

Eratosthenes Meets the GPS

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Multiple representations of a problem encourage students to make and strengthen connections across mathematics. This article presents three solution methods that students can use to estimate the Earth's circumference. All solution methods involve proportional reasoning, require physical measurements, and draw upon basic concepts of geography, notably longitude and latitude. They range from a geometric representation first proposed by ancient Eratosthenes using simple measuring devices of meter sticks and protractors (with and without use of trigonometry) to the use of modern Global Positioning System (GPS) units.

Introduction

Using multiple representations of a mathematics problem encourages students to think about the problem through various lenses, make connections across solutions, and view areas of mathematics as belonging to an integrated whole. The following discussion presents three methods to solve a problem from antiquity, utilizing representations involving geometry, trigonometry, and geography. Each method requires that students make measurements of the physical world, with measuring devices ranging from a simple meter stick to a Global Positioning System. Though two of the solution methods have existed for centuries, while the third is only decades old, together they provide rich mathematical experiences in a setting ripe for making comparisons of multiple representations of a physical problem.

How Big Is the Earth?

One of the great questions about our physical world that the ancients pondered was *How big is the Earth? What is its Circumference?* The first accurate estimate of the polar circumference of the Earth was provided by the Greek Eratosthenes in the third century BCE. True to the Greek philosophical nature of the time, Eratosthenes was a scholar with a ubiquity of interests. He served as head librarian at the Museum

in Alexandria, developed the “Sieve of Eratosthenes,” a method to identify prime numbers, and created the first scaled map of the entire known world of his time from the British Isles to Ceylon, and from the Caspian Sea to Ethiopia (Nicastro, 2008). But perhaps his most well known contribution to the understanding of our world was his measurement of the polar circumference of the Earth, the circumference of a Great Circle that intersects the North and South Poles. The basic method that Eratosthenes used was to create a proportion comparing the angle and distance between two locations on a common line of longitude to the entire 360° and circumference of a Great Circle through that line of longitude [see Figure 1].

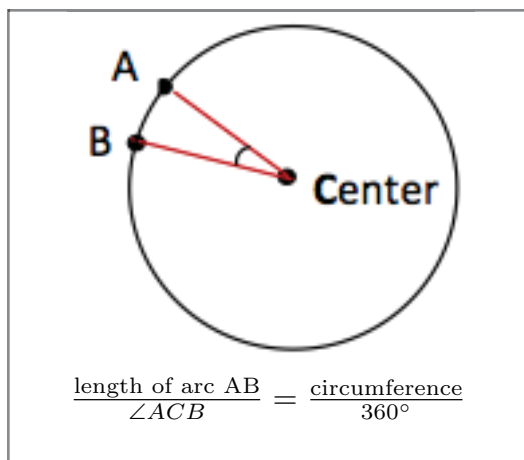


Fig 1 Proportion used to estimate the circumference of the Earth

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Whereas Eratosthenes already had a method to estimate the distance between locations, he needed to devise a method to determine the angle of latitude between these two locations. This paper discusses several methods for finding that angle, and in so doing, provides the student with a comparison of accessible solutions drawing upon various concepts of mathematics and geography.

Angle of Latitude

Eratosthenes had heard of a well in Syene, Upper (southern) Egypt where the sun's rays reached the water at the bottom on only one day of the year. He reasoned that the sun's position (relative to the Earth) reached directly overhead Syene on only this day of the year. As Egypt lies north of the equator, this event would occur on the summer solstice. Indeed Syene lies close to the Tropic of Cancer, the northernmost latitude at which the Sun can appear directly overhead at noon. Taking measurements on the summer solstice, and using Syene as the zero of his angle, he devised a method to determine the angle between Syene and Alexandria, which he believed were on a common line of longitude. His method involved measuring the angle of a shadow falling from a stake set perpendicular to the ground in Alexandria.

If one assumes, as Eratosthenes did, that the sun's rays intersecting Syene and Alexandria were for all practical purposes parallel, then Figure 2 illustrates that the angle formed by the stake and shadow, and the angle formed by the rays emanating from the center of the Earth and intersecting Syene and Alexandria are alternating interior angles, and thus equivalent. In general, the angle formed by a stake and its shadow is equivalent to the angle with vertex at the center of the Earth and whose rays intersect the stake and the point at which the sun lies directly overhead, both

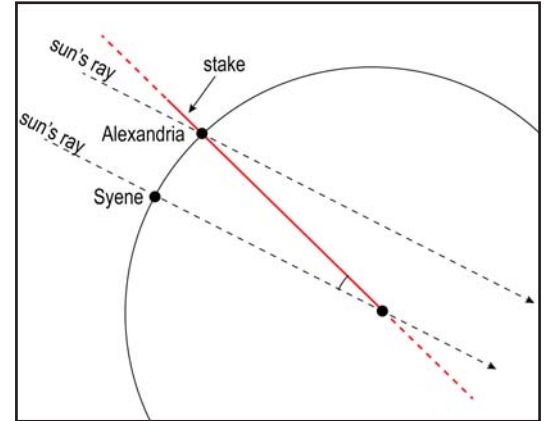


Fig 2 A transversal cuts the parallel rays of the sun creating alternate interior angles (not drawn to scale)

lying on the same line of longitude. On the autumnal equinox my small group of high school students set out to estimate the polar circumference by finding the angle defined by three points: their current location, the point along the equator on same line of longitude, and the vertex at the center of the Earth (see Figure 3).

Performing the Calculations in Class

We chose to perform this experiment on an equinox instead of Eratosthenes' choice of a solstice since school was in session at this time and the weather could be predicted to be fair. At our latitude, it likely would have been uncomfortably chilly to work on

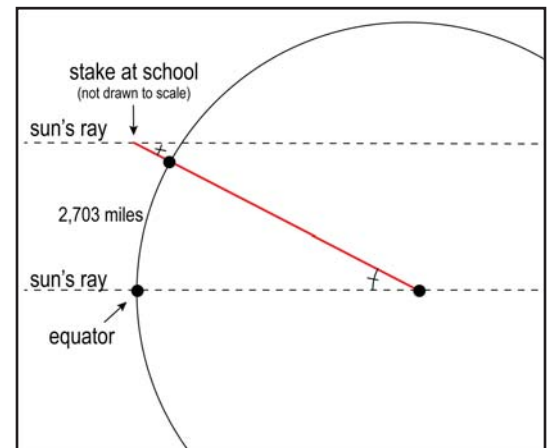


Fig 3 Angle between student's location and equator (not drawn to scale)

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the winter solstice. In order to perform this experiment the students additionally needed to know their distance from the equator and exactly when to measure the shadow of the angle cast by the stake. Though Eratosthenes had other means to determine these measures, the unknowns presented us with a conundrum as our method to determine distance relied upon knowledge of our latitude location, the targeted unknown of our experiment. The timing of the experiment can be determined from one's longitudinal position, the latitudinal partner. I decided to provide the students with both the distance and time to perform the experiment, shielding them from premature knowledge of their latitude location. Using the conversion that one degree of latitude is approximately 69 miles or 111 kilometers (the Earth's circumference divided by 360 degrees), I provided our calculated distance to the equator (2,703 miles). Using the school's latitudinal and longitudinal measurements found from Google Earth, the National Oceanic and Atmospheric Administration's (NOAA) website <http://www.srrb.noaa.gov/highlights/sunrise/sunrise.html> provided the exact time that the sun, traveling along the equator on this equinox day, would pass overhead at our noted line of longitude. [This website actually provides both the solar noon and solar declination which can be used to adjust the zero of the angle, enabling this experiment to be performed on any day.] As far as the students were concerned, the only remaining unknown was the angle of the shadow.

Instead of using a stake set in the ground, we chose to vertically position a four foot long level, marking its base position on cement with chalk. At the designated time that the sun, traveling along the equator, was directly overhead their line of longitude, the students marked the furthest reach of the stake's shadow. Using a protractor, they measured the angle created by the stake and

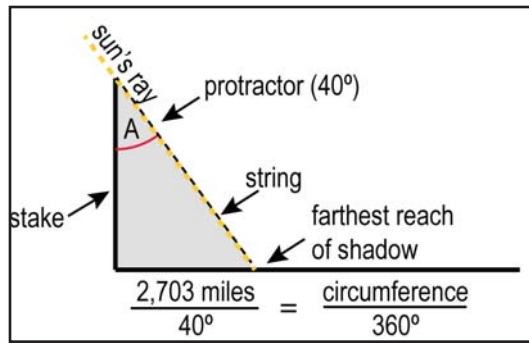


Fig 4 Using string and protractor to measure the angle formed by stake and shadow

a string that they stretched from the top of the stake to the furthestmost reach of the shadow to be 40 degrees (Figure 4). Using this measurement to solve the proportion in Figure 1, students estimated the circumference of the Earth to be 24,327 miles.

The second approach the students used to measure the angle created by the stake and the sun's ray along the outermost edge of the shadow involved trigonometry (Figure 5), yet to be developed in Eratosthenes' day. The length of our shadow measured 40.0 inches. Solving $\tan(A) = \frac{\text{length of shadow}}{\text{height of stake}} = \frac{40 \text{ in}}{48 \text{ in}}$ students found angle A to measure 39.8°.

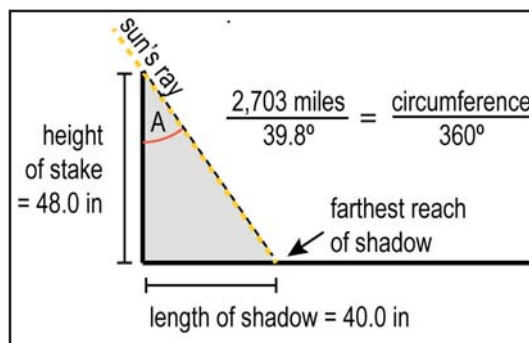


Fig 5 Using trigonometry to calculate the angle formed by stake and shadow

Using this angle measurement, along with the distance from the equator, they again solved the proportion in Figure 1. Their second method of estimating the circumference of the Earth yielded a measurement of 24,449 miles. An important connection that some students began to surmise from

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experiences that they had about latitude and longitude measurements from maps of their region of the U.S. was that the angle that they had estimated, first using a protractor and later using trigonometry, which they knew to be equivalent to the corresponding angle whose vertex was at the center of the Earth with rays intersecting their location and the equator, was simply the latitude of their location.

The Advantage of GPS

The traditional difficulty in this general approach to estimating the polar circumference rests with being able to determine two unknowns: 1) the angle between any two locations on the same line of longitude; Syene and Alexandria for Eratosthenes, our school's location and the equator for the students; and 2) the distance from the zero of the angle along the line of longitude. With the advent of Global Positioning Systems (GPS) the angle can be easily measured at any time, and distance from the equator is no longer necessary to find. Triangulating from a minimum of three satellites, a GPS can report one's location in longitude and latitude. After making sure that our GPS settings measured angles in decimals as compared to minutes and seconds (for example the Washington Monument is located at latitude 38.889426° N or $38^\circ 53' 22''$ N), students practiced taking latitude and longitude readings. To be able to estimate the circumference of the Earth using a GPS, the students again needed to compare the angle between two locations on the same longitude. However, unlike Eratosthenes, and their previous methods, the students did not need to perform this activity on an equinox or solstice (or know the angle of the sun's declination to adjust), nor did they need to know their distance from the equator. Using aerial images from *Google Earth* the students located a viable location for the activity, choosing a large

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open area on the school's parking lot across which students could easily maneuver a trundle wheel on a north-south line of longitude. Students placed a marking cone at their initial location (see point A in Figure 6). Then using their GPS compass (or directional setting), they walked due north to the other end of the parking lot and placed a second marking cone (see point B in Figure 6). Using a trundle wheel they measured the distance between cones to be 264 feet. To find the angle between their marking stakes (points A and B) with vertex at the center of the Earth, the students simply needed to find the difference in latitude of the stakes' positions. This was not immediately apparent to all students in the pre-activity discussion and thus an added benefit of this activity developed with the opportunity to question and discuss what is meant by a latitude reading. The students first found the latitude reading at point A to be 39.18955° N. One understanding of

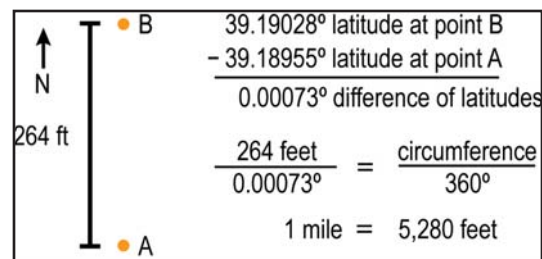


Fig 6 Using GPS to estimate circumference

this measurement is that point A lies on a line of latitude 39.18955° north from the equator. In perhaps more revealing words, the angle formed by the vertex at the center of the Earth and rays intersecting point A and the equator on a common line of longitude measured 39.18955° .

Students then used the GPS unit to determine the latitude measurement at point B to be 39.19028° N. They found the difference in latitudes to be 0.00073° (see Figure 6). Note the difference in latitude readings was very small. This very small difference in latitude caused some students

to question if they had made an error in measurement. One student countered, “What did you expect? The whole circle is 360° . We just walked a little bit.” Once again they were able to solve the proportion (see Figure 1) relating the length of the arc and angle between points A and B with vertex at the center of the Earth to the circumference of the Earth and 360° , estimating the polar circumference to be 130,191,781 feet or 24,658 miles.

Enhancing the Experiment

More accurate results are likely to be obtained if points A and B are separated by greater distance. The following suggested extensions address the issue of greater accuracy and maintain the benefits of not prematurely assuming the measure of latitude to determine distance from the equator, or time needed to make measurement on the solstice / equinox. If your school lies within a town or city whose streets are laid out on a north-south east-west grid, students could select a north-south road to serve as their line of longitude, using a GPS to measure latitude for two points whose distance between could be measured by a trundle wheel. Though students would no longer be physically immersed in the problem, they could achieve even greater accuracy by modeling the problem entirely within *Google Earth*. Students can readily select and “push-pin” mark locations along a common line of longitude separated by great distance, noting their latitude measurements. *Google Earth*’s ruler tool can be used to find the distance between “push-pin” points. With the length of the arc and angle between points [difference in latitudes] known, students can once again solve the proportion from Figure 1. Indeed, if you are willing to provide students with the conversion that one degree latitude is roughly 111 kilometers (69 miles) so that the distance to the equator can be

approximated, students need only find their latitude reading (using either GPS or entering their street address in *Google Earth*) to solve the proportion in Figure 1.

Comparing Results

The true polar circumference of the Earth is roughly 40,000 km (24,860 miles). Eratosthenes estimated the circumference to be 24,607 miles. This series of activities provided three methods to find the angle between points on a common line of longitude, each resulting with an estimate for the circumference of the Earth (24,327 miles, 24,449 miles, and 24,658 miles). Having found the measure of the angle using three different representations, students discussed what they viewed to be the pros and cons of each method. All three methods involved finding the length and angle of an arc on a line of longitude to set up a proportion that could be used to find the polar circumference. Students recognized the GPS and trigonometry methods to be much easier to implement. It was a challenge to measure an angle whose one ray was modeled by an unstable string, regardless how taut it was held. Students were intrigued with the novelty of the GPS method and enjoyed learning how to use this technology to solve the problem. Though the mathematical (geometric) reasoning that it required was not as sophisticated as either the trigonometric or protractor methods, it did have the surprising benefit of connecting and clarifying the use of latitude within geography. Additionally, students recognized that they could use the GPS to measure the distance between any two points on a common longitude, whereas the other methods relied upon given distance from the equator. Finally, students recognized that the GPS’s other great advantage was that, unlike the other two methods, its use was not restricted to specific days of the year (equinoxes and sol-

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stices). Being able to “see” a mathematics problem through multiple lenses disavows the view that there is only one solution to a problem and encourages students to be open-minded while seeking solutions.

A Final Challenge

A final challenge was posed to the students, “If you were unable to find your distance from the equator, did not know your latitude, and were unable to use electronic measuring devices, how could you modify Eratosthenes’ experiment to estimate the polar circumference?” Students demonstrated their ability to synthesize representations by integrating aspects of the GPS and non-GPS representations. Their solution is summarized as follows. Using a compass, walk due north from point A to point B, measuring the distance between points. As the sun passes over the corresponding line of longitude (approximated by noon on the equinox not adjusted for daylight savings), simultaneously use either the protractor or trigonometric method to find the angles (latitude of points A and B). Using the difference

in latitude measurements and the distance between point A and point B, once again solve the basic proportion from Figure 1. Ω

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But a number of manufacturers say that even if demand surges, they will never bring back many of the lower-skilled jobs, and that training is not yet delivering the skilled employees they need. Here in this suburb of Cleveland, supervisors at Ben Venue Laboratories, a contract drug maker for pharmaceutical companies, have reviewed 3,600 job applications this year and found only 47 people to hire at \$13 to \$15 an hour, or about \$31,000 a year.

Motoko, R. (2010, July 1). Factory jobs return, but employers find skills shortage. *New York Times*. Retrieved from <http://tinyurl.com/jobs-nytimes>