delta\psi_i = \psi_{i+1} - \psi_i$$

In Table 1 values for the above sum J, from $\psi = \pi$ down to $\psi = \psi_o$, are given. The following intervals $\Delta\psi$ were taken:

$180^\circ < \psi < 10^\circ$	$\Delta\psi = 5^\circ$
$10^\circ < \psi < 2^\circ$	$\Delta\psi = 0.5^\circ$
$2^\circ < \psi < 0.1^\circ$	$\Delta\psi = 0.1^\circ$

Table 1

The integral $J(\psi_0) = \int_{\psi_0}^{\pi} S^2 \sin\psi \, d\psi$ is obtained by numerical integration.

ψ_0	$J(\psi_0)$	ψ_0	$J(\psi_0)$
180°	0.0	30°	5.2
165°	0.3	15°	7.4
150°	0.9	10°	9.7
135°	1.4	8°	10.9
120°	1.6	6°	12.8
105°	1.6	4°	15.2
90°	2.2	2°	19.0
75°	3.3	1°	22.4
60°	4.5	0.5°	25.6
45°	5.1	0.1°	32.7

Fig. 1 shows a graphical representation of J as a function of ψ_0. Note that the function $J(\psi_0)$ hardly changes in the zones between 30° and 50° and between 110° and 130°. This corresponds to the fact that Stokes' function has zeros near 40° and 120° and indicates that gravity anomalies in these zones have little influence on the geoid height ([2], p. 280).

Molodensky ([8], p. 157) gives an explicit evaluation of the integral (6):

$$\int_{\psi_0}^{\pi} S^2(\psi)\sin\psi d\psi = \pi^2 + 12 \sum_{r=1}^{\infty}(-1)^r \frac{t^r}{r^2} - 8\ln 2 - 6 \ln^2 t -$$

$$-12 \ln t + (6-18t^2+36t^4-24t^6)\ln^2(t+t^2)+$$
$$+(8+12t-24t^2-56t^3+84t^4+48t^5-64t^6)\ln(t+t^2)+$$
$$+(12t+2t^2-\frac{188}{3}t^3+30t^4+64t^5-\frac{136}{3}t^6).$$

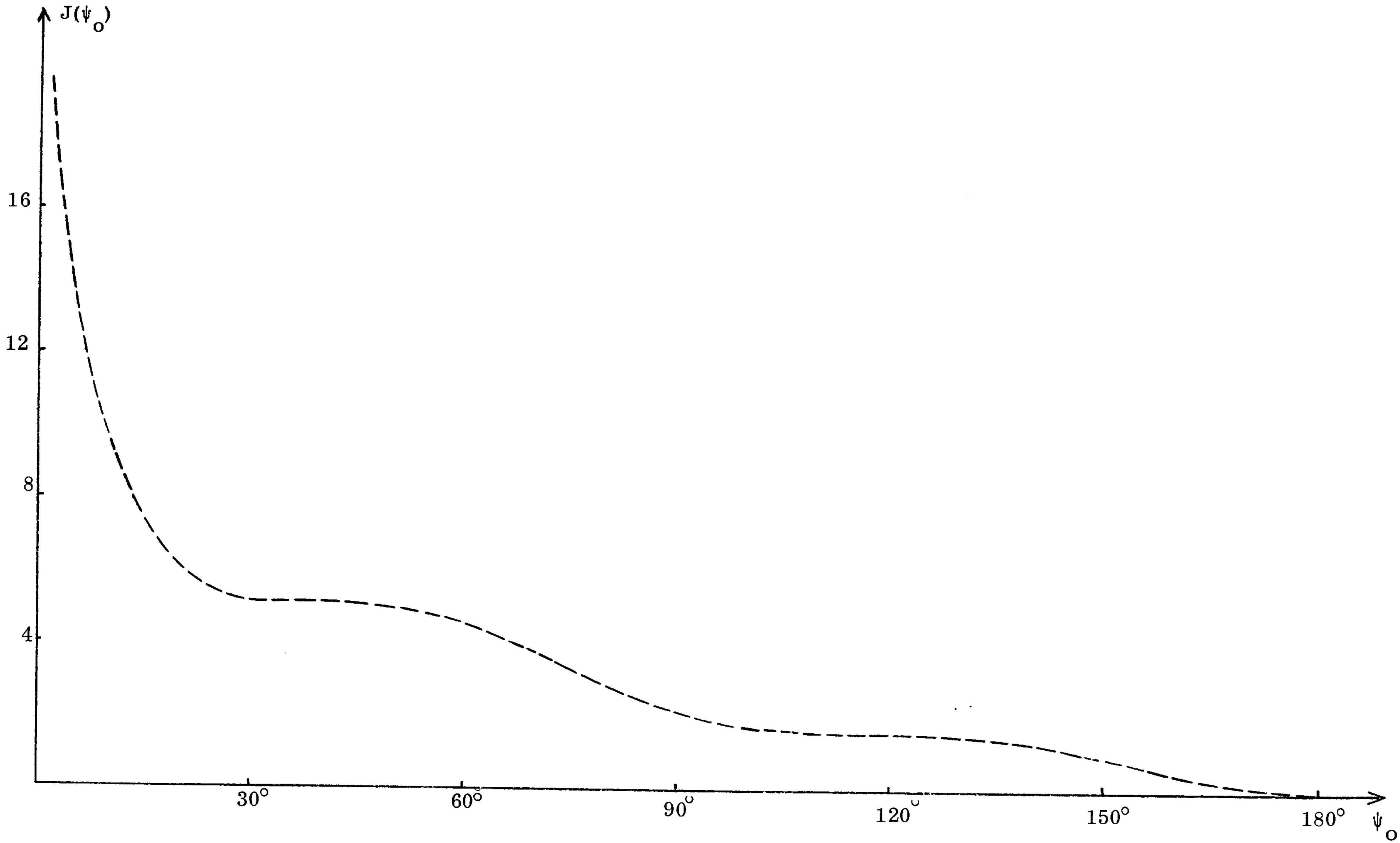

Figure 1

The integral $J(\psi_o) = \int_{\psi_o}^{\pi} S^2 \sin\psi \, d\psi$ as a function of ψ_o

where

$$t = \sin \frac{\psi_o}{2}$$

This formula can be used as a check of the above numerical integration; for $\psi_o = 1^\circ$ we get J = 22.3, which agrees well with the corresponding value 22.5 of Table 1.

Numerical values of the error integral L, or the quotient L/R^2 (R=6371 km), can be found in [11], p. 14. From Table 3 of this paper we extract the values of Table 2:

Table 2

$\frac{L}{R^2}$ in mgal^2

Block	$1^\circ \times 1^\circ$	$2^\circ \times 2^\circ$	$5^\circ \times 5^\circ$	$10^\circ \times 10^\circ$
Point	0.040	0.52	4.5	18.7
Profile	0.027	0.21	1.2	2.5

These values of L/R^2 were computed on the assumption that there is, over the whole earth, one gravity station or one gravity profile in every rectangular area of the size of a $1^\circ \times 1^\circ$, $2^\circ \times 2^\circ$, $5^\circ \times 5^\circ$, or $10^\circ \times 10^\circ$ block in 45° latitude (heading "Block"). For the values in the line headed "Point," the gravity station was assumed to be located arbitrarily within the rectangle ("Average Representation" of [11], Table 3). For the line headed "Profile," an east-west gravity profile through the center of the block was assumed (Representation" of [11], Table 3).

Figures 2 and 3 show the standard errors m(N) of the geoid height N as derived from eq. (5), plotted against the radius ψ_o. They repre-

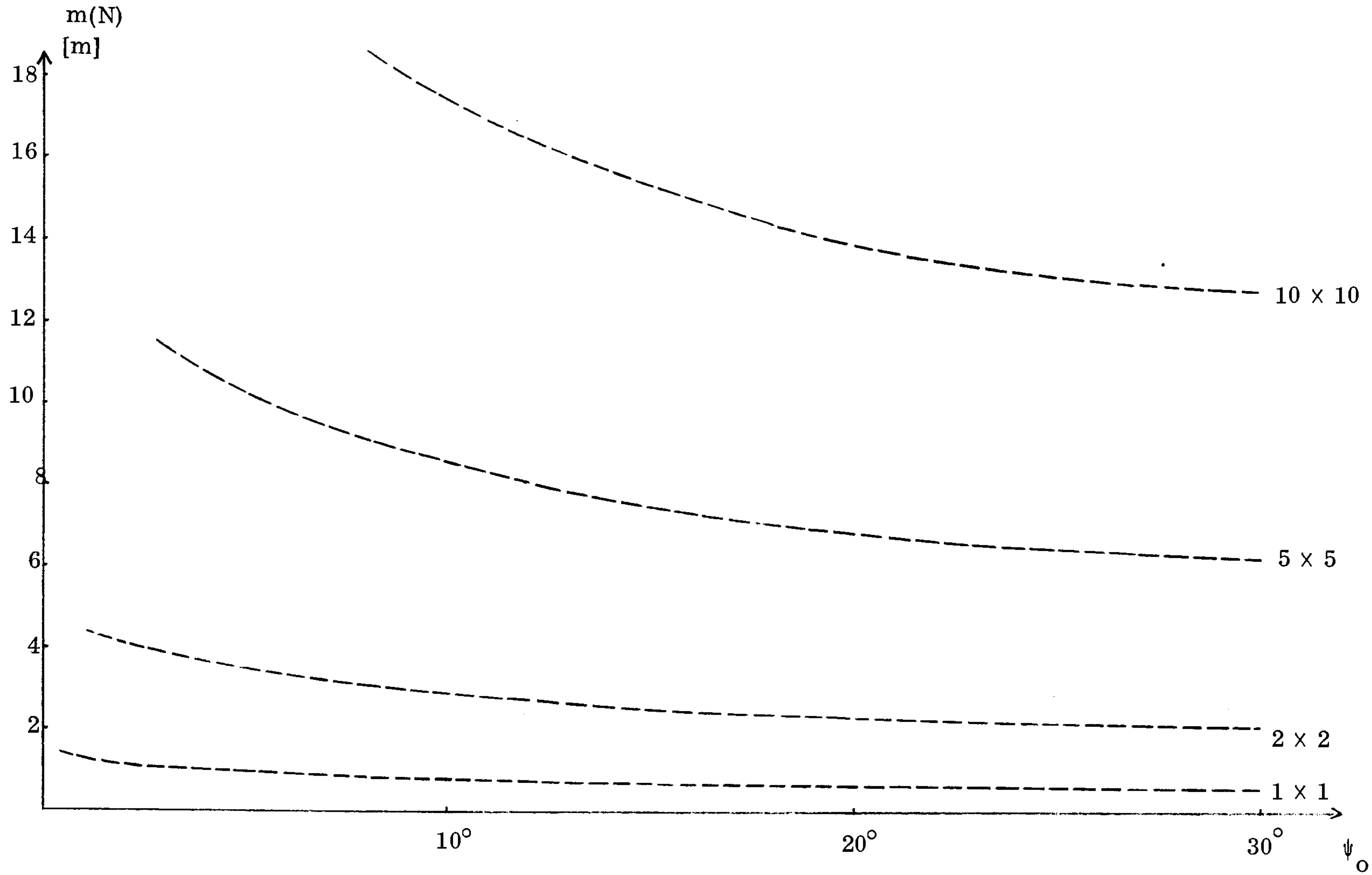

Figure 2

m(N) from point measurements as a function of ψ_o.

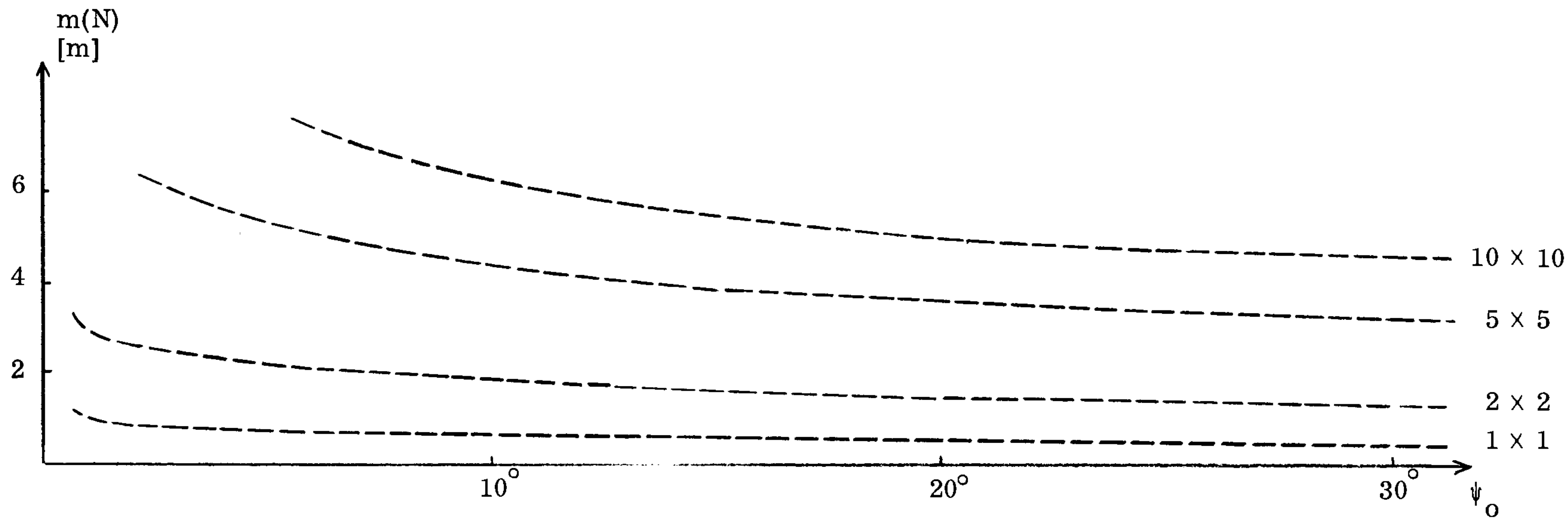

Figure 3

m(N) from profile measurements as a function of ψ_o.

sent the effect of the zone outside a circle of the small radius ψ_o.

B3. EFFECT OF THE CENTRAL ZONE.

According to eq. (1), the effect of the central zone ($\psi<\psi_o$) on the geoid height N is given by

$$\Delta N = \frac{R}{4\pi G} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_o} \Delta g(\psi,\alpha)\; S(\psi)\; \sin\psi \; d\psi d\alpha \, .$$

Since for small ψ,

$$S(\psi) \doteq \frac{2}{\psi}, \quad \sin\psi \doteq \psi$$

we have

$$\Delta N = \frac{R}{2\pi G} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_o} \Delta g(\psi,\alpha) d\psi d\alpha$$

If we can assume Δg to be approximately constant within the circle $\psi' = \psi_o$, we have

$$\Delta N = \frac{R\psi_o}{G} \Delta g$$

and therefore

$$m^2(\Delta N) = \frac{R^2 \psi_o^2}{G^2} m^2 (\Delta g) \tag{7}$$

or $$m^2(\Delta N) = 0.0129 \; \psi_o^2 \; m^2(\Delta g)$$

[$m(\Delta N)$ in meters, ψ_o in degrees, and $m(\Delta g)$ in mgals].

For numerical evaluation of this formula we need $m(\Delta g)$, the (average) standard interpolation error of Δg. We cannot be too far wrong if we identify the average error variance $m^2(\Delta g)$ with the error variance of the average block anomaly which we have denoted by $\sigma[0,0]$ in [11]. From [11], Tables 1 and 2, we take the error variances $\sigma[0,0] \doteq m^2(\Delta g)$ of Table 3:

Table 3

Error variances $\sigma\,[0,0] \doteq m^2(\Delta g)$ in $mgal^2$

Block	$1^\circ \times 1^\circ$	$2^\circ \times 2^\circ$	$5^\circ \times 5^\circ$	$10^\circ \times 10^\circ$
Point	356	609	825	934
Profile	80	165	180	125

The arrangment of this table corresponds to that of Table 2.

B4. TOTAL STANDARD ERROR OF N: CENTRAL AND OUTER ZONE COMBINED.

As the standard errors due to the outer and the central zones are only weakly correlated, we may neglect the correlation altogether, finding by the usual formula of error propagation

$$M^2(N) = m^2(N) + m^2\ (\Delta N). \qquad (8)$$

Here we have denoted the total standard error of N (central and outer zone combined) by M(N).

M(N) should obviously be independent of the way how the sphere is divided into a central and an outer zone, i.e., independent of ψ_o. Both m(N) and m(ΔN) depend on ψ_o, by (5) and (7), but if we combine them according to (8), the result should not depend on ψ_o.

If we actually add $m^2(N)$ according to (5) and $m^2(\Delta N)$ according to (7), we do not get a constant. The situation is shown in Fig. 4. The full lines represent the true functions $\underline{m}^2(N)$ and $\underline{m}^2(\Delta N)$; their sum $\underline{M}^2(N)$ is indeed constant(horizontal line). The dotted lines represent the functions $m^2(N)$ and $m^2(\Delta N)$ according to (5) and (7); their sum $M^2(N)$ has a minimum that is slightly greater than the true value $\underline{M}^2(N)$.

Closer investigations show that $M^2(N)$ deviates from $\underline{M}^2(N)$ only by a few per cent so that it canwell be used instead of the unknown $\underline{M}^2(N)$. It can be found graphically, by plotting the curves $m^2(N)$ and $m^2(\Delta N)$ according to Fig. 4, or better numerically. Adding (5) and (7) we have

$$M^2(N) = \frac{L}{8\pi G^2}\, J + \frac{R^2}{G^2}\, \psi_o^2\, m^2(\Delta g) = \frac{L}{8\pi G^2}\, [J(\psi_o) + \varkappa\, \psi_o{}^2] \qquad (9)$$

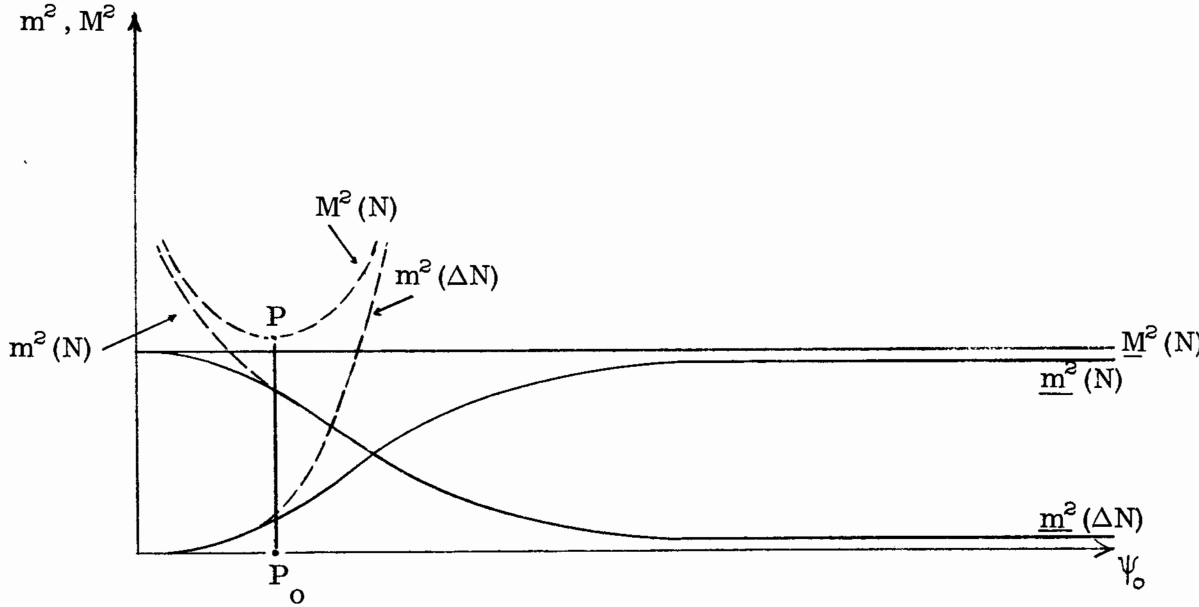

Figure 4

Continuation of the effects of central and outer zones.

where

$$\varkappa = \frac{8\pi m^2(\Delta g)}{L/R^2} \tag{10}$$

We have to find that value ψ_o for which M^2 is a minimum. Setting the derivative with respect to ψ_o equal to zero we have

$$J'(\psi_o) + 2\varkappa\psi_o = 0$$

or, by (6),

$$S^2(\psi_o) \sin\psi_o = 2\varkappa\psi_o . \tag{11}$$

From this equation ψ_o can be easily determined with sufficient accuracy, and (9) then yields $M^2(N)$.

Table 4 shows numerical results obtained in this way from the numerical data of Tables 2 and 3.

Table 4

Standard errors, in meters, of N for idealized gravity nets.

Block	$1^o \times 1^o$	$2^o \times 2^o$	$5^o \times 5^o$	$10^o \times 10^o$
Point	±1.5 m	±4.9 m	± 13 m	± 25 m
Profile	±1.2 m	±3.1 m	± 6.8 m	± 9.2 m

Arrangement and meaning correspond to Tables 2 and 3.

B. 5 COMPUTATION BY SPHERICAL HARMONICS: TOTAL STANDARD ERROR.

By a development in orthogonal functions it is possible to replace an integral transformation by a linear transformation in infinitely many variables that is usually simpler. This is also true for Stokes' integral; the relevant orthogonal functions are the spherical harmonics.

Write the spherical harmonics development of Δg in the form

$$\Delta g(\varphi,\lambda) = \sum_{i=1}^{\infty} f_i \psi_i(\varphi,\lambda) \tag{12}$$

where ψ_i are the fully normalized harmonics numbered by one index i only, as in [12], so that

$$\frac{1}{4\pi}\int_{\lambda=0}^{2\pi}\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_i(\varphi,\lambda)\,\psi_j(\varphi,\lambda)\cos\varphi\, d\varphi\, d\lambda = \delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases} \tag{13}$$

If the corresponding development of N is written

$$N(\varphi,\lambda) = \sum_{i=1}^{\infty} f_i^* \psi_i(\varphi,\lambda) \tag{14}$$

then it is well known that

$$f_i^* = \frac{R}{G}\frac{1}{n-1} f_i \tag{15}$$

where n is the degree of the spherical harmonic $\psi_i(\varphi,\lambda)$.

Apply the error propagation formula (10c) of [12],

$$f_i^* = \sum_{k=1}^{\infty} h_{ik} f_k, \quad \sigma_{ij}^* = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} h_{ik} h_{j\ell} \sigma_{k\ell} \tag{16}$$

where, by (15)

$$h_{ik} = \frac{R}{G}\frac{1}{n-1}\delta_{ik}$$

So we get from (16) at once (if f_i and f_j have the same degree n)

$$\sigma_{ij}^* = \left(\frac{R}{G}\right)^2 \frac{1}{(n-1)^2}\sigma_{ij}$$

By [12], eq. (19) we have

$$\sigma_{ij} = L\,\delta_{ij}/4\pi R^2 \tag{17}$$

so that

$$\sigma_{ij}^* = \frac{L}{4\pi G^2}\frac{\delta_{ij}}{(n-1)^2}. \tag{18}$$

This is our desired result. A few words of explanation are in order. σ_{ij} is the error covariance matrix of the coefficients f_i of Δg; σ_{ij}^* is the error covariance matrix of the corresponding coefficients f_i^* of N, i.e.

$$\sigma_{ii}^* = \frac{L}{4\pi G^2}\frac{1}{(n-1)^2} = m^2(f_i^*) \tag{19}$$

is the variance (square of standard error) of the coefficient f_i^*, whereas

$$\sigma_{ij}^* = 0 \quad \text{if } i \neq j$$

means that the coefficients f_i^* are uncorrelated.

Let us now apply this to compute the standard error $M(N)$ of N. From (14) we immediately obtain

$$M^2(N) = \sum_{i=1}^{\infty} \sigma_{ii} \psi_i^2 (\varphi,\lambda).$$

Since the result must be the same for every point (φ,λ), for reasons of symmetry, we may as well average over the whole sphere, obtaining

$$M^2(N) = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{i=1}^{\infty} \sigma_{ii}^* \psi_i^2(\varphi,\lambda) \cos\varphi \, d\varphi \, d\lambda .$$

Interchanging integration and summation and considering (13) we have

$$M^2(N) = \sum_{i=1}^{\infty} \sigma_{ii}^* = \frac{L}{4\pi G^2} \sum_{i=1}^{\infty} \frac{1}{(n-1)^2}$$

[by(18)]. Since we have $2n + 1$ different harmonics ψ_i of degree n, we can collect these and replace the sum over i by a sum over n:

$$M^2(N) = \frac{L}{4\pi G^2} \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2} . \tag{20}$$

This is our final result for $M(N)$; the sum begins with $n=2$ rather than $n=0$ because zero and first degree harmonics are omitted in the usual way.

We have chosen this somewhat lengthy derivation in order to make use of the results of [12] and to introduce formulas we shall need in the following. A much simpler way is to start directly from (4). As

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n (\cos \psi)$$

(P_n are the conventional Legendre polynomals) we have, formally,

$$\int_0^\pi S^2(\psi)\sin\psi d\psi = \int_0^\pi \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} \frac{2n+1}{n-1} \frac{2n'+1}{n'-1} P_n(\cos\psi) P_{n'}(\cos\psi) \sin\psi d\psi$$

Interchanging integration and summation and noting that

$$\int_0^\pi P_n(\cos\psi)\, P_{n'}(\cos\psi)\, \sin\psi d\psi = \begin{cases} \frac{2}{2n+1} & \text{if } n' = n \\ 0 & \text{if } n' \neq n \end{cases}$$

we have

$$\int_0^\pi S^2(\psi)\sin\psi d\psi = \sum_{n=2}^{\infty} \frac{2(2n+1)}{(n-1)^2}$$

and, by (4),

$$M^2(N) = \frac{L}{4\pi G^2} \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)^2}$$

in agreement with (20). Since both derivations are based on different principles, we obtain in this way a check of the formulas of [12],too.

Unfortunately the series (20) behaves for large n like the series

$$\Sigma \frac{1}{n}$$

which is divergent. So (20) would give the value ∞. This is quite similar to the behavior of the integral (4), and related to it. The reason is that eq. (17) is only valid for not too large n; for large n, σ_{ij} is much smaller than (17).

The best way is, therefore, to cut off the higher degree terms and set

$$M^2(N) = \frac{L}{4\pi G^2} \sum_{n=2}^{n_0} \frac{2n+1}{(n-1)^2} \tag{21}$$

where the upper limit n_0 is suitably chosen.

This upper limit n_0 can be obtained with sufficient accuracy by the following reasoning. In exactly the same way as we derived (20) from (18), we can derive the (average) standard error $M(\Delta g)$ of the gravity anomalies Δg, just by replacing σ^*_{ij} by σ_{ij} (17):

$$M^2(\Delta g) = \frac{L}{4\pi R^2} \sum_{2}^{\infty} (2n+1), \tag{22}$$

This series is still more divergent than (20), but $M^2(\Delta g)$ must have a finite value, such as given by Table 3. Again replacing the upper limit by n_o we have

$$M^2(\Delta g) = \frac{L}{4\pi R^2} \sum_{n=2}^{n_o} (2n+1) = \frac{L}{4\pi R^2}(n_o+3)(n_o-1) \tag{23}$$

by the well-known formula for the sum of an arithmetic progression. Writing the above equation in the form

$$n_o^2 + 2n_o - 3 - \frac{4\pi M^2(\Delta g)}{L/R^2} = 0 \tag{24}$$

we see that it is a quadratic equation that can be solved for n_o. The right hand side is computed by means of the numerical values of Tables 2 and 3.

In this way we find the following rounded-off values of n_o (Table 5, arrangement as before)

Table 5

Upper limits n_o

Block	$1^\circ \times 1^\circ$	$2^\circ \times 2^\circ$	$5^\circ \times 5^\circ$	$10^\circ \times 10^\circ$
Point	330	120	50	25
Profile	190	100	40	25

The value of M(N) according to (21) is very insensitive with respect to n_o, as Fig. 5 shows. n_o can even be replaced by $2n_o$ or $n_o/2$, without essential changes of M(N).

Evaluating eq. (21) using Tables 2 and 5 we obtain the values of M(N) of Table 6.

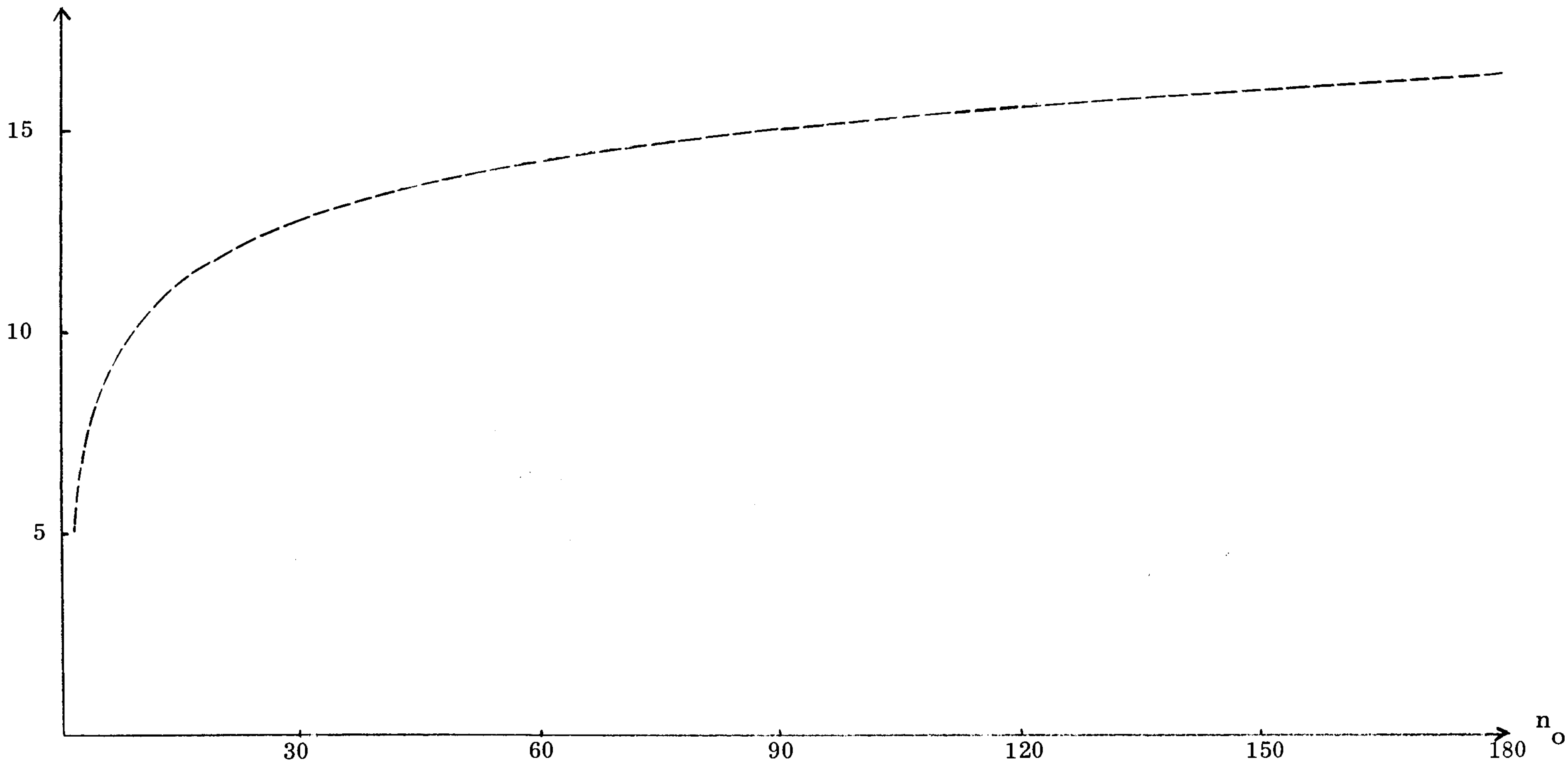

Figure 5

$\sum_{2}^{n_o} \frac{2n+1}{(n-1)^2}$ as a function of n_o

Table 6

Standard errors, in meters, cf N for idealized gravity nets, computed by means of spherical harmonics

Block	$1^\circ \times 1^\circ$	$2^\circ \times 2^\circ$	$5^\circ \times 5^\circ$	$10^\circ \times 10^\circ$
Point	± 1.5 m	± 5.2 m	±14 m	± 28 m
Profile	± 1.2 m	± 3.3 m	±7.4 m	±10.2 m

These values can be directly compared to those of Table 4; the agreement is satisfactory.

Evaluation of the sum. It should be noted that, instead of actually evaluating the sum (21), it is more convenient to apply certain asymptotic formulas. From [1], pp. 125 and 131, we take the formulas

$$\sum_{1}^{n_o} \frac{1}{n} = \ln n_o + C + \frac{1}{2n_o} + O\left(\frac{1}{n_o^2}\right)$$

$$\sum_{1}^{n_o} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{1}{n_o} + O\left(\frac{1}{n_o^2}\right) \tag{25}$$

where C is Euler's constant:

$$C = \lim_{n_o \to \infty} \left(\sum_{n=1}^{n_o} \frac{1}{n} - \ln n_o\right) = 0.5772 \cdots \tag{26}$$

So we obtain from

$$\sum_{n=2}^{n_o} \frac{2n+1}{(n-1)^2} = 2\sum_{n=2}^{n_o} \frac{1}{n-1} + 3\sum_{n=2}^{n_o} \frac{1}{(n-1)^2} = 2\sum_{n=1}^{n_o-1} \frac{1}{n} + 3\sum_{n=1}^{n_o-1} \frac{1}{n^2} ,$$

The convenient expression

$$\sum_{n=2}^{n_o} \frac{2n+1}{(n-1)^2} = 2\ln (n_o - 1) + 2C + \frac{\pi^2}{2} - \frac{2}{n_o - 1} + O\left(\frac{1}{n_o^2}\right). \tag{27}$$

The actual computation was performed by direct summation (Fig. 5) and checked by (27).

B.6 SPHERICAL HARMONICS: OUTER ZONE.

This section solves the problem of sec. B2 in spherical harmonics. It is of less practical importance but a method due to Molodensky [8] can be used, which is of considerable theoretical interest.

Two different spherical harmonics developments connected with Stokes' function can be used. First consider the (discontinuous) function

$$\bar{S}(\cos\psi) = \begin{cases} 0 & \text{if } 0 \leqq \psi < \psi_o \\ S(\cos\psi) & \text{if } \psi_o \leq \psi < \pi \end{cases} \quad ^{1)} \tag{28}$$

(see Fig. 6). Develop it in a series of zonal harmonics:

$$\bar{S}(\cos\psi) = \sum_{0}^{\infty} \frac{2n+1}{2} Q_n P_n(\cos\psi). \tag{29}$$

By well-known formulas the coefficients Q_n are given by

$$Q_n = \int_0^\pi \bar{S}(\cos\psi) P_n(\cos\psi) \sin\psi \, d\psi = \int_{\psi_o}^{\pi} S(\cos\psi) P_n(\cos\psi) \sin\psi \, d\psi .$$

Setting

$$\cos\psi = y$$

we have

$$Q_n = \int_{-1}^{1} \bar{S}(y) P_n(y) \, dy = \int_{-1}^{\cos\psi_o} S(y) P_n(y) \, dy . \tag{30}$$

Then we get smilarly as before

$$\int_{\psi_o}^{\pi} S^2(\cos\psi) \sin\psi \, d\psi = \int_0^{\pi} \bar{S}^2(\cos\psi) \sin\psi \, d\psi = \int_{-1}^{1} \bar{S}^2(y) \, dy =$$

$$= \int_{-1}^{1} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{2n+1}{2} \frac{2n'+1}{2} Q_n Q_{n'} P_n(y) P_{n'}(y) \, dy =$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n^2$$

1) In this section we write $S(\cos\psi)$ instead of $S(\psi)$ for Stokes' function.

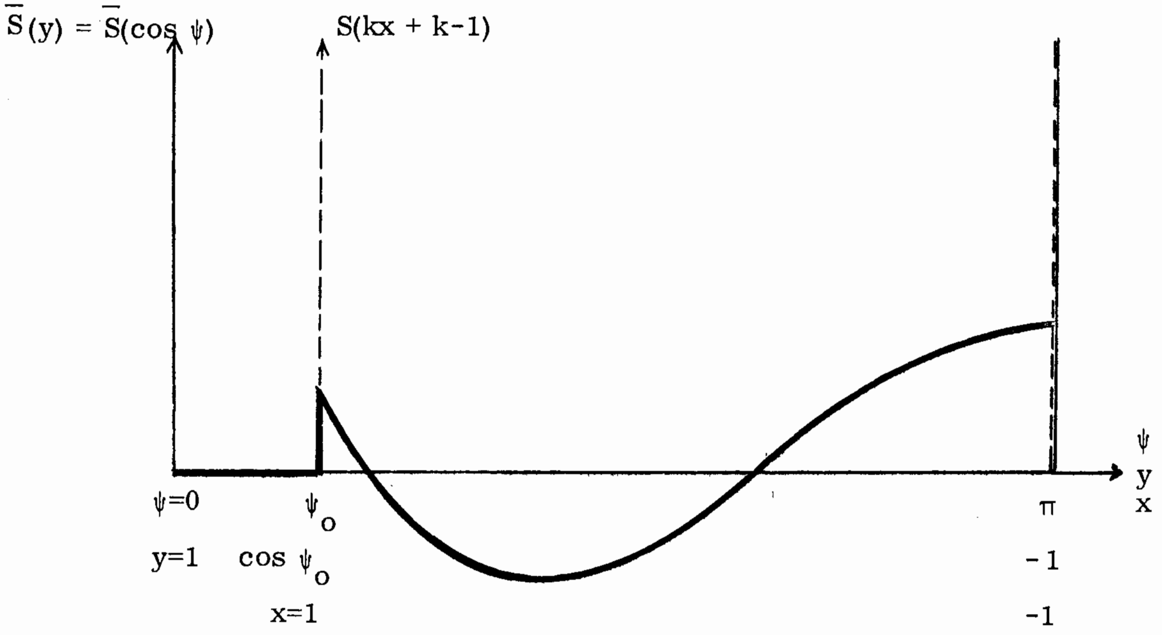

Figure 6

The function $\bar{S}(y)$

so that, by (5)

$$m^2(N) = \frac{L}{16\pi G^2} \sum_{n=0}^{\infty} (2n+1)\, Q_n^2 . \tag{31}$$

A series that converges better is obtained in the following way. Perform the linear transformation

$$y = k\,x + k - 1 \text{ where } k = \cos^2 \frac{\psi_0}{2} \tag{32}$$

which takes the point $y = \cos\psi_0$ ($\psi=\psi_0$) into the point $x = 1$ and the point $y = 1$ ($\psi=\pi$) into the point $x = -1$. Develop the function

$$S(\cos\psi) = S(y) = S\,(k\,x + k - 1) \tag{33}$$

in the interval $\psi_0 \leqq \psi \leqq \pi$, i.e., $-1 \leqq x \leqq 1$, in a series of zonal harmonics with argument x:

$$S = \sum_0^{\infty} \frac{2n+1}{2} K_n P_n(x) \tag{34}$$

so that

$$K_n = \int_{-1}^{1} S(k\,x + k - 1)\, P_n(x) dx . \tag{35}$$

Then we have, because $dy = k\,dx$,

$$\int_{\psi=\psi_0}^{\pi} S^2(\cos\psi)\sin\psi\, d\psi = \int_{y=-1}^{\cos\psi_0} S^2(y) dy = k\int_{x=-1}^{1} S^2(kx+k-1)dx =$$

$$= k\int_{x=-1}^{1} \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} \frac{2n+1}{2}\,\frac{2n'+1}{2} K_n K_{n'} P_n(x) P_{n'}(x)\, dx$$

$$= k \sum_{n=0}^{\infty} \frac{2n+1}{2} K_n^2 .$$

Hence,

$$m^2(N) = \frac{kL}{16\pi G^2} \sum_{n=0}^{\infty} (2n+1)\, K_n^2 . \tag{36}$$

This formula is completely equivalent to (31); both equations differ in the factor k and in the use of Q_n or K_n. Eq. (36) converges better than (31), so that fewer terms are sufficient for numerical computation.

Let us once more stress the conceptual difference in using the Q_n and the K_n. The Q_n are the coefficients of the development of the discontinuous function

$$\bar{S}(y) = \begin{cases} S(y) & \text{if } -1 \leq y < \cos\psi_o \\ 0 & \text{if } \cos\psi_o < y \leq 1 \end{cases}$$

in Legendre polynomials $P_n(y)$ with argument y in the complete interval $-1 \leq y \leq 1$. The K_n are the coefficients of the development of that part of Stokes' function

$$S(y) = S(k\,x + k - 1)$$

whose arguments lie within the partial interval $\psi_o \leq \psi \leq \pi$ or $-1 \leq y \leq \cos\psi_o$ or $-1 \leq x \leq 1$ (Fig. 6), in a series of Legendre Polynomials $P_n(x)$ with argument x. Within this interval, but not outside, the functions $S(y)$, $\bar{S}(y)$, and $S(kx+k-1)$ coincide.

Formulas and tables of the Q_n and K_n as functions of ψ_o are given in [8].

Table 7 shows some numerical values of m(N), as functions of ψ_o, computed by means of the K_n from eq. (36), neglecting $n \geq 9$.

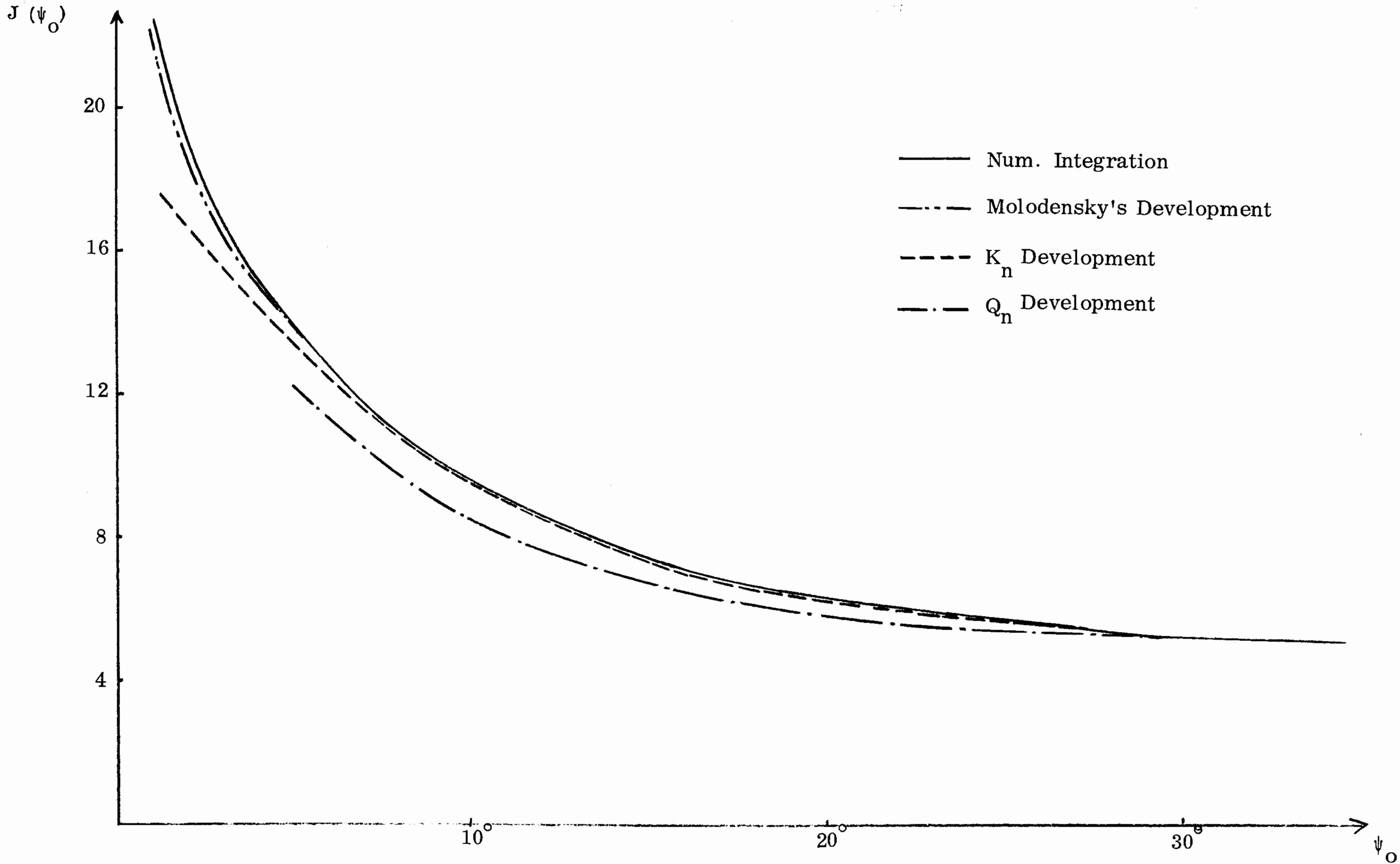

Figure 7

$$J(\psi_o) = \int_{\psi_o}^{\pi} S^2 \sin\psi \, d\psi$$ evaluated by different methods.

Table 7

m(N), in meters, computed by means of K_n

	$\psi_o=1^o10'$	$\psi_o=11.5^o$	$\psi^o=23.1^o$	$\psi^o=34.9^o$
Point measurements				
$1^o \times 1^o$	1.1	0.8	0.6	0.6
$2^o \times 2^o$	3.9	2.8	2.2	2.1
$5^o \times 5^o$	11.6	8.2	6.6	6.2
$10^o \times 10^o$	23.6	16.6	13.4	12.7
Profile measurements				
$1^o \times 1^o$	0.9	0.6	0.5	0.5
$2^o \times 2^o$	2.5	1.8	1.4	1.3
$5^o \times 5^o$	6.0	4.2	3.4	3.2
$10^o \times 10^o$	8.6	6.1	4.9	4.7

These values are in good agreement with those of Figs. 2 and 3.

Fig. 7 shows the function of ψ_o

$$\int_{\psi_o}^{\pi} S^2 \sin\psi \, d\psi = \sum_{o}^{\infty} \frac{2n+1}{2} Q_n^{\,2} = k\sum_{o}^{\infty} \frac{2n+1}{2} K_n^{\,2} \qquad (37)$$

as computed by numerical integration of $S^2 \sin\psi$, from the Q_n (up to n = 8), and from the K_n (up to n = 8). The agreement is sufficient.

C. ACCURACY OF DEFLECTIONS OF THE VERTICAL

It is known that for the computation of deflections of the vertical we need a dense gravity net around the station. Deflections computed in the case that we have only one gravity station or profile in each $5^o \times 5^o$ block, say, are quite meaningless.

We therefore assume that around the station (central, $\psi \cdot \psi_o$) we have a gravity net which is so dense that no error is caused by this central zone. Outside, for $\psi > \psi_o$, we assume the usual, less dense, coverage of one point or one profile in each 1°, 2°, 5°, or 10° block all around the earth.

The components ξ and η of the deflection of the vertical are given by

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \int_{\alpha=o}^{2\pi} \int_{\psi=o}^{\pi} \Delta g \, S'(\psi) \left\{ \begin{matrix} \cos \alpha \\ \sin \alpha \end{matrix} \right\} \sin \psi \, d\psi \, d\alpha \tag{38}$$

where

$$S'(\psi) = \frac{dS}{d\psi} .$$

Hence, by the formula for error propagation,

$$\left\{ \begin{matrix} m^2(\xi) \\ m^2(\eta) \end{matrix} \right\} = \frac{1}{16\pi^2 G^2} \int_{\alpha=o}^{2\pi} \int_{\psi=o}^{\pi} \int_{\alpha'=o}^{2\pi} \int_{\psi'=o}^{\pi} \sigma(\psi,\alpha,\psi',\alpha') S'(\psi) S'(\psi') .$$

$$\cdot \left\{ \begin{matrix} \cos \alpha \cos \alpha' \\ \sin \alpha \sin \alpha' \end{matrix} \right\} \sin\psi \sin \psi' \, d\psi d\alpha d\psi' d\alpha' . \tag{40}$$

According to our assumption the error function $\sigma(\psi,\alpha,\psi',\alpha')$ is zero for the central zone $\psi < \psi_o$. We thus can in the above equation replace the integration

$$\int_{\alpha=o}^{2\pi} \int_{\psi=o}^{\pi} \int_{\alpha'=o}^{2\pi} \int_{\psi'=o}^{\pi} \quad \text{by} \quad \int_{\alpha=o}^{2\pi} \int_{\psi=\psi_o}^{\pi} \int_{\alpha'=o}^{2\pi} \int_{\psi'=\psi_o}^{\pi} .$$

If we now apply the same trick as with eq. (2) we get

$$\left\{ \begin{matrix} m^2(\xi) \\ m^2(\eta) \end{matrix} \right\} = \frac{L}{16\pi^2 G^2 R^2} \int_{\psi=\psi_o}^{2\pi} \int_{\psi=\psi_o}^{\pi} S'^2(\psi) \cdot \left\{ \begin{matrix} \cos^2 \alpha \\ \sin^2 \alpha \end{matrix} \right\} \sin \psi \, d\psi d\alpha$$

and, performing the integration with respect to α,

$$m^2(\xi) = m^2(\eta) = \frac{L}{16\pi G^2 R^2} \int_{\psi_o}^{\pi} S'^2(\psi) \sin\psi \, d\psi . \tag{41}$$

The total standard error $m(\theta)$ of the deflection of the vertical thus is

$$m^2(\theta) = m^2(\xi) + m^2(\eta) = \frac{L}{8\pi G^2 R^2} \int_{\psi_0}^{\pi} S'^2(\psi) \sin\psi \, d\psi \tag{42}$$

This equation is completely analogous to eq.(5) and applies to the same distribution of gravity stations (accurate gravity net for $\psi<\psi_0$).

In the case of the geoid undulations it was also practically meaningful to assume that there is in the central zone no better gravity coverage than one point or one profile in each $1^\circ \times 1^\circ$, etc., block, as outside. This is not so in the case of the vertical deflection, so that (42) is our final result (no $M(\theta)$ corresponding to $M(N)$ of sec. B4).

As in the case of (5) it is not possible to set $\psi_0=0$, because the singularity of Vening Meinesz function $S'(\psi)$ at the origin is still worse than that of Stokes' function $S(\psi)$.

The integral

$$J_1(\psi_0) \equiv \int_{\psi_0}^{\pi} S'^2(\psi) \sin\psi \, d\psi \tag{43}$$

can be evaluated in several different ways, as in the case of the corresponding integral of Stokes' function.

The first possibility is numerical integration of

$$S'^2(\psi) \sin\psi.$$

Secondly, Molodensky gives an explicit formula ([8], p. 162):

$$\int_{\psi_0}^{\pi} S'^2(\psi)\sin\psi d\psi = 2\pi^2 + 24 \sum_{r=1}^{\infty} \frac{(-t)^r}{r^2} -$$

$$\begin{aligned} &-12\, \ln^2 t - 26\, \ln t + (12-36t^4+24t^6)\ln^2(t+t^2) + \\ &+(48t^2 + 56t^3 - 168t^4 - 48t^5 + 112t^6)\, \ln(t+t^2) + 34\, \ln\frac{1+t}{2} + \\ &+\frac{1}{2t^2} + \frac{6}{t} - \frac{29}{2} - 24t + 56t^2 + \frac{392}{3}t^3 - 176t^4 - 112t^5 + \frac{400}{3}t^6 \end{aligned} \tag{44}$$

where

$$t = \sin \frac{\psi_o}{2}$$

Finally, we may use the Q_n: From

$$\bar{S}'(\psi) = \sum_{n=o}^{\infty} \frac{2n+1}{2} Q_n \frac{dP_n}{d\psi} \tag{45}$$

we have

$$\int_{\psi=\psi_o}^{\pi} S'^2(\psi) \sin \psi \, d\psi = \int_{\psi=o}^{\pi} \bar{S}'^2(\psi) \sin \psi \, d\psi =$$

$$= \sum_{n=o}^{\infty} \sum_{n'=o}^{\infty} \frac{2n+1}{2} \frac{2n'+1}{2} Q_n Q_{n'} \int_{y=-1}^{1} \left(\frac{dP_n}{d\psi}\right)^2 dy =$$

$$= \sum_{n=o}^{\infty} \left(\frac{2n+1}{2}\right)^2 \frac{2n(n+1)}{2n+1} Q_n^2$$

or

$$\int_{\psi_o}^{\pi} S'^2(\psi) \sin \psi \, d\psi = \sum_{n=o}^{\infty} \frac{1}{2} n(n+1)(2n+1) Q_n^2 \tag{46}$$

This corresponds to eq. (37). The use of the K_n seems to lead to more complicated formulas.

The numerical integration of $S'^2 \sin \psi$ was done using Sollins' tables for Vening Meinesz' function [13]. In Table 8 we give some values of $J_1(\psi_o)$ as a function of ψ_o according to eq. (43); Fig. 8 gives a graphical representation. The numerical method was the same as used in integrating (6). The following intervals were taken:

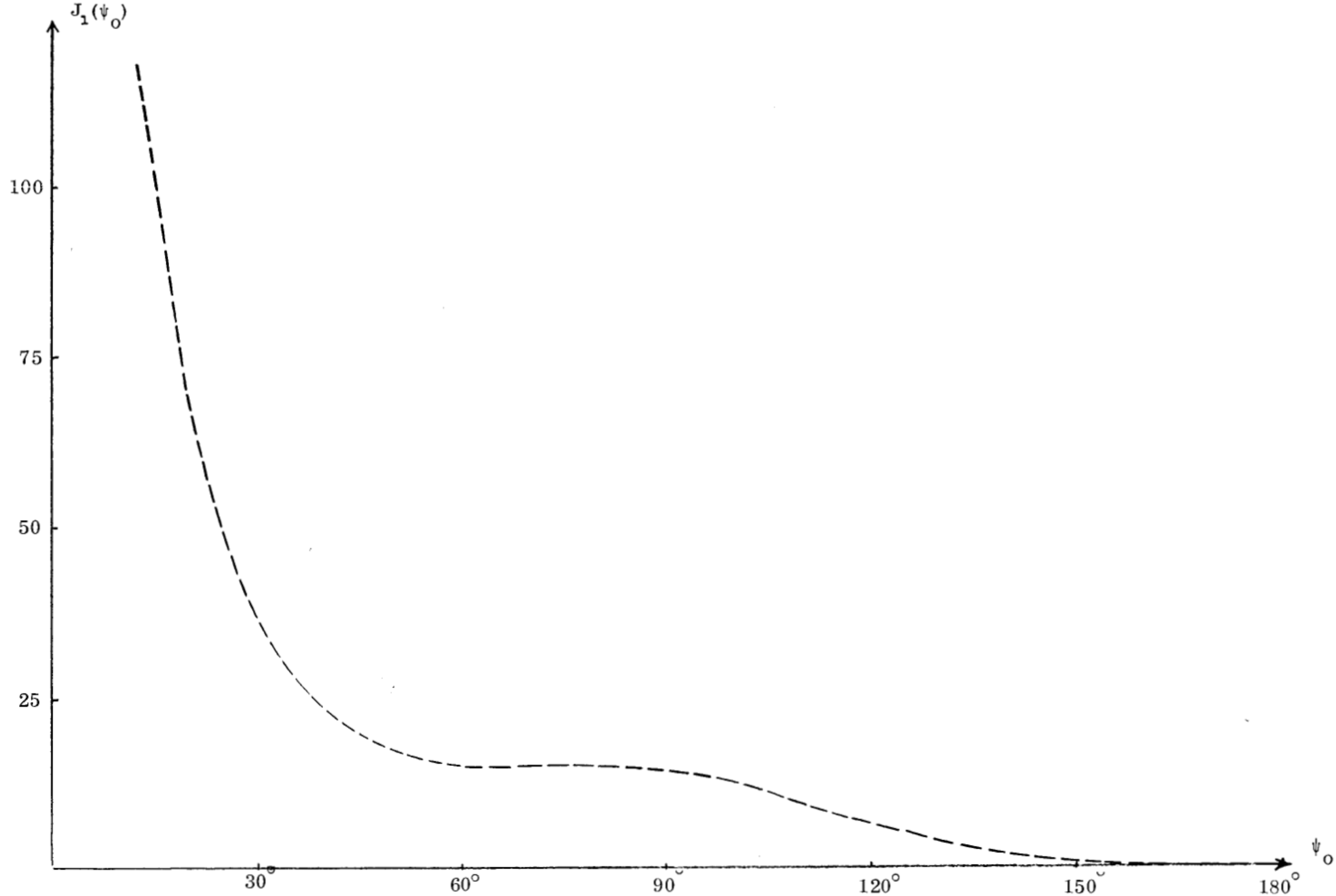

Figure 8

The integral $J_1(\psi_o) = \int_{\psi_o}^{\pi} S'^2 \sin\psi \, d\psi$ as a function of ψ_o.

$180^\circ > \psi < 10^\circ$ $\Delta\psi = 5^\circ$

$10^\circ < \psi < 1^\circ$ $\Delta\psi = 0.5^\circ$

Table 8

The integral $J_1(\psi_o) = \int_{\psi_o}^{\pi} S'^2 \sin\psi \, d\psi$ as obtained by numerical integration

ψ_o	$J_1(\psi)$	ψ_o	$J_1(\psi_o)$
180°	0.0	30°	38
160°	0.2	20°	70
140o	2.0	10°	184
120°	6.6	8°	245
100°	12.2	6°	363
80°	14.8	4°	662
60°	15.3	2°	2135
50°	17.5	1°	8164
40°	23.8		

For small ψ the numerical integration becomes somewhat less accurate, but the accuracy is still sufficient. The check by means of Molodensky's formula (44) yields for $\psi_o = 10^\circ$: $J_1 = 173$ and for $\psi_o = 1^\circ$: $J_1 = 7357$. The deviation from the corresponding values of Table 8 is less than 10°/o . The evaluation by means of (46) up to n = 8 gave no satisfactory result because (46) converges much more slowly than (31); one would have to go much farther than n=8.

Using the numerical values for the error integral of Table 2 we can compute $m(\theta)$ for point and profile measurements. The results are shown in Figs. 9 and 10 as functions of ψ_o; for $\psi < \psi_o$ an errorless gravity is assumed.

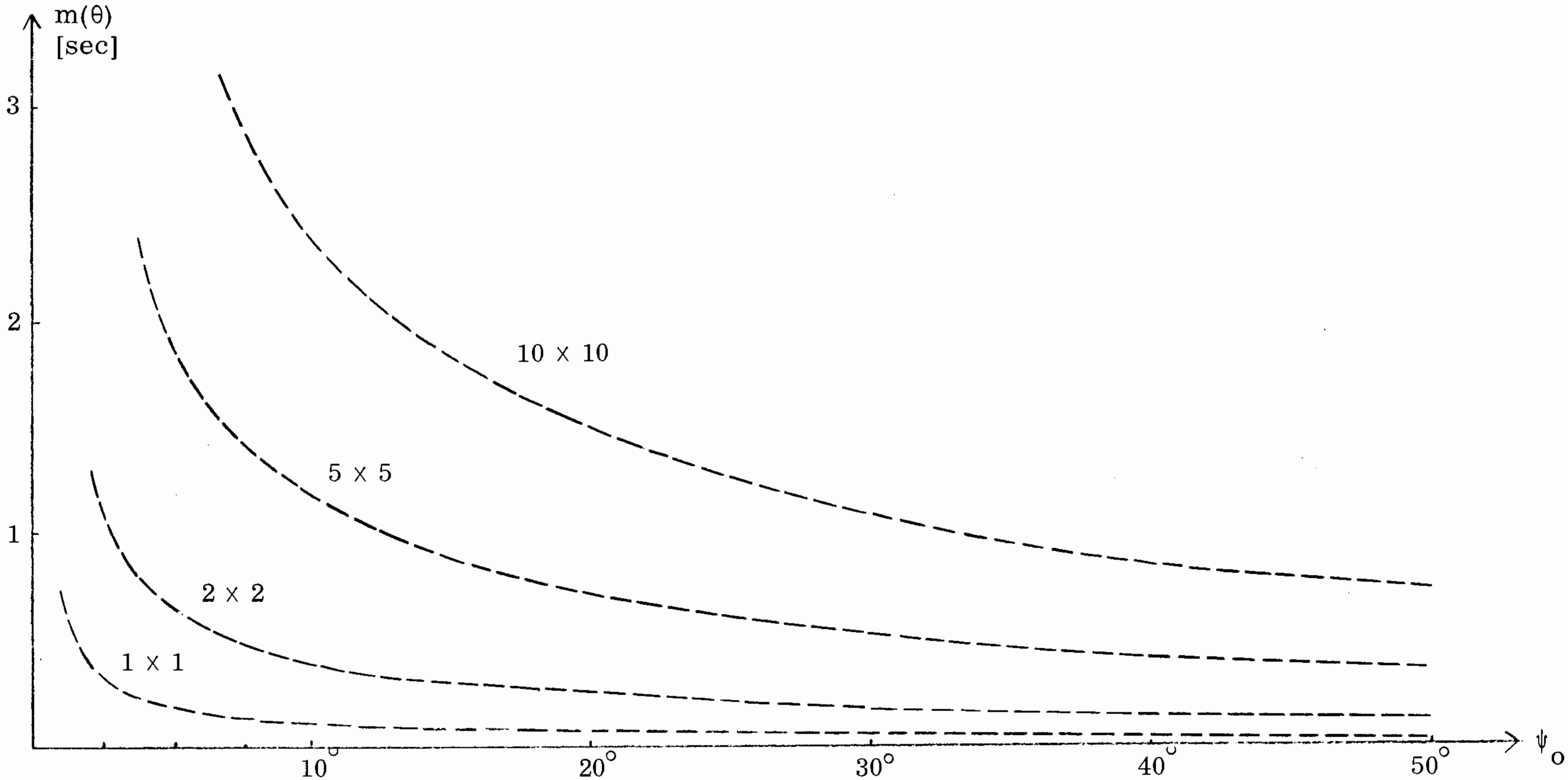

Figure 9

m(θ) from point measurements as a function of ψ_o.

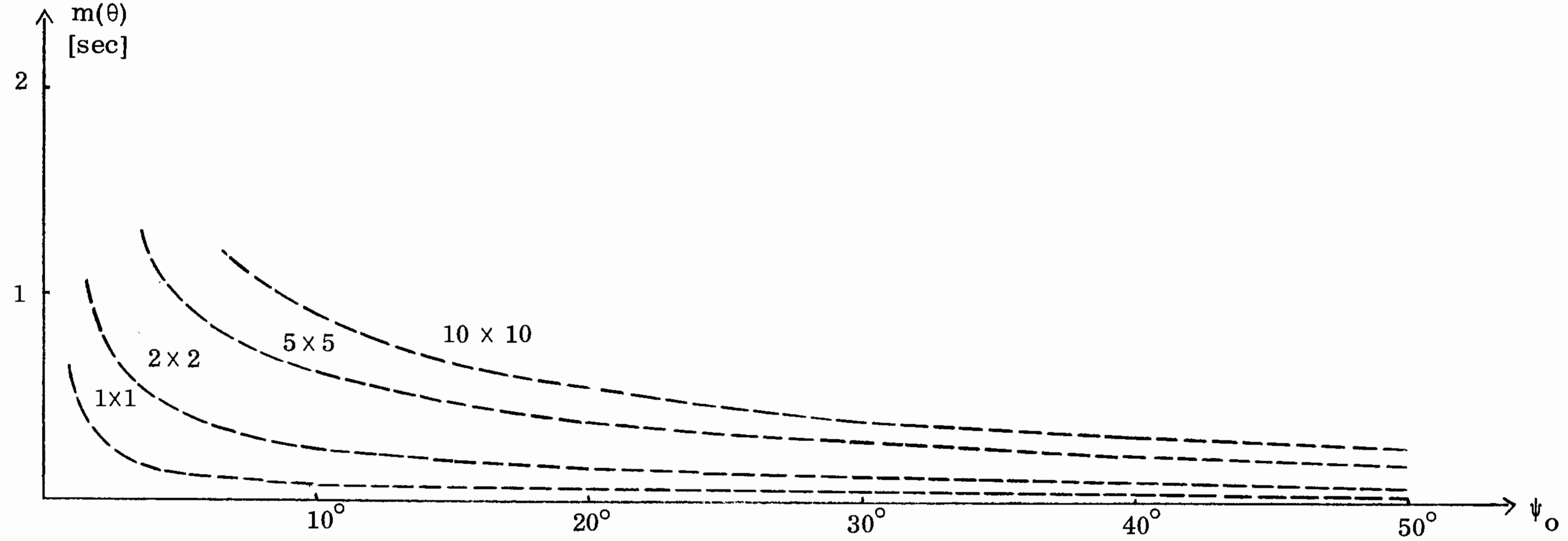

Figure 10

m(θ) from profile measurements as a function of ψ_o.

D. CONCLUSION

The results of Table 4 and Table 6 for the accuracy of the geoid undulation N are in good agreement with each other. Concluding we may state the following results for the practically most important assumptions: one gravity station in each $1^\circ \times 1^\circ$ or $2^\circ \times 2^\circ$ block, and one central gravity profile in each $5^\circ \times 5^\circ$ or $10^\circ \times 10^\circ$ block (Table 9).

Table 9

Standard errors of N in meters

Measurement	Block	M(N)
Point	$1^\circ \times 1^\circ$	± 1.5
	$2^\circ \times 2^\circ$	± 5
Profile	$5^\circ \times 5^\circ$	± 7
	$10^\circ \times 10^\circ$	± 9

In the case of deflections of the vertical, the situation is different. A uniform gravity net of, say, one gravity station in each $2^\circ \times 2^\circ$ block is both unsufficient and unnecessary, because here we need a much more accurate gravity net around the station and a much less accurate gravity net farther off than in the case of N.

In the immediate neighborhood of the deflection station we need gravity with an average standard interpolation error of

$$m(\Delta g) \doteq G\, m(\theta) \qquad (G = 980 \text{ gal})$$

if its rms effect on the deflection of the vertical is prescribed to be $m(\theta)$ according to an approximate formula of Molodensky ([8], p. 174). If we want $m(\theta) = \pm 0.2''$, then $m(\Delta g) = \pm 1$ mgal. To achieve this, an average density of one gravity station per 10 km distance or better is necessary.

The effect of different distributions of gravity measurements farther off can be seen from Figures 9 and 10.

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