

Spectrum and thermodynamic properties of two-dimensional $\mathcal{N} = (1, 1)$ super Yang-Mills theory with fundamental matter and a Chern-Simons term

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We consider $\mathcal{N} = (1, 1)$ super Yang-Mills theory in $1 + 1$ dimensions with fundamentals at large N_c . A Chern-Simons term is included to give mass to the adjoint partons. Using the spectrum of the theory, we calculate thermodynamic properties of the system as a function of the temperature and the Yang-Mills coupling. In the large- N_c limit there are two noncommunicating sectors, the glueball sector, which we presented previously, and the mesonlike sector that we present here. We find that the mesonlike sector dominates the thermodynamics. Like the glueball sector, the meson sector has a Hagedorn temperature T_H , and we show that the Hagedorn temperature grows with the coupling. We calculate the temperature and coupling dependence of the free energy for temperatures below T_H . As expected, the free energy for weak coupling and low temperature grows quadratically with the temperature. Also the ratio of the free energies at strong coupling compared to weak coupling, r_{s-w} , for low temperatures grows quadratically with T . In addition, our data suggest that r_{s-w} tends to zero in the continuum limit at low temperatures.

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I. INTRODUCTION

The free energy of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory at a large number of colors N_c is larger at strong coupling by a factor $4/3$ compared to weak coupling [1,2]. The weak-coupling result is calculable in perturbation theory, while the strong-coupling result can be derived from black-hole thermodynamics. In the light of this finding, one can ask if other SYM theories exhibit a similar behavior. An analytic calculation of both the strong- and the weak-coupling limit of a field theory is generally not possible, although there have been a number of proposals for methods to obtain solutions of finite-temperature supersymmetric quantum field theories [3]. We will use a numerical approach which is based on supersymmetric discrete light-cone quantization (SDLCQ) [4,5], thereby preserving supersymmetry exactly. Currently, this is the only method available for numerically solving strongly coupled SYM theories. Conventional lattice methods have difficulty with supersymmetric theories because of the asymmetric way that fermions and bosons are treated, and progress [6] in supersymmetric lattice gauge theory has been relatively slow.

Previously we calculated the thermodynamic properties of pure glue $\mathcal{N} = (1, 1)$ SYM theory in $1 + 1$ dimensions [7]. Here we extend the calculations to include a sector with fundamental partons. In the large- N_c limit the bound states in this sector of the theory are chains in color space with a fundamental parton at each end. The links in the chain are adjoint partons. Bound states of this type will be called mesonic because they have two fundamental partons, whereas solutions with only adjoint partons will be called glueballs. We have also extended the calculation to include a Chern-Simons (CS) term, which gives mass to the adjoint partons. This serves two purposes. First, it

renders the model slightly more relevant, emulating a mass hierarchy. Second, we can study thermodynamic properties as a function of one of the couplings; the other will merely set the overall scale.

The mesons and glueballs constitute two sectors of the same theory; both contribute to the thermodynamics. In the large- N_c limit the sectors decouple. Because of the cyclic redundancy of single-trace glueball states, there are many meson states, which are in turn likely to dominate the thermodynamic properties of the system. In previous work on the glueball sector [7] we found that the system possesses a Hagedorn temperature.

Recall that SDLCQ makes use of light-cone coordinates, with $x^+ = (x^0 + x^3)/\sqrt{2}$ the time variable and $p^- = (p^0 - p^3)/\sqrt{2}$ the energy. One must be careful in defining thermodynamic quantities on the light cone. It seems natural to define the partition function [8] on the light cone as $e^{-\beta_{LC} p^-}$. However, as discussed in Refs. [9,10], the above prescription leads to singular results for well-known quantities which are finite in the equal-time approach. Their argument is based on the fact that using $e^{-\beta_{LC} p^-}$ as the partition function implies that the physical system is in contact with a heat bath that has been boosted to the light-cone frame. However, this is *not* equivalent to the physics of a system in contact with a heat bath at rest. This can be realized in a more direct way by noting that, since the light-cone momentum

$$p^+ = \frac{(p^0 + p^3)}{\sqrt{2}}$$

is conserved, the partition function must include the conserved quantity and is of the form

$$Z = e^{-\beta_{LC}(p^- + \mu p^+)},$$

where μ is the chemical potential corresponding to the

conserved quantity. The physical interpretation of the chemical potential is that of a rotation of the quantization axis. Thus $\mu = 1$ corresponds to quantization in an equal-time frame, where the heat bath is at rest and the inverse temperature is $\beta = \sqrt{2}\beta_{\text{LC}}$, and $\mu \neq 1$ corresponds to quantization in a boosted frame where the heat bath is not at rest. Thus μ corresponds to a continuous rotation of the axis of quantization, and $\mu = 0$ would correspond to rotation all the way to the light-cone frame.

A rotation from an equal-time frame to the light-cone frame is not a Lorentz transformation. It is known that such a transformation can give rise to singular results for physical quantities. This appears to be consistent with the results found in [9]. A number of related issues have been extensively discussed by Weldon [11]. The method has also recently been applied to the Nambu–Jona-Lasinio model [12].

The difficulties are avoided if we compute the equal-time partition function $Z = e^{-\beta p^0}$, as was proposed much earlier by Elser and Kalloniatis [13]. The computation may, of course, still use light-cone coordinates. Elser and Kalloniatis did this with ordinary DLCQ [14,15] as a numerical approximation to $(1+1)$ -dimensional quantum electrodynamics.

Here we will follow a similar approach using SDLCQ to calculate the spectrum of $\mathcal{N} = (1, 1)$ super Yang-Mills theory in $1+1$ dimensions [16]. Though this calculation is done in $1+1$ dimensions, it is known that SDLCQ can be extended in a straightforward manner to higher dimensions [17–19]. As is customary, we will assume that the single-trace bound states of our large- N_c approximation are single-particle states.

We have discussed the SDLCQ numerical method in a number of other places, and we will not present a detailed discussion of the method here; for a review, see [5]. For those familiar with DLCQ [14,15], it suffices to say that SDLCQ is similar; both impose periodic boundary conditions on a light-cone box $x^- \in [-L, L]$ and have discrete momenta and cutoffs in momentum space. In $1+1$ dimensions the discretization is specified by a single integer $K = (L/\pi)P^+$, the resolution [14], such that longitudinal momentum fractions are integer multiples of $1/K$. However, SDLCQ is formulated in such a way that the theory is also exactly supersymmetric. Exact supersymmetry brings a number of very important numerical advantages to the method; in particular, theories with enough supersymmetry are finite. We have also seen greatly improved numerical convergence in this approach.

The calculation of thermodynamic quantities requires summing over the spectrum of available states, which we represent by a density of states (DoS). We will use a new numerical approach to estimate the density of states. The new approach is more efficient than the method used in previous work [7], because it allows us to extract the density of states without fully diagonalizing the

Hamiltonian, a computationally challenging task. This innovation will enable us to pursue calculations at higher values of the resolution K . The improvement will also be relevant for future work on thermodynamic properties, since it is applicable to generic theories in arbitrary dimensions. That being said, it might be hard to handle other issues such as renormalization associated with higher-dimensional theories.

We find that the two-dimensional SYM theory with fundamentals and a Chern-Simons term exhibits a Hagedorn temperature T_H [20], and we calculate T_H for several values of the resolution K and Yang-Mills coupling g . Extrapolating to the continuum limit, we obtain T_H as a function of the coupling. The Hagedorn temperature is used as an upper limit for the temperatures we can use to calculate the thermodynamic properties of the system.

In Sec. II we provide a review of the formulation of super Yang-Mills theory with fundamental matter and a Chern-Simons term in $1+1$ dimensions. In Sec. III we discuss some of the properties of the SDLCQ spectra in some limiting cases and provide comparisons between glueball and mesonic sectors of the theory. The discussion in Sec. IV presents the methods for estimating the density of states and the Hagedorn temperature. In Sec. V, we summarize our formulation of the thermodynamics and the formulas we use to calculate the free energy. We then present the numerical results for the free energy, which we obtained using the DoS approximation to the spectrum, at various values of SYM coupling g up to the Hagedorn temperature. Finally, in Sec. VI we conclude by summarizing our results and the prospects for future work using these methods.

II. SUPER YANG-MILLS THEORY WITH FUNDAMENTAL MATTER AND CHERN-SIMONS TERM

A. Formulation of the theory and its supercharges

We start by considering $\mathcal{N} = 1$ supersymmetric gauge theory in $2+1$ dimensions coupled to fundamental matter and a Chern-Simons three-form. The action is

$$S_{2+1} = S_{\text{YM}} + S_{\text{f.matter}} + S_{\text{CS}}, \quad (2.1)$$

with

$$S_{\text{YM}} = \int dx^3 \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\Lambda} \Gamma^\mu D_\mu \Lambda \right), \quad (2.2a)$$

$$S_{\text{f.matter}} = \int dx^3 (D_\mu \xi^\dagger D^\mu \xi + i \bar{\Psi} D_\mu \Gamma^\mu \Psi - g [\bar{\Psi} \Lambda \xi + \xi^\dagger \bar{\Lambda} \Psi]), \quad (2.2b)$$

$$S_{\text{CS}} = \int dx^3 \frac{\kappa}{2} \text{Tr} \left(\epsilon^{\mu\nu\lambda} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right) + 2 \bar{\Lambda} \Lambda \right). \quad (2.2c)$$

The SYM part of the action describes a system of gauge bosons A_μ and their superpartners, the Majorana fermions Λ . Both fields are $(N_c \times N_c)$ matrices transforming under the adjoint representation of $SU(N_c)$; hereafter, unless indicated otherwise, we treat these fields as matrices, and thus we suppress the color indices (i, j, k) . Additionally, we have two complex fields, a scalar ξ and a Dirac fermion Ψ , all transforming according to the fundamental representation of the gauge group. In matrix notation the covariant derivatives and the gauge field strength are defined as follows:

$$\begin{aligned} D_\mu \Lambda &= \partial_\mu \Lambda + ig[A_\mu, \Lambda], & D_\mu \xi &= \partial_\mu \xi + igA_\mu \xi, \\ D_\mu \Psi &= \partial_\mu \Psi + igA_\mu \Psi, & D_\mu \xi^\dagger &= \partial_\mu \xi^\dagger - ig\xi^\dagger A_\mu, \\ D_\mu \Psi^\dagger &= \partial_\mu \Psi^\dagger - ig\Psi^\dagger A_\mu, & F_{\mu\nu} &= \partial_{[\mu} A_{\nu]} + ig[A_\mu, A_\nu]. \end{aligned} \quad (2.3)$$

The action (2.1) is invariant under supersymmetry transformations parametrized by a constant two-component Majorana spinor $\varepsilon \equiv (\varepsilon_1, \varepsilon_2)^T$; $\bar{\varepsilon} \equiv \varepsilon^T \Gamma^0$:

$$\begin{aligned} \delta A_\mu &= \frac{i}{2} \bar{\varepsilon} \Gamma_\mu \Lambda, & \delta \Lambda &= \frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \varepsilon, \\ \delta \xi &= \frac{i}{2} \bar{\varepsilon} \Psi, & \delta \xi^\dagger &= -\frac{i}{2} \bar{\Psi} \varepsilon, \\ \delta \Psi &= -\frac{1}{2} \Gamma^\mu \varepsilon D_\mu \xi, & \delta \bar{\Psi} &= -\frac{1}{2} D_\mu \xi^\dagger \bar{\varepsilon} \Gamma^\mu, \end{aligned} \quad (2.4)$$

where $\Gamma^{\mu\nu}$, the spinor generator of the Lorentz group, is written as

$$\Gamma^{\mu\nu} = \frac{1}{2} [\Gamma^\mu, \Gamma^\nu] = i \epsilon^{\mu\nu\lambda} \Gamma_\lambda; \quad (\epsilon^{-+2} = 1).$$

Using standard Noether techniques, we construct the spinor supercurrent corresponding to the above supersymmetric field variations,

$$\begin{aligned} \bar{\varepsilon} q^\mu &= \frac{i}{4} \bar{\varepsilon} \Gamma^{\alpha\beta} \Gamma^\mu \text{Tr}(\Lambda F_{\alpha\beta}) + \frac{i}{2} D^\mu \xi^\dagger \bar{\varepsilon} \Psi \\ &+ \frac{i}{2} \xi^\dagger \bar{\varepsilon} \Gamma^{\mu\nu} D_\nu \Psi - \frac{i}{2} \bar{\Psi} \varepsilon D^\mu \xi + \frac{i}{2} D_\nu \bar{\Psi} \Gamma^{\mu\nu} \varepsilon \xi. \end{aligned} \quad (2.5)$$

For the remainder of the paper we assume that the fields are independent of the spacelike dimension x^2 , i.e. $\partial_2(\dots) = 0$, thereby dimensionally reducing the theory to two dimensions. Thus the $\mathcal{N} = 1$ supersymmetry in $2 + 1$ dimensions is naturally expressed in terms of $\mathcal{N} = (1, 1)$ supersymmetry in $1 + 1$ dimensions.

We will implement light-cone quantization, which means that initial conditions as well as canonical (anti)-commutation relations will be imposed on the lightlike surface $x^+ = \text{const}$. In particular, we construct the supercharge by integrating the supercurrent (2.5) over the lightlike surface,

$$\begin{aligned} \bar{\varepsilon} Q &= \int dx^- \left(\frac{i}{4} \bar{\varepsilon} \Gamma^{\alpha\beta} \Gamma^+ \text{Tr}(\Lambda F_{\alpha\beta}) + \frac{i}{2} D_- \xi^\dagger \bar{\varepsilon} \Psi \right. \\ &+ \frac{i}{2} \xi^\dagger \bar{\varepsilon} \Gamma^{+\nu} D_\nu \Psi - \frac{i}{2} \bar{\varepsilon} \Psi^\dagger D^+ \xi \\ &\left. - \frac{i}{2} \bar{\varepsilon} \Gamma^{+\nu} D_\nu \Psi^\dagger \xi \right). \end{aligned} \quad (2.6)$$

Note that because we have taken the fields to be independent of x^2 , the integration over this coordinate resulted in a constant factor, which rescaled our original fields.

By choosing the following imaginary (Majorana) representation for the Dirac matrices in three dimensions:

$$\Gamma^0 = \sigma_2, \quad \Gamma^1 = i\sigma_1, \quad \Gamma^2 = i\sigma_3, \quad (2.7)$$

the Majorana spinor field Λ is manifestly real, i.e. $\Lambda^\dagger = \Lambda^T$. At this point it is convenient to introduce the component form for the spinors:

$$\Lambda = (\lambda, \tilde{\lambda})^T, \quad \Psi = (\psi, \tilde{\psi})^T, \quad Q = (Q^+, Q^-)^T. \quad (2.8)$$

In terms of this decomposition, the superalgebra is realized explicitly in its $\mathcal{N} = (1, 1)$ form, namely

$$\begin{aligned} \{Q^+, Q^+\} &= 2\sqrt{2}P^+, & \{Q^-, Q^-\} &= 2\sqrt{2}P^-, \\ \{Q^+, Q^-\} &= 0, \end{aligned} \quad (2.9)$$

where Q^+ (Q^-) are left (right) Majorana-Weyl spinors, each characterizing the smallest spinor representation in $1 + 1$ dimensions.

To readily eliminate the nondynamical fields, we impose the light-cone gauge ($A^+ = A_- = 0$). In this case the supercharges can be read off (2.6) and are given by

$$\begin{aligned} Q^+ &= \frac{1}{2} \int dx^- \left(\text{Tr}(\lambda \partial_- A^2) + \frac{i}{2} \partial_- \xi^\dagger \psi - \frac{i}{2} \psi^\dagger \partial_- \xi \right. \\ &\left. - \frac{i}{2} \xi^\dagger \partial_- \psi + \frac{i}{2} \partial_- \psi^\dagger \xi \right), \end{aligned} \quad (2.10)$$

$$\begin{aligned} Q^- &= \frac{1}{\sqrt{2}} \int dx^- \left(\text{Tr}(\lambda \partial_- A^-) - i\xi^\dagger D_2 \psi + iD_2 \psi^\dagger \xi \right. \\ &\left. - \frac{i}{\sqrt{2}} \partial_- (\tilde{\psi}^\dagger \xi - \xi^\dagger \tilde{\psi}) \right). \end{aligned} \quad (2.11)$$

Notice that the right movers ($\tilde{\psi}$) appear in the supercharge Q^- only in the total derivative term. This is a consequence of the light-cone formulation, which singles out the non-dynamical fermion degrees of freedom, leaving in the expression only the physical spinor fields (λ and ψ). Among the equations of motion that follow from the action (2.1), in the light-cone gauge, three serve as constraints rather than as dynamical equations. Namely, for $\tilde{\lambda}$ and $\tilde{\psi}$, respectively, we have

$$\partial_- \tilde{\lambda} = -\frac{ig}{\sqrt{2}} ([A^2, \lambda] + i\xi \psi^\dagger - i\psi \xi^\dagger - i\kappa\lambda), \quad (2.12)$$

$$\partial_- \tilde{\psi} = -\frac{ig}{\sqrt{2}} A^2 \psi + \frac{g}{\sqrt{2}} \lambda \xi, \quad (2.13)$$

while for A^- we obtain

$$\partial_-^2 A^- = gJ, \quad (2.14)$$

with

$$J = -i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}} \{\lambda, \lambda\} - i(\partial_- \xi) \xi^\dagger + i\xi(\partial_- \xi^\dagger) + \sqrt{2} \psi \psi^\dagger + \frac{\kappa}{g} \partial_- A^2. \quad (2.15)$$

Note that the field A^- has to be eliminated from the supercharges, in favor of the physical degrees of freedom. This can be done by inverting (2.14).

The only contribution from the Chern-Simons term enters into the supercharges via Eq. (2.15), because $\delta \mathcal{L}_{\text{CS}} = 0$ under the supersymmetry transformations (2.4). The inclusion of a Chern-Simons term in our theory is important, since it effectively generates mass for the adjoint partons proportional to the coupling κ .

B. Bound-state eigenvalue problem

The bound-state spectrum is obtained by solving the following mass eigenvalue equation:

$$\begin{aligned} 2P^+ P^- |\varphi\rangle &= \sqrt{2} P^+ (Q^-)^2 |\varphi\rangle \\ &= \sqrt{2} P^+ (g(Q_{\text{SYM}}^- + Q_{\text{f.matter}}^-) + i\kappa Q_{\text{CS}}^-)^2 |\varphi\rangle \\ &\equiv M^2 |\varphi\rangle, \end{aligned} \quad (2.16)$$

where the various pieces of Q^- , after dropping the surface terms and eliminating the nondynamical fields using the constraint (2.14), may be expressed as follows:

$$Q_{\text{SYM}}^- = \frac{ig}{\sqrt{2}} \int dx^- \left([A^2, \partial_- A^2] + \frac{i}{\sqrt{2}} \{\lambda, \lambda\} \right) \frac{1}{\partial_-} \lambda, \quad (2.17a)$$

$$\begin{aligned} Q_{\text{f.matter}}^- &= \frac{g}{\sqrt{2}} \int dx^- \left(i(\partial_- \xi) \xi^\dagger - i\xi(\partial_- \xi^\dagger) \right. \\ &\quad \left. - \sqrt{2} \psi \psi^\dagger \right) \frac{1}{\partial_-} \lambda + \xi^\dagger A^2 \psi + \psi^\dagger A^2 \xi, \end{aligned} \quad (2.17b)$$

$$Q_{\text{CS}}^- = \frac{i\kappa}{\sqrt{2}} \int dx^- (\partial_- A^2) \frac{1}{\partial_-} \lambda, \quad (2.17c)$$

where a trace over color space is understood.

The strategy for solving Eq. (2.16) is to cast it as a matrix eigenvalue problem. This is achieved by employing a discrete basis where the longitudinal light-cone momentum P^+ is diagonal. The discrete basis is introduced by first discretizing the supercharge¹ Q^- and then constructing P^- from the square of the supercharge: $P^- = (Q^-)^2 / \sqrt{2}$. The

¹Note the relative phase between Q_{SYM}^- and Q_{CS}^- . Q_{SYM}^- is defined as Hermitian and Q_{CS}^- is defined to be anti-Hermitian such that Q^- remains Hermitian.

two-dimensional theory is compactified on a lightlike circle ($-L < x^- < L$), and periodic boundary conditions are imposed on all dynamical degrees of freedom. This leads to the following field mode expansions:

$$\begin{aligned} A_{ij}^2(0, x^-) &= \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_{ij}(n) e^{-in\pi x^-/L} \\ &\quad + a_{ji}^\dagger(n) e^{in\pi x^-/L}), \end{aligned} \quad (2.18)$$

$$\begin{aligned} \lambda_{ij}(0, x^-) &= \frac{1}{2^{1/4} \sqrt{2L}} \sum_{n=1}^{\infty} (b_{ij}(n) e^{-in\pi x^-/L} \\ &\quad + b_{ji}^\dagger(n) e^{in\pi x^-/L}), \end{aligned} \quad (2.19)$$

$$\begin{aligned} \xi_i(0, x^-) &= \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (c_i(n) e^{-in\pi x^-/L} \\ &\quad + \tilde{c}_i^\dagger(n) e^{in\pi x^-/L}), \end{aligned} \quad (2.20)$$

$$\begin{aligned} \psi_i(0, x^-) &= \frac{1}{2^{1/4} \sqrt{2L}} \sum_{n=1}^{\infty} (d_i(n) e^{-in\pi x^-/L} \\ &\quad + \tilde{d}_i^\dagger(n) e^{in\pi x^-/L}). \end{aligned} \quad (2.21)$$

In the above expressions² we introduced the discrete longitudinal momenta $k^+ \equiv k$ as fractions $nP^+/K = n\pi/L$ ($n = 1, 2, 3, \dots$) of the total longitudinal momentum P^+ , where K is the integer that determines the resolution of the discretization. The color indices were made explicit as well. Because light-cone longitudinal momenta are always positive, K and each n are positive integers. The number of constituents is thus bounded by K . The continuum limit is reached by letting $K \rightarrow \infty$.

The time direction in the light-cone formalism is taken to be the x^+ direction. Thus the (anti)commutation relations between fields and their conjugate momenta are assumed on the surface $x^+ = 0$. Quantization is achieved by imposing the following relations:

$$\begin{aligned} [A_{ij}^2(0, x^-), \partial_- A_{kl}^2(0, y^-)] &= i \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \\ &\quad \times \delta(x^- - y^-), \end{aligned} \quad (2.22)$$

$$\begin{aligned} \{\lambda_{ij}(0, x^-), \lambda_{kl}(0, y^-)\} &= \sqrt{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \\ &\quad \times \delta(x^- - y^-), \end{aligned} \quad (2.23)$$

$$[\xi_i(0, x^-), \partial_- \xi_j(0, y^-)] = i \delta_{ij} \delta(x^- - y^-), \quad (2.24)$$

$$\{\psi_i(0, x^-), \psi_j(0, y^-)\} = \sqrt{2} \delta_{ij} \delta(x^- - y^-). \quad (2.25)$$

²The inclusion of zero modes is beyond the scope of the present paper.

The above (anti)commutators can also be expressed, with the help of Eqs. (2.18), (2.19), (2.20), and (2.21), in terms of creation-annihilation operators,

$$\begin{aligned} [a_{ij}(k), a_{kl}^\dagger(k')] &= \{b_{ij}(k), b_{kl}^\dagger(k')\} \\ &= \left(\delta_{ik} \delta_{jl} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \delta(k - k'), \end{aligned} \quad (2.26)$$

$$\begin{aligned} [c_i(k), c_j^\dagger(k')] &= [\tilde{c}_i(k), \tilde{c}_j^\dagger(k')] = \{d_i(k), d_j^\dagger(k')\} \\ &= \{\tilde{d}_i(k), \tilde{d}_j^\dagger(k')\} = \delta_{ij} \delta(k - k'). \end{aligned} \quad (2.27)$$

The expansion of the supercharge Q^- in terms of creation and annihilation operators is a straightforward exercise. For instance, the decomposition of Q_{CS}^- in terms of Fourier modes gives the following expression:

$$Q_{\text{CS}}^- = \left(-\frac{i\kappa\sqrt{L}}{2^{5/4}\sqrt{\pi}} \right) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_{ij}^\dagger(n)b_{ij}(n) + b_{ij}^\dagger(n)a_{ij}(n)). \quad (2.28)$$

Similarly, for the supercharge, $Q_{\text{f.matter}}^-$, that controls the behavior of the fundamental matter fields, we obtain

$$\begin{aligned} Q_{\text{f.matter}}^- &= \left(-\frac{i g \sqrt{L}}{2^{5/4} \pi} \right) \sum_{n_1, n_2, n_3=1}^{\infty} \left\{ \frac{(n_2 + n_3)}{2n_1 \sqrt{n_2 n_3}} (\tilde{c}_i^\dagger(n_3) \tilde{c}_j(n_2) b_{ji}(n_1) - \tilde{c}_i^\dagger(n_2) b_{ij}^\dagger(n_1) \tilde{c}_j(n_3) + b_{ji}^\dagger(n_1) c_i^\dagger(n_2) c_j(n_3) \right. \\ &\quad - c_i^\dagger(n_3) b_{ij}(n_1) c_j(n_2)) + \frac{1}{\sqrt{2}n_1} (\tilde{d}_i^\dagger(n_2) b_{ij}^\dagger(n_1) \tilde{d}_j(n_3) + \tilde{d}_i^\dagger(n_3) \tilde{d}_j(n_2) b_{ji}(n_1) + b_{ij}^\dagger(n_1) d_j^\dagger(n_2) d_i(n_3) \\ &\quad + d_i^\dagger(n_3) b_{ij}(n_1) d_j(n_2)) + \frac{i}{2\sqrt{n_2 n_3}} (c_i^\dagger(n_3) a_{ij}(n_2) d_j(n_1) + a_{ij}^\dagger(n_2) d_j^\dagger(n_1) c_i(n_3) + \tilde{d}_j^\dagger(n_1) a_{ji}^\dagger(n_2) \tilde{c}_i(n_3) \\ &\quad + \tilde{c}_i^\dagger(n_3) \tilde{d}_j(n_1) a_{ji}(n_2)) + \frac{i}{2\sqrt{n_1 n_2}} (a_{ji}^\dagger(n_2) c_i^\dagger(n_1) d_j(n_3) + d_j^\dagger(n_3) a_{ji}(n_2) c_i(n_1) + \tilde{d}_j^\dagger(n_3) \tilde{c}_i(n_1) a_{ij}(n_2) \\ &\quad \left. + \tilde{c}_i^\dagger(n_1) a_{ij}^\dagger(n_2) \tilde{d}_j(n_3)) \right\} \delta_{n_3, n_1 + n_2}. \end{aligned} \quad (2.29)$$

Our computer code carries out these expansions automatically.

Apart from supersymmetry, the theory we set out to explore possesses another symmetry,³ which may be used to reduce the size of the Hamiltonian matrix we need to produce and diagonalize. Namely, we have a \mathbf{Z}_2 symmetry \mathcal{T} that is associated with the orientation of the large- N_c string of partons in a state [21,22]. It gives a sign when the gauge group indices are permuted,

$$a_{ij}(k) \xrightarrow{\mathcal{T}} -a_{ji}(k), \quad b_{ij}(k) \xrightarrow{\mathcal{T}} -b_{ji}(k). \quad (2.30)$$

In this paper we will discuss numerical results obtained in the large- N_c limit; i.e. terms of order $1/N_c$ in the above expressions are dropped. Note that corrections on the order of $1/N_c$ are expected to lead to interesting effects [23]; however, they are beyond the scope of this work.

III. MESON AND GLUEBALL SPECTRA

A. Limiting cases

We first investigate the strong-coupling ($g \gg \kappa$) and weak-coupling ($g \rightarrow 0$) limits of the theory. In the strong-coupling limit, we previously found [24] that there

³Note that when $\kappa = 0$, parity (\mathcal{P}) is also conserved.

are approximate Bogomol'nyi-Prasad-Sommerfield⁴ (aBPS) glueball bound states with masses (squared),

$$M_{\text{aBPS}}^2 = n^2 \kappa^2, \quad n = 2, 3, \dots, (K - 1).$$

In the meson sector under investigation in the present paper, nearly all the masses grow with g . However, we see evidence for a state that remains near zero mass as $g \rightarrow \infty$.

The free theory can be solved analytically, and the results are shown in Fig. 1. Its free-meson spectrum, Fig. 1(a), has $K - 1$ massless states in each symmetry sector. Each such state is made out of two fundamental partons. All states in the corresponding glueball sector, Fig. 1(b), are massive. The mass scale for all states is set by the CS coupling κ . To obtain the spectrum, we consider sets of free partons that form mesonic color-singlet multi-parton combinations. There are many other combinations of such free partons that belong to the nonsinglet sector of the Hilbert space, which we can omit in the large- N_c limit; for $g \neq 0$ the only viable states are color singlets. In other

⁴Exactly massless BPS states occur in the pure $\mathcal{N} = 1$ SYM system [24] and are annihilated by one of the two supercharges, namely, Q^- . The mass of BPS states is independent of the gauge coupling, as opposed to the rest of the bound states. A Chern-Simons coupling will cause them to acquire mass, while they remain independent of the gauge coupling. The latter states we call approximate BPS states.

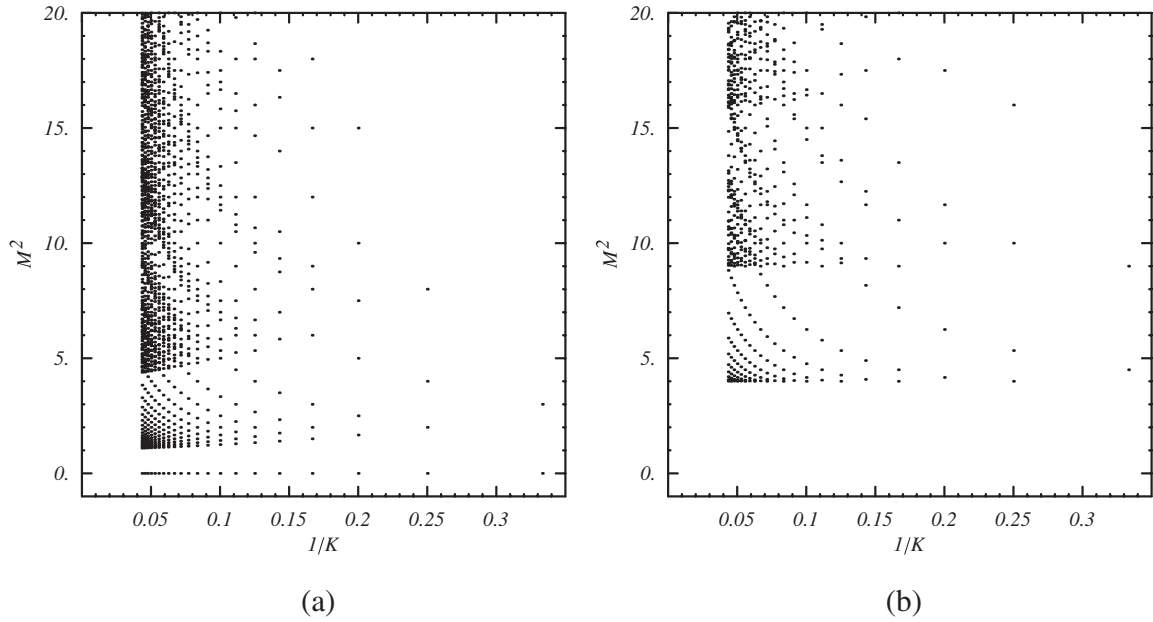


FIG. 1. Mass spectra for (a) mesonic bound states and (b) glueball bound states as a function of the inverse resolution for $3 \leq K \leq 23$ when $\kappa = 1$ and $g = 0$ (free theory).

words, the free color-singlet combinations are expected to become bound states as soon as the coupling is turned on. Mesonic multiparton color-singlet states are of the form

$$f_i^\dagger(m_1)\tilde{f}_i^\dagger(m_2)|0\rangle, f_i^\dagger(m_1)\mathcal{O}_{ij}^\dagger(n_1)\tilde{f}_j^\dagger(m_2)|0\rangle, \\ f_i^\dagger(m_1)\mathcal{O}_{ik}^\dagger(n_1)\mathcal{O}_{kl}^\dagger(n_2)\tilde{f}_l^\dagger(m_2)|0\rangle, \dots,$$

where the operators $f^\dagger, \tilde{f}^\dagger$ create fundamental partons, while \mathcal{O}^\dagger creates adjoint partons: fermions or bosons. In our theory we have *four* types of fundamental partons and *two* types of adjoint partons. In the large- N_c limit we can have the following four types of mesonic multiparton color-singlet states:

$$c_i^\dagger(\text{adj})_{ij}\tilde{c}_j^\dagger|0\rangle, c_i^\dagger(\text{adj})_{ij}\tilde{d}_j^\dagger|0\rangle, \\ d_i^\dagger(\text{adj})_{ij}\tilde{d}_j^\dagger|0\rangle, d_i^\dagger(\text{adj})_{ij}\tilde{c}_j^\dagger|0\rangle,$$

where $(\text{adj})_{ij}$ can be any string of adjoint partons. Only the adjoint partons contribute to the mass of a state and do so proportional to the CS coupling κ . Note also that the first pair of fundamental partons above forms a massless combination; there are $4(K-1)$ of these at each value of the resolution.

To construct the spectrum and find the degeneracies of each mass state, we utilize combinatorics and the DLCQ multiparticle formula

$$M_{j_{\max}}^2(K) = \kappa^2 K \sum_{l=1}^{j_{\max}} \frac{1}{n_l}. \quad (3.1)$$

Here $1 \leq j_{\max} \leq (K-2)$ is the number of adjoint partons. Specifically, we calculate the compositions

$$C_j^K = \binom{K-1}{j-1}$$

(i.e. ordered partitions) of the integer K into $j = (2 + j_{\max})$ parts, where the factor 2 counts the two massless fundamental partons in a particular state. For example, at $K = 5$ the compositions that give massive multiparticle states are $\{C_3^5, C_4^5, C_5^5\}$. The first composition in the list, C_3^5 , yields (modulo parton type) three states with $M_1^2 = 5\kappa^2$, two states with $M_1^2 = \frac{5}{2}\kappa^2$, and one state with $M_1^2 = \frac{5}{3}\kappa^2$. Thus, taking into account the several types of partons that can form C_3^5 , the total number of states we get for this case is $2^3(3+2+1) = 48$. The total number of states—including the massless ones—as a function of K is thus

$$N(K) = \sum_{k=2}^K 2^k C_k^K. \quad (3.2)$$

The dimension $N(K)$ of the Hilbert space of states grows exponentially with K , e.g. $N(16) = 28, 697, 812$.

As an example, consider a four-parton state, consisting of two fundamental and two adjoint partons, with mass

$$M_4^2 = \frac{n+m}{nm}K, \quad n, m = 1, 2, 3, \dots$$

The two adjoint partons have mass unity ($\kappa^2 = 1$), while the two fundamentals are massless. Thus at $K = 7$, we have four-parton states starting at $M^2 = \frac{35}{6}$.

The glueball spectrum is evaluated in a similar fashion. However, the free glueball multiparticle color-singlet states form closed loops made out of adjoint fermion (b_{ij}^\dagger) and boson (a_{ij}^\dagger) partons, with a mass easily obtained

by (3.1). Obviously, there are no massless states in the glueball sector. The cyclic symmetry of the color trace reduces the total number of states that are available, so the free mesons will dominate the free energy. Because of supersymmetry it suffices to count only the fermions. Using the combinatorics above, we arrive at the number of fermionic states with j partons [25],

$$N_f(K; j) = \sum_{q=0}^{\infty} \frac{(2q+1)}{j} \tilde{C}_f\left(\frac{K}{2q+1}; \frac{j}{2q+1}\right). \quad (3.3)$$

The function \tilde{C}_f is defined recursively as

$$\tilde{C}_f(K; j) = 2^{j-1} C_j^K - \sum_{q=1}^{\infty} \tilde{C}_f\left(\frac{K}{2q+1}; \frac{j}{2q+1}\right).$$

Note that $\tilde{C}_f(\frac{K}{2q+1}; \frac{j}{2q+1})$ is zero if none of its arguments is an integer. The total number of states at a specific K is found by summing over the number of partons j . For example, at $K = 5$ and $j = 3$, we have a total of 16 states. Eight of these states have mass squared $M^2 = 10\kappa^2$ and the other eight have $M^2 = \frac{35}{3}\kappa^2$.

B. Comparison of meson and glueball spectra

The generic meson and glueball spectra for nonzero g are shown in Figs. 2 and 3. In the glueball sector with finite coupling but vanishing CS coupling, there are $2(K-1)$ massless BPS states. The number of partons in these states grows with the resolution K , and there is a mass gap between these massless states and the lowest massive states that decreases with increasing resolution. When the CS coupling is not vanishing ($g \neq 0$, $\kappa \neq 0$), the BPS massless glueball bound states become approximate BPS states [24], with bound-state masses nearly independent of the

gauge coupling. The masses of the remaining states in this sector grow rapidly with the coupling.

In earlier work [7] we studied the thermodynamics of this sector with vanishing CS coupling. Here, we are considering the mesonic sector of this SYM theory. From a previous work [26,27] we know that for nonzero coupling there is a mass gap in the low-mass sector. The low-mass sector consists of the states that become massless bound states of two fundamental partons in the limit that the coupling goes to zero. This mass gap decreases as the resolution increases. Of the $K-1$ massless states in each symmetry sector at vanishing coupling, only one remains at finite coupling.

In the large- N_c limit, the mesonic and glueball sectors decouple. The thermodynamics of the theory is generated by the partition function which is the product of the partition functions of the two sectors, and the free energy is the sum of the two free energies. The glueball bound states are closed loops in color space; their cyclic symmetry greatly reduces the number of basis states. Therefore, the number of glueball states relative to the meson states at a particular K is very small. One would thus expect the mesonic bound states to dominate the thermodynamics.

There are several ways that the glueball bound states may affect the thermodynamics of the full theory. At very low temperature, the thermodynamics will be dominated by the very low-mass states. At small coupling there are many more light mesonic states than glueballs. At strong coupling there are many approximate BPS glueball bound states, while only one of the mesonic states remains massless. Thus at strong coupling and at temperature high enough to be influenced by the approximate BPS states, the thermodynamics will eventually be dominated by the glueball sector.

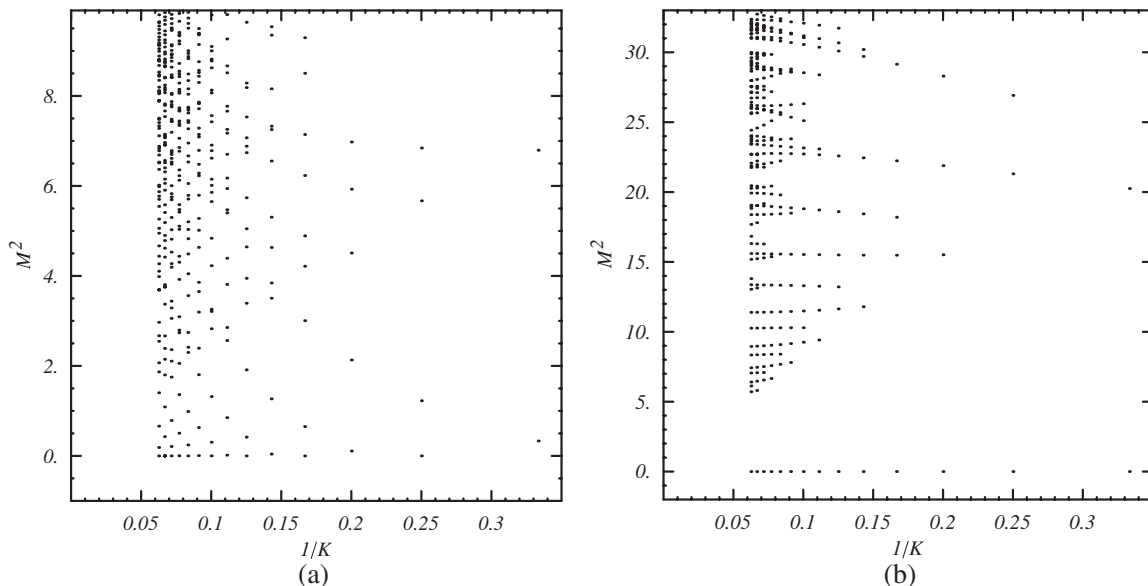


FIG. 2. Mass spectra for (a) the meson (\mathcal{P} -even, \mathcal{T} -even) sector and (b) the glueball (\mathcal{P} -even, \mathcal{T} -even) sector as a function of the inverse resolution for $3 \leq K \leq 16$ when $\kappa = 0$ and $g = 1$.

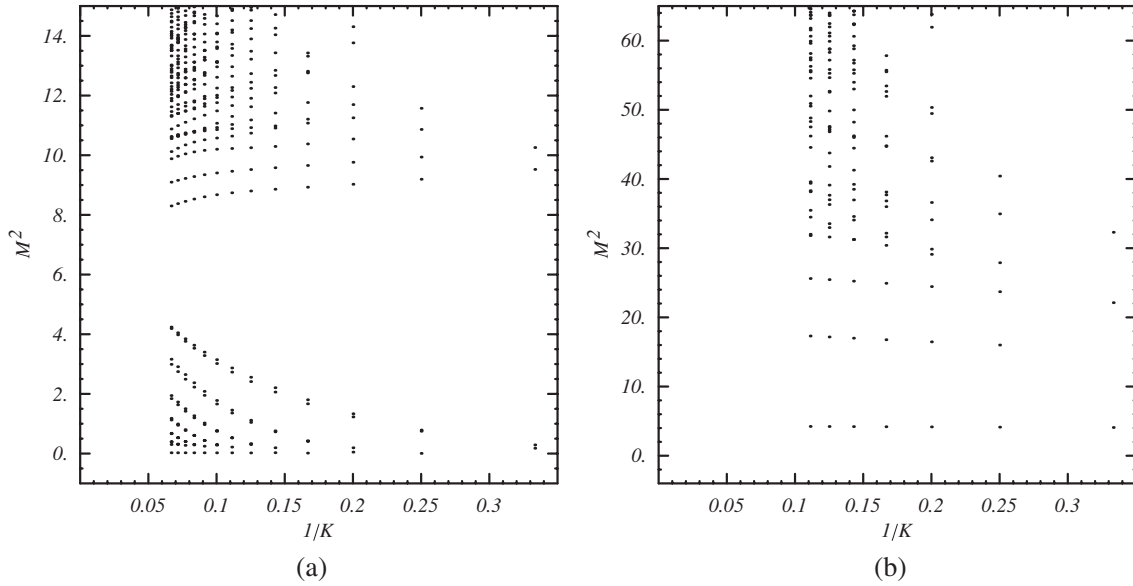


FIG. 3. Mass spectra for (a) the \mathcal{T} -even meson sector as a function of the inverse resolution for $3 \leq K \leq 15$ and (b) the \mathcal{T} -even glueball sector as a function of the inverse resolution for $3 \leq K \leq 9$. In both cases $\kappa = 1$ and $g = 1$. We note also a small splitting in the masses due to the presence of a CS term which breaks explicitly the \mathcal{P} symmetry.

IV. DENSITY OF STATES AND HAGEDORN TEMPERATURES

We calculate thermodynamic quantities from the partition function, which we express as a sum of Boltzmann factors weighted by the DoS, $\rho(M^2, K)$. The discrete spectrum is estimated numerically, and a fit to the data is used to calculate the DoS. The discrete spectrum is used to calculate the cumulative distribution function (CDF), $N(M^2, K)$, which is the number of states with mass squared below M^2 at resolution K . The DoS is related to the CDF by

$$\rho(M^2, K) \equiv \rho_K(M^2) = \frac{dN(M^2, K)}{dM^2}, \quad (4.1)$$

with dimensions of L^2 .

A comment regarding the units of the invariant mass squared eigenvalues M^2 is in order. From (2.16) it is inferred that the Hamiltonian is of the form

$$P^- = g^2 A_n + \kappa g B_n + \kappa^2 C_n = \kappa^2 \left(\frac{g^2}{\kappa^2} A_n + \frac{g}{\kappa} B_n + C_n \right), \quad (4.2)$$

where g stands for $g\sqrt{N_c/\pi}$. Thus it is a function of a dimensionless ratio, g/κ , and the dimensionful⁵ parameter κ . The latter sets the mass scale. Here we simply fix the value of κ to unity, while we numerically investigate the

⁵Choosing the CS coupling to set the mass scale is quite natural for the problem at hand, since among others we investigate the case where $g = 0$ and $\kappa = 1$, where the mass squared eigenvalues are proportional to κ^2 ; see (3.1). In general, we have the freedom of choosing the parameters such that they suit the problem.

spectra for several values of g . So the quantities we calculate are expressed in units where $\kappa = 1$.

It is interesting that for g large and $k = 1$ we have $M_i^2 = g^2 A_i$, so the eigenvalues scale with g^2 . Therefore, we may determine the strong-coupling properties of the theory from the solution of the $g = 1, \kappa = 0$ theory, a numerically much simpler problem. For some associated numerical results, see Sec. V C.

A. Estimating the density of states

The DoS can be estimated by diagonalizing P^- , computing the CDF from the spectrum, and differentiating a smooth fit to the CDF. This is what was done in previous work [7]. The size of the matrix representation of P^- increases with K and with the number of fields. Eventually, the computational cost becomes too high. To ameliorate the situation, we adapted a Lanczos-based algorithm to estimate the CDF directly.

We start by writing the density of states as

$$\rho(M^2) = \sum_n d_n \delta(M^2 - M_n^2), \quad (4.3)$$

where d_n is the degeneracy of the mass eigenvalue M_n . The CDF is just

$$N(\text{mass}^2 \leq M^2) = \int^{M^2} d\bar{M}^2 \rho(\bar{M}^2). \quad (4.4)$$

The density can be written in the form of a trace over $e^{-iP^- x^+}$ as follows:

$$\begin{aligned}
\rho(M^2) &= \frac{1}{2P^+} \sum_n d_n \delta(M^2/2P^+ - P_n^-) \\
&= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+ / 2P^+} \sum_n d_n e^{-iP_n^- x^+} dx^+ \\
&= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+ / 2P^+} \text{Tr} e^{-iP^- x^+} dx^+. \quad (4.5)
\end{aligned}$$

To approximate the trace, we use an average over a random sample of vectors [28]. Define a local density for a single vector $|s\rangle$ as

$$\rho_s(M^2) = \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+ / 2P^+} \langle s | e^{-iP^- x^+} | s \rangle dx^+, \quad (4.6)$$

so that the average can be written

$$\rho(M^2) \simeq \frac{1}{S} \sum_{s=1}^S \rho_s(M^2). \quad (4.7)$$

The sample eigenstates $|s\rangle$ can be chosen as random phase vectors [29], meaning that the coefficient of each Fock state in the basis is a random number of modulus one.

The matrix element $\langle s | e^{-iP^- x^+} | s \rangle$ can be approximated by Lanczos iteration [30]. Let D be the square of the norm of $|s\rangle$, and define $|u_1\rangle = \frac{1}{\sqrt{D}} |s\rangle$ as the initial Lanczos vector. Then we have

$$\rho_s(M^2) = \frac{D}{4\pi P^+} \int e^{iM^2 x^+ / 2P^+} \langle u_1 | e^{-iP^- x^+} | u_1 \rangle dx^+, \quad (4.8)$$

and $\langle u_1 | e^{-iP^- x^+} | u_1 \rangle$ can be approximated by the (1, 1) element of the exponentiation of the Lanczos tridiagonalization of P^- . Let P_s^- be this tridiagonal matrix, and solve the eigenvalue problem

$$P_s^- \tilde{c}_n^s = \frac{M_{sn}^2}{2P^+} \tilde{c}_n^s. \quad (4.9)$$

A diagonal matrix Λ is related to P_s^- by the usual similarity transformation $P_s^- = U \Lambda U^{-1}$, where $U_{ij} = (c_j^s)_i$ and $\Lambda_{ij} = \delta_{ij} \frac{M_{sn}^2}{2P^+}$. This means that the (1, 1) element is given by

$$(e^{-iP_s^- x^+})_{11} = \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2 x^+ / 2P^+}. \quad (4.10)$$

The local density is

$$\begin{aligned}
\rho_s(M^2) &\simeq \frac{D}{4\pi P^+} \int e^{iM^2 x^+ / 2P^+} \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2 x^+ / 2P^+} dx^+ \\
&\simeq \frac{D}{4\pi P^+} \sum_n |(c_n^s)_1|^2 2\pi \delta(M^2/2P^+ - M_{sn}^2/2P^+) \\
&\simeq \sum_n w_{sn} \delta(M^2 - M_{sn}^2), \quad (4.11)
\end{aligned}$$

where $w_{sn} \equiv D |(c_n^s)_1|^2$ is the weight of each Lanczos eigenvalue. Note that only the extreme Lanczos eigenvalues are good approximations to eigenvalues of the original

P^- ; however, the other Lanczos eigenvalues and eigenvectors provide a smeared representation of the full spectrum.

The contribution to the cumulative distribution function is

$$N_s(M^2) \equiv \int^{\bar{M}^2} d\bar{M}^2 \rho(\bar{M}^2) \simeq \sum_n w_{sn} \theta(M^2 - M_{sn}^2). \quad (4.12)$$

The full CDF is then approximated by the average

$$N(M^2) \simeq \frac{1}{S} \sum_s N_s(M^2). \quad (4.13)$$

In forming the full CDF, one has to decide how to combine theta functions. This is done by using the first sample run as a template for values M_{1n}^2 at which to evaluate N . The contributions of the other samples to N at these values are estimated by linear interpolation in cases where the Lanczos eigenvalues M_{sn}^2 are not the same as those in the first set. Also, in cases where duplicate eigenvalues are generated by the Lanczos iterations, only one is included in the template and the associated weights are added together.

The convergence of the approximation is dependent on the number of Lanczos iterations per sample, as well as the number S of samples. Test runs indicate that the recommended value [28] of 20 samples is sufficient. The number of Lanczos iterations is kept at 1000 per sample; using only 100 leaves errors on the order of 1%–2%.

A check for the validity of this approach is the comparison between the CDFs for the free theory, where the analytic solution is available. Figure 4 clearly shows that the numerical technique introduced here gives a CDF al-

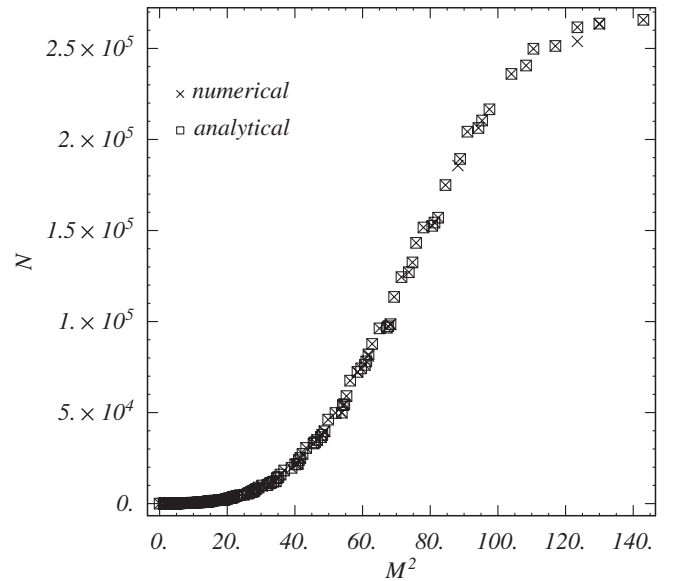


FIG. 4. CDF of the free ($g = 0$) mesonic \mathcal{T} -odd (\mathcal{T}^-) sector at $K = 13$. Crosses (boxes) refer to numerical (analytical) calculation of the bound-state masses.

most identical to the one obtained by the analytical calculation. The very few points that appear to be extraneous have no impact on the fitting algorithm we use to calculate the fits to the CDFs.

B. Fits to the spectrum

In an earlier work related to thermodynamics [7], we split the spectrum into low- and high-mass regions, separated by the mass gap. The bound-state spectrum for this problem has similar characteristics. For instance, for small values of the coupling, namely $g \lesssim 1$, the mass gap separates the $K - 1$ nearly massless color-singlet states evolving from the massless states of the free theory from the rest of the spectrum. For those values of the coupling these states have $M^2 \lesssim 1$. Thus our density of states, $\rho_K(M^2)$, is zero in the mass gap (M_1^2, M_2^2), and the CDF has the following generic form:

$$N(M^2, K) = \begin{cases} N_1(M^2, K) & \text{for } M_{\min}^2 \leq M^2 \leq M_1^2, \\ \text{const.} & \text{for } M_1^2 < M^2 < M_2^2, \\ N_2(M^2, K) & \text{for } M_1^2 \leq M^2 \leq M_{\max}^2. \end{cases} \quad (4.14)$$

We fit the low-lying mass spectrum of the CDF using the following function,

$$N_1(M^2, K) = \sum_{p=0}^{p(K)} \alpha_p M^2{}^p, \quad (4.15)$$

while the logarithm of the CDF for higher masses is fit to the following function,

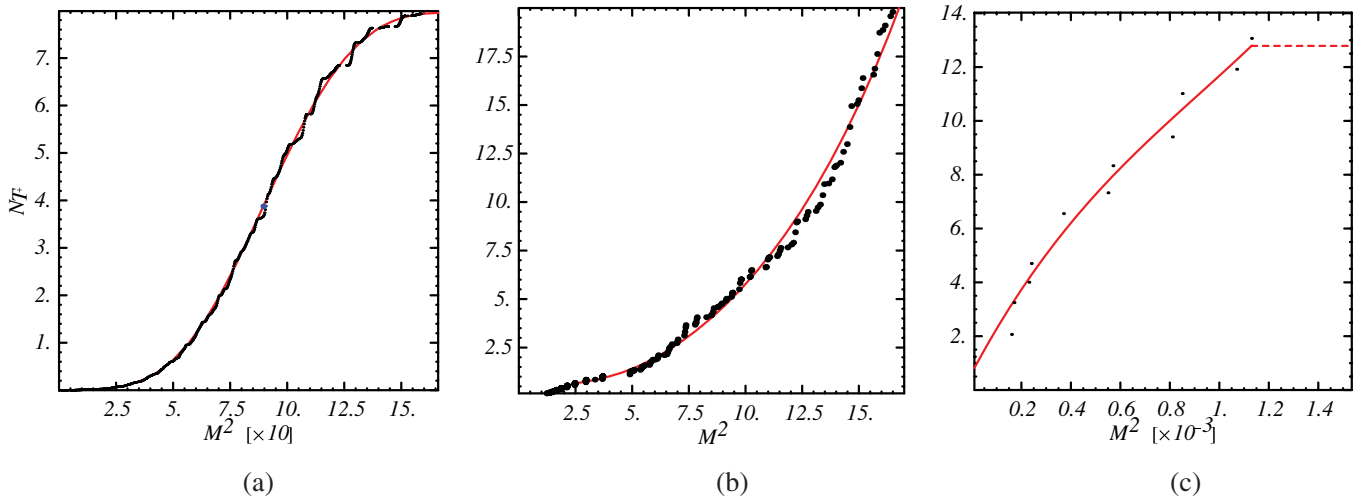


FIG. 5 (color online). CDF of the \mathcal{T} -even (\mathcal{T}^+) sector at $K = 14$ and $g = 0.1$. Shown are data (dots) and a fit to the data: (a) full spectrum with CDF in units of 10^5 states, (b) range of masses just above the mass gap with CDF in units of 10^2 states, (c) states below the mass gap with CDF in units of 1 state.

$$\ln[N_2(M_h^2, K)] = (x_h + a_1)^\gamma \exp[-b_1 M_h^{2\delta}] \sum_{p=0}^{p(K)} \alpha_p M_h^{2p}. \quad (4.16)$$

The other parameters in our fit functions are computed using standard nonlinear fit algorithms. Typical results are shown in Figs. 5–10.

The spectrum exhibits some structure at relatively small M^2 , as we can see from Figs. 5(b) and 5(c). These states dominate the thermodynamics in the range of temperatures at which we perform the free-energy calculations. Noteworthy is also the role of the point of inflection in the CDF plot and the role of the peak in the DoS plot. The data beyond this point show the effect of the cutoff imposed by the resolution K .

From the plots of the CDF and the DoS, and, in particular, the figures that depict the DoS for $g \gtrsim 0.5$, one may predict the result for the free energy \mathcal{F} . The main contribution to \mathcal{F} comes from the one nearly massless state of the spectrum. This state exists only in the \mathcal{T} -even sector, and consequently this sector dominates the thermodynamics at low temperatures. Therefore, we do not expect the fit [e.g., see Fig. 10(c)] to give us an accurate result for the free energy, because it essentially leaves out the contribution from the nearly massless state. We will return to this point when we discuss the numerical results for the free energy in Sec. V C.

C. Hagedorn temperature

We find that the physical spectrum in the theory grows approximately exponentially with the mass of the state

$$\rho_H(M^2) \sim \exp(M/T_H), \quad (4.17)$$

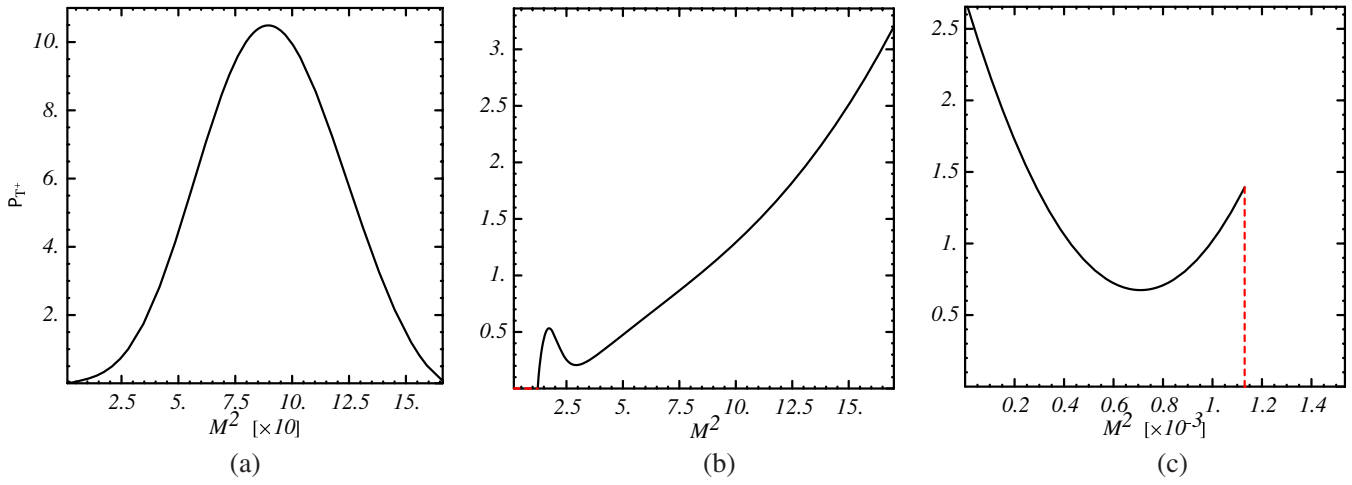


FIG. 6 (color online). DoS corresponding to the plots of Fig. 5, in units of (a) 10^3 states, (b) 10^2 states, and (c) 1 state.

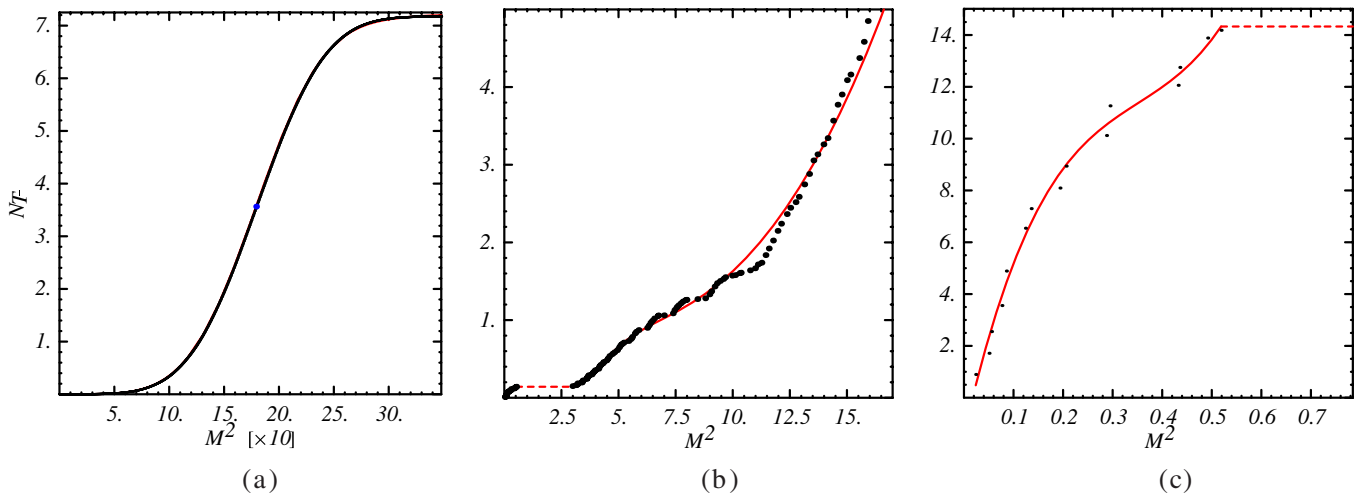


FIG. 7 (color online). CDF of the \mathcal{T} -odd (\mathcal{T}^-) sector at $K = 16$ and $g = 0.5$. Shown are data (dots) and a fit to the data: (a) full spectrum with CDF in units of 10^6 states with point of inflection; (b) range of masses just above the mass gap in units of 10^2 states; (c) states below the mass gap with CDF in units of 1 state. The relatively poor fit near $M^2 = 10$ in (b) does not have a significant effect on the results.

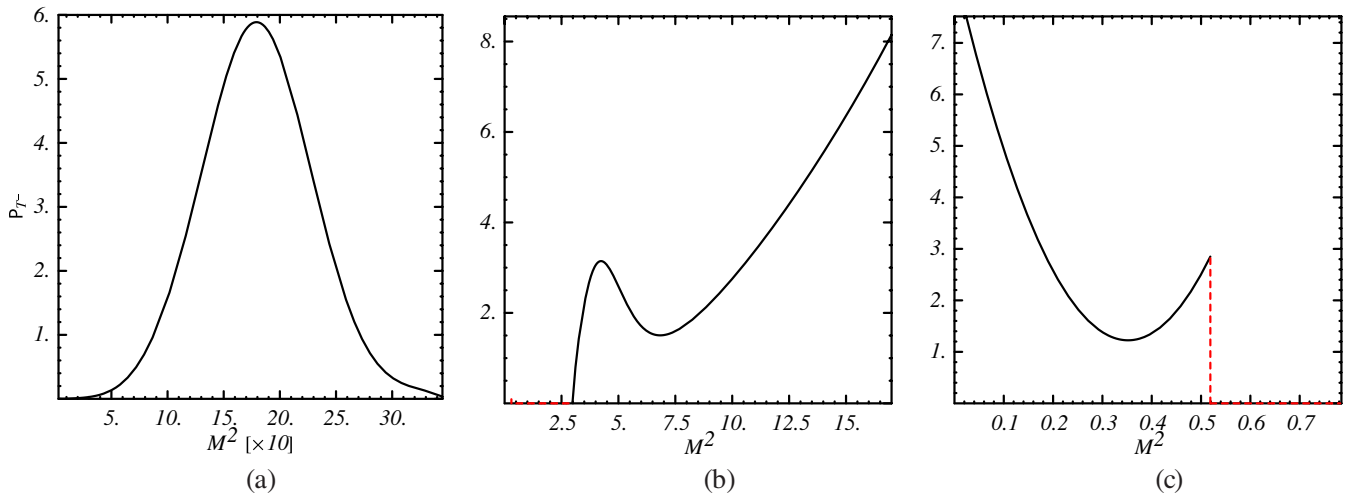


FIG. 8 (color online). DoS corresponding to the plots of Fig. 7, in units of (a) 10^4 states, and (b) and (c) 10 states.

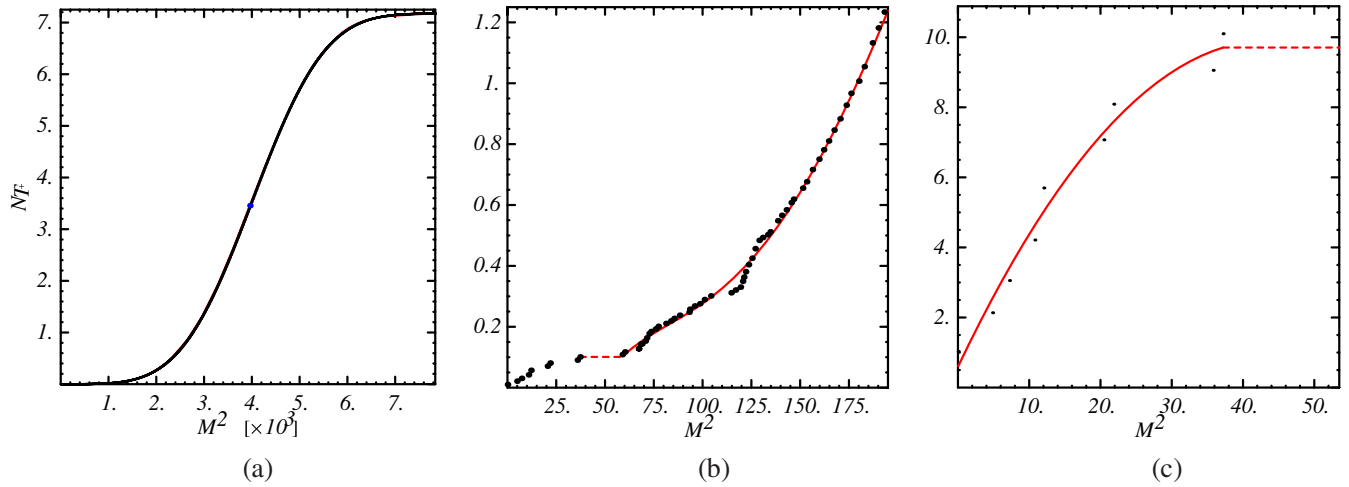


FIG. 9 (color online). CDF of the \mathcal{T} -even sector at $K = 16$ and $g = 4.0$. Shown are data (dots) and a fit to the data: (a) full spectrum with CDF in units of 10^6 states; (b) range of masses just above the mass gap in units of 10^2 states; (c) states below the mass gap with CDF in units of 1 state.

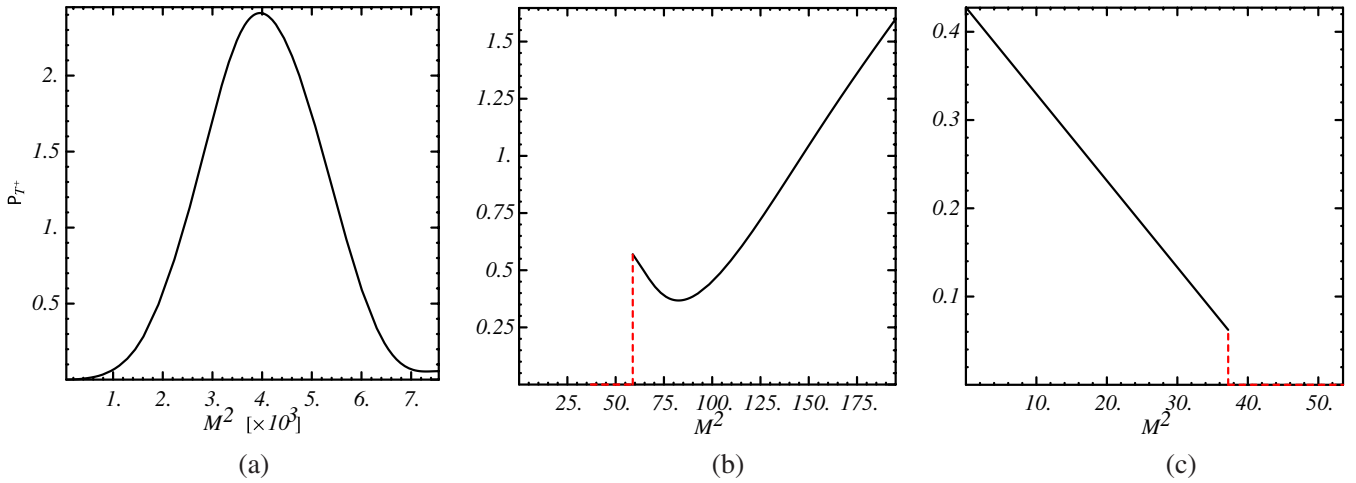


FIG. 10 (color online). DoS corresponding to the plots of Fig. 9, in units of (a) 10^3 states, and (b) and (c) 1 state.

and therefore has a Hagedorn temperature T_H . The partition function has the following general form:

$$Z \propto \int^M d\bar{M} \rho_H(\bar{M}) \exp\left(\frac{-\bar{M}}{T}\right) \propto T_H T \frac{\exp\left[M\left(\frac{1}{T_H} - \frac{1}{T}\right)\right]}{T - T_H}. \quad (4.18)$$

The partition function Z obviously diverges as $T \rightarrow T_H$, and T_H sets the region of validity for the calculation of thermodynamic properties. Thus T_H serves as an upper limit for the temperatures we can use to calculate the thermodynamic functions.

We will calculate T_H by fitting the CDF with an exponential function, and then determining its exponent as a function of the resolution and coupling. At fixed coupling we extrapolate to infinite resolution and obtain the continuum Hagedorn temperature as a function of the coupling. We fit the CDF in the region lying above the mass gap and

below the point of inflection. The number of states below the mass gap is closely related to the number of free massless states and therefore not a factor in the Hagedorn domain. The number of states above the point of inflection is significantly reduced because of the cutoff imposed by the finite resolution and is therefore not useful in determining the Hagedorn temperature.

In Fig. 11(a) we show a typical fit to the CDF in this region of M^2 . The data are fit well with an exponential. The particular figure deals with the \mathcal{T} -odd sector of the spectrum at coupling $g = 0.1$ and $K = 13$. A similar behavior occurs for the \mathcal{T} -even sector and other values of K . It is clear from our data that the spectral CDF and the DoS exhibit a Hagedorn behavior. We plot the logarithm of the CDF versus the bound-state mass obtained from our numerical calculations. Then we estimate the range of the mass values M where the plot is approximately linear, and fit this region to a linear fit of the form $\alpha M + T_H^{-1}$. The

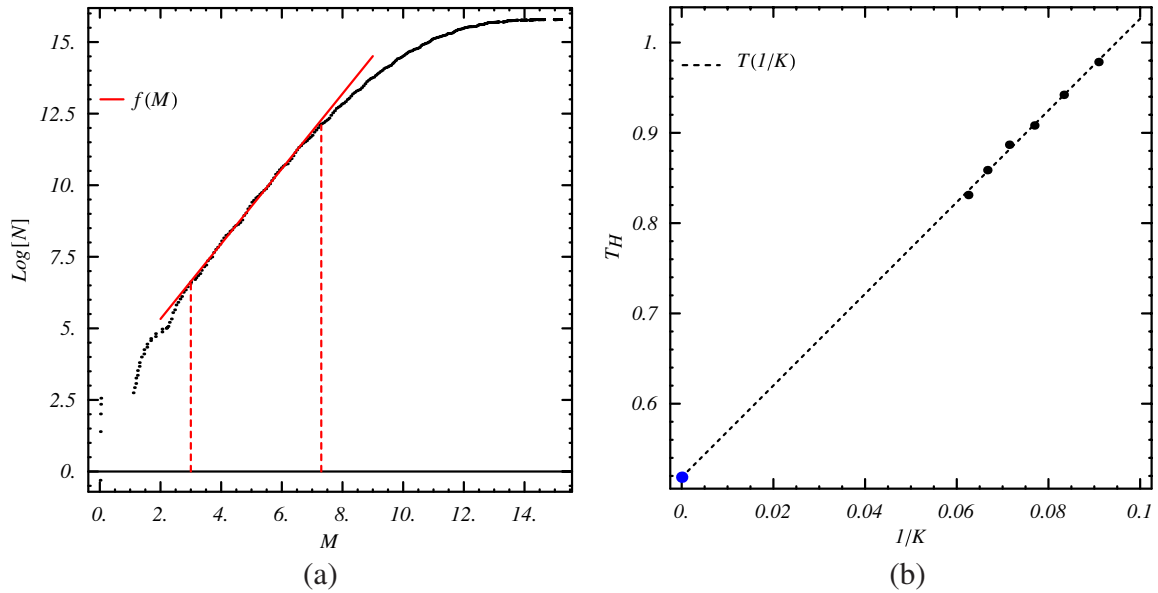


FIG. 11 (color online). (a) Logarithm of the CDF versus M . The approximately linear part of this logarithmic CDF is fit to $f(M) = \alpha M + \frac{1}{T_H}$. (b) Extrapolated Hagedorn temperature for coupling $g = 0$ with $T_H^\infty(0) \approx 0.52\kappa$.

nonlinear part of the distribution, for high values of M , is cut off because of the finite resolution K . The extrapolated result for the \mathcal{T} -even sector is very similar to the result of the \mathcal{T} -odd sector, so we take the average of the two values, for each K .

In Fig. 11(b) we show the Hagedorn temperature as a function of the inverse resolution for the massive, free theory ($g = 0$, $\kappa = 1$). The data appear to be following a

straight line and have been extrapolated to the continuum, where we found $T_H(g = 0) \approx 0.52$ in units where $\kappa = 1$. We show the Hagedorn temperature in Fig. 12 for several cases where ($g \neq 0$, $\kappa = 1$) and for resolution $K \in [11, 16]$. As a check we note that for the cases $g = 0$ and $g = 0.1$ one expects the corresponding extrapolated Hagedorn temperatures to be comparable.

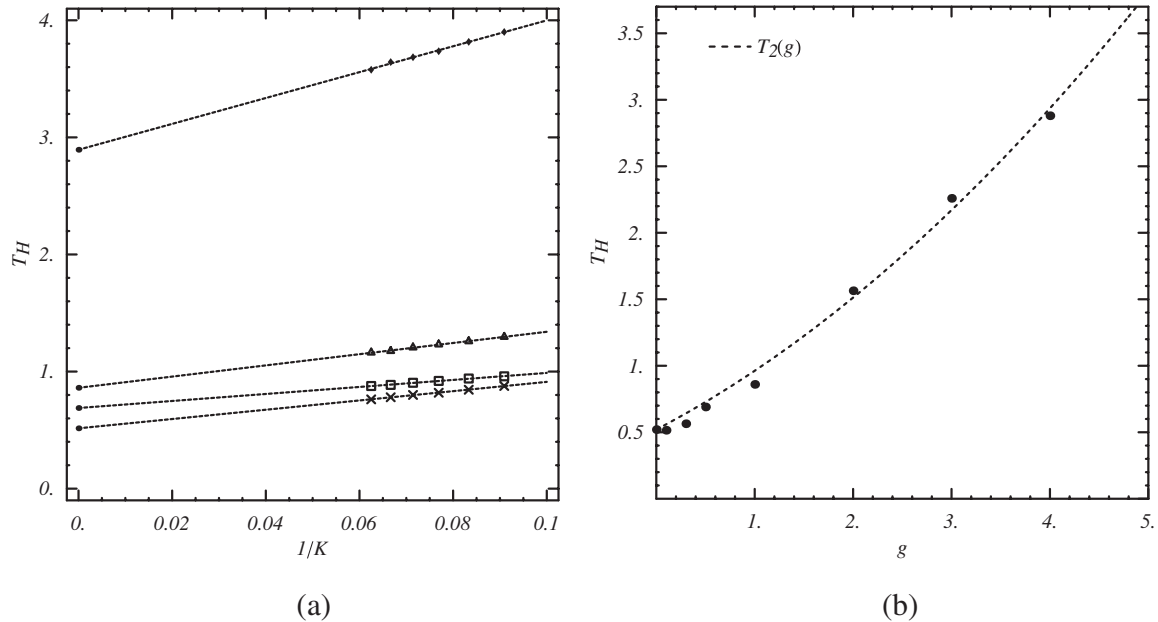


FIG. 12. Hagedorn temperature (a) plotted versus $1/K$ at couplings $g = 0.1$ (crosses), 0.5 (boxes), 1.0 (triangles), 4.0 (diamonds) in units where $\kappa = 1$, and (b) extrapolated in K as a function of g with a fit to $T_2(g) = 0.52 + 0.39g + 0.054g^2$ (dashed line). In (a), the dots at $1/K = 0$ are the continuum values. For clarity we have included only four representative values of g in (a).

For values of $g \in \{0, 4.0\}$ that are considered here, the upper bound for temperatures is set by the Hagedorn temperature of the free theory, $T_H^\infty(0) \approx 0.52\kappa$, and thus we calculate the thermodynamic properties of the theory below this limit. From Fig. 12 we glean that $T_H^\infty(g)$ grows with the coupling. For larger values of g we may therefore access a significantly larger region in T . However, we will leave the discussion of these cases for future work.

V. FINITE-TEMPERATURE RESULTS IN 1 + 1 DIMENSIONS

A. The free energy

We now introduce the basic formulation necessary for our finite-temperature calculations. Note that our approach here deviates slightly from our earlier work [7], mainly in the way the free energy and the mass squared are normalized. We consider a system with constant volume which is in contact with a heat bath of constant temperature. The free energy,⁶ in units where $k_B = 1$, is given by

$$\mathcal{F}(T, V) = -T \ln Z. \quad (5.1)$$

For the large- N_c system at hand, the thermodynamics is described by a canonical ensemble of *noninteracting* glueball and mesonlike states. The bound states of the theory constitute a supersymmetric two-dimensional free gas. The canonical free energy specific for such an idealistic noninteracting gas (due to large N_c) in D space-time dimensions is given by⁷

$$\mathcal{F}_b = T \sum_{n=1}^{\infty} \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \ln(1 - e^{-(1/T)\sqrt{\mathbf{p}^2 + M_n^2}}), \quad (5.2)$$

$$\mathcal{F}_f = -T \sum_{n=1}^{\infty} \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \ln(1 + e^{-(1/T)\sqrt{\mathbf{p}^2 + M_n^2}}) \quad (5.3)$$

for bosons and fermions, respectively, where M_n^2 in the expression for \mathcal{F}_b (\mathcal{F}_f) is the invariant bosonic (fermionic) mass spectrum. The integral is performed by expanding the logarithm and using the integral representation of the modified Bessel function of the second kind $\mathbf{K}_\nu(x)$, to find

$$\mathcal{F}_{b,M} = -2V_{D-1} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \left(\frac{M_n T}{2\pi q}\right)^{D/2} \mathbf{K}_{D/2}\left(\frac{qM_n}{T}\right), \quad (5.4)$$

⁶Although our focus is the free energy \mathcal{F} , it is straightforward to use our numerical techniques to calculate other thermodynamic functions, such as the internal energy $\mathcal{E}(T, V) = T^2 \left(\frac{\partial \ln Z}{\partial T}\right)_V$, the entropy $S = (\mathcal{E} - \mathcal{F})/T$, and the heat capacity $\mathcal{C}_V(T, V) = \left(\frac{\partial \mathcal{E}}{\partial T}\right)_V$, cf. [7].

⁷Note that for a finite- N_c scenario the relations (5.2) and (5.3) are no longer valid.

$$\mathcal{F}_{f,M} = -2V_{D-1} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} (-1)^{q+1} \left(\frac{M_n T}{2\pi q}\right)^{D/2} \mathbf{K}_{D/2}\left(\frac{qM_n}{T}\right). \quad (5.5)$$

Here V_{D-1} is the volume in $D - 1$ space dimensions. The total free energy is obtained by adding these two expressions. Because the spectrum is supersymmetric, the sums over masses M_n traverse the same spectrum, and the total free energy takes the form

$$\mathcal{F}_{\text{tot},M} = -4V_{D-1} \sum_{n=1}^{\infty} \sum_{q=0}^{\infty} \left(\frac{M_n T}{2\pi(2q+1)}\right)^{D/2} \times \mathbf{K}_{D/2}\left(\frac{(2q+1)M_n}{T}\right). \quad (5.6)$$

For our calculations we use a rescaled form of Eq. (5.6), with $D = 2$,

$$\begin{aligned} \tilde{\mathcal{F}} &\equiv -\frac{\mathcal{F}_{\text{tot},M}}{4(K-1)L} \\ &= \frac{1}{2(K-1)} \sum_{n=1}^{\infty} \sum_{q=0}^{\infty} \frac{M_n T}{(2q+1)\pi} \mathbf{K}_1\left(\frac{(2q+1)M_n}{T}\right) \\ &= \frac{1}{2(K-1)} \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} \frac{d_k M_k T}{(2q+1)\pi} \mathbf{K}_1\left(\frac{(2q+1)M_k}{T}\right). \end{aligned} \quad (5.7)$$

In the last line we introduced the factor d_k which counts degeneracies of mass eigenvalues. This equation is most efficient in the present calculation, because it expresses the free energy solely as a function of the numerically evaluated bound-state masses M_k . We have chosen to normalize by $(-4(K-1)L)^{-1}$, since $2n_0 \equiv 4(K-1)$ is the total number of massless states of the free, massive theory (with $g = 0$ and $\kappa = 1$). In practice, we can truncate the sum over Bessel functions at $q = 10$ due to fast convergence. Obviously, the sum over states is finite at any finite K .

The contribution of the massless states to the free energy in $D = 2$ dimensions can be calculated analytically to be

$$\mathcal{F}_{b,0} = -\frac{n_b L}{\pi} \int_0^\infty dp_o \frac{p_o}{e^{p_o/T} - 1} = -n_0 L T^2 \frac{\pi}{6}, \quad (5.8)$$

$$\mathcal{F}_{f,0} = -\frac{n_f L}{\pi} \int_0^\infty dp_o \frac{p_o}{e^{p_o/T} + 1} = -n_0 L T^2 \frac{\pi}{12}, \quad (5.9)$$

$$\tilde{\mathcal{F}}_0 = n_0 T^2 \frac{\pi}{16(K-1)} \quad (5.10)$$

for bosons, fermions, and the contribution to the rescaled total, respectively. Thus one may separate this contribution from the rest of Eq. (5.7).

Finally, the sum over the states is replaced by an integral over the density of states, and Eq. (5.7) becomes

$$\tilde{\mathcal{F}} = \frac{1}{2(K-1)} \int^{M^2} d\bar{M}^2 \rho(\bar{M}^2) \times \left\{ \sum_{q=0}^{\infty} \frac{\bar{M}T}{(2q+1)\pi} \mathbf{K}_1\left(\frac{(2q+1)\bar{M}}{T}\right) \right\}, \quad (5.11)$$

where $\rho(M^2) = \rho_b(M^2) = \rho_f(M^2)$ for supersymmetric systems. The \mathcal{T} symmetry splits the bosonic and fermionic sectors into halves. Thus when calculating the free energy, or other thermodynamic properties, we can write

$$\begin{aligned} \tilde{\mathcal{F}} &= (\tilde{\mathcal{F}}_b + \tilde{\mathcal{F}}_f)_{\mathcal{T}^+} + (\tilde{\mathcal{F}}_b + \tilde{\mathcal{F}}_f)_{\mathcal{T}^-} \\ &= \tilde{\mathcal{F}}_{\text{tot}\mathcal{T}^+} + \tilde{\mathcal{F}}_{\text{tot}\mathcal{T}^-}. \end{aligned} \quad (5.12)$$

B. An analytic result: The free energy for the free theory

Let us start by exploring the free, massive theory ($g = 0$, $\kappa = 1$), which can be solved analytically. We will compare the contributions of the meson and glueball sectors to the free energy. In particular, using the free-meson sector we can check the validity of our approach to replace the sums over discrete spectra with a density of states function and check how good our numerical results are compared to an analytic calculation.

First, we compare the free energy obtained by the means of the analytic method outlined above, $\tilde{\mathcal{F}}_{\text{spect}}$, to the free energy extracted from the numerical approach, $\tilde{\mathcal{F}}_{\text{fit}}$, using the DoS. The graph presented in Fig. 13(a) compares analytic and numerical results at $K = 16$ for temperatures

TABLE I. The free energy as a function of the temperature in the meson and the glueball sectors in the free theory.

T/κ	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
$\tilde{\mathcal{F}}_{\text{meson}}/\kappa^2$	0.009	0.016	0.026	0.040	0.059	0.089	0.133	0.199
$\tilde{\mathcal{F}}_{\text{glueball}}/\kappa^2$	0.000	0.000	0.001	0.004	0.011	0.026	0.053	0.101

$0.015 \leq T \leq 0.5$, in units where $\kappa = 1$. We deduce from the plot that the agreement is within 1%, which is a typical result. Cutoff dependence is very mild: $\tilde{\mathcal{F}}_{\text{spect}}/\tilde{\mathcal{F}}_{\text{fit}}(K = 13) = 1.012$, while $\tilde{\mathcal{F}}_{\text{spect}}/\tilde{\mathcal{F}}_{\text{fit}}(K = 16) = 1.015$. In Fig. 13(b) we show the free energy of the free, massive theory as a function of the inverse resolution at different temperatures. It seems that the free energy $\tilde{\mathcal{F}}$ converges for low and intermediate temperatures, while it diverges for temperatures close to the Hagedorn transition in the continuum limit, as expected.

We can extract the contributions to the free energy of different parts of the spectrum by using Eq. (5.7). It is interesting to compare the contributions of the two non-interacting sectors of the theory. At temperature $T = 0.5\kappa$ and $K = 5$, the three-parton meson states contribute $1.9 \times 10^{-2}\kappa^2$ to the free energy, while the corresponding $K = 5$, three-parton glueball state contributes only $3.9 \times 10^{-4}\kappa^2$. Results at different temperatures are listed in Table I. The free energy $\tilde{\mathcal{F}}_{\text{mesons}}$ associated with the free-meson sector dominates the corresponding $\tilde{\mathcal{F}}_{\text{glueball}}$. This is a consequence of the fact that the mesonic spectrum has $4(K-1)$ massless states that contribute $\frac{\pi}{8}T^2$ to $\tilde{\mathcal{F}}$ and that the

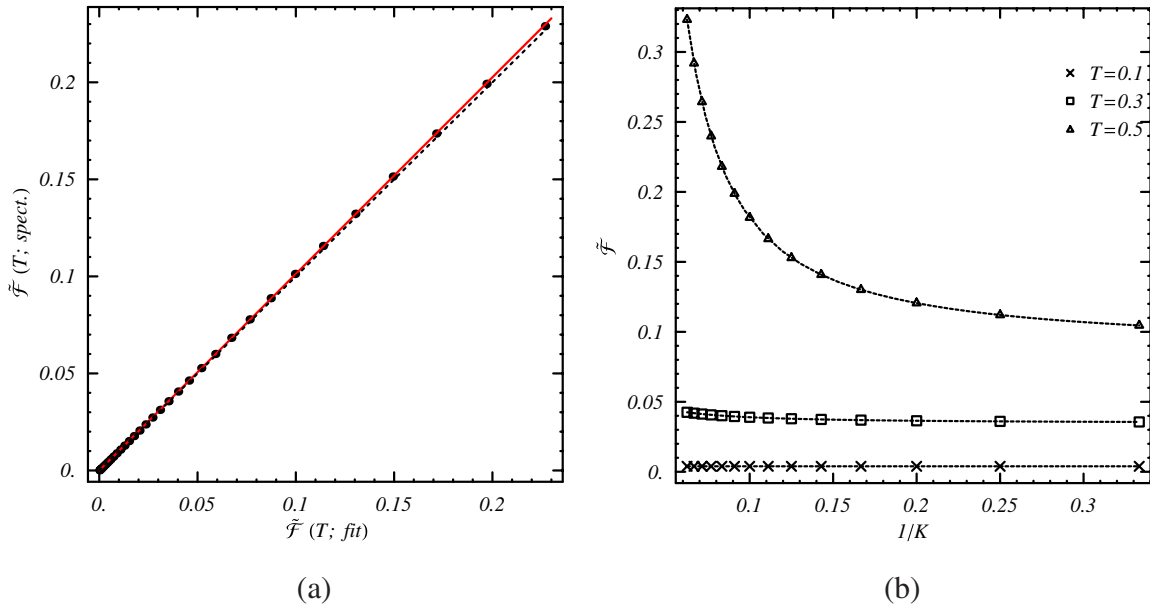


FIG. 13 (color online). The free energy $\tilde{\mathcal{F}}$ for the free theory ($g = 0$, $\kappa = 1$) as (a) compared between the analytic and numerical results and (b) a function of $1/K$. In (a) the temperature ranges from 0.015 to 0.5, in units where $\kappa = 1$, by steps of $\Delta T = 0.015\kappa$. The dashed line represents an exact match relation, the solid line the actual relation between $\tilde{\mathcal{F}}_{\text{spect}}$ and $\tilde{\mathcal{F}}_{\text{fit}}$. In (b) the temperatures are $T = 0.1\kappa$ (crosses), 0.3κ (boxes), 0.5κ (triangles).

meson sector possesses many more states than the glueball sector, especially low-mass states which are important in the present calculation.

C. Numerical results for nonzero coupling

We now discuss numerical results for the free energy at finite values of g . These results were obtained by applying the DoS method, i.e. replacing the sum over states by an integral over the bound-state masses times a density of states computed from a fit to the numerically obtained CDF. First, we discuss the temperature dependence and then the coupling dependence of the free energy $\tilde{\mathcal{F}}$, for temperatures that lie below the zero-coupling Hagedorn temperature $T_H(g=0) \approx 0.52\kappa$.

1. Temperature dependence of the free energy

The DoS method works also for the interacting theory ($g > 0$). Using either the discrete spectrum approach [sum over the states; (5.7)] or the DoS fit to the spectrum yields good agreement, at least for weak to medium couplings. The disagreement between the two approaches is typically below 1%, as seen in Table II. This table also presents contributions to the free energy from about a thousand states up to $M^2 = 9.10283\kappa^2$, which belong to the \mathcal{T} -even sector at resolution $K = 14$ and weak coupling $g = 0.1$. We also consider, in Table III, the free energy at resolution $K = 16$ for large coupling $g = 4.0$.

It is verified from these tables that, at low temperatures, the major contribution to the free energy comes from the

TABLE II. Free energy as a function of temperature T at $K = 14$ in the \mathcal{T} -even sector for weak coupling, $g = 0.1$: $\tilde{\mathcal{F}}_{\text{spect}}$ is obtained by summing over the eigenvalues in the interval $M^2 \in [0.00001, 9.10283]\kappa^2$; $\tilde{\mathcal{F}}_{\text{fit}}$ is obtained by the DoS method described in Sec. IVA. $\tilde{\mathcal{F}}_{\text{fit}}^<$ and $\tilde{\mathcal{F}}_0$ are the contributions to the latter of the states below the mass gap (i.e., $M^2 < 0.00113\kappa^2$) and of a single supersymmetric massless state, respectively.

$T (\kappa)$	$\tilde{\mathcal{F}}_{\text{spect}} (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_{\text{fit}} (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_{\text{fit}}^< (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_0 (10^{-2}\kappa^2)$
0	0	0	0	0
0.025	0.00658	0.00654	0.00654	0.0009
0.05	0.03459	0.03448	0.03448	0.0038
0.075	0.08522	0.08499	0.08499	0.0085
0.1	0.15852	0.15810	0.15809	0.0151
\vdots	\vdots	\vdots	\vdots	\vdots
0.225	0.89782	0.89487	0.86245	0.0765
0.25	1.14656	1.14269	1.07109	0.0944
0.275	1.44823	1.44337	1.30232	0.1142
0.3	1.81661	1.81078	1.55613	0.1359
\vdots	\vdots	\vdots	\vdots	\vdots
0.425	5.33793	5.33113	3.16396	0.2728
0.45	6.54702	6.54176	3.55327	0.3059
0.475	7.98870	7.98597	3.96515	0.3408
0.5	9.69247	9.69351	4.39962	0.3776

TABLE III. Results for free energy as a function of temperature T at $K = 16$ and strong coupling, $g = 4.0$. $\tilde{\mathcal{F}}$ corresponds to the overall free energy including both symmetry sectors. The third column shows the overall contribution of the \mathcal{T} -even sector, the fourth the contribution from the single nearly massless state, $M^2 \approx 0.0362\kappa^2$, and the last column is the contribution that a supersymmetric massless state would make, if it were present.

$T (\kappa)$	$\tilde{\mathcal{F}} (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_{\mathcal{T}^+} (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_1 (10^{-2}\kappa^2)$	$\tilde{\mathcal{F}}_0 (10^{-2}\kappa^2)$
0	0	0	0	0
0.025	0	0	0	0.0008
0.05	0.0002	0.0002	0.0002	0.0033
0.075	0.0011	0.0011	0.0011	0.0074
0.1	0.0033	0.0033	0.0033	0.0131
\vdots	\vdots	\vdots	\vdots	\vdots
0.225	0.0384	0.0383	0.0383	0.0663
0.25	0.0505	0.0503	0.0503	0.0818
0.275	0.0644	0.0641	0.0639	0.0990
0.3	0.0803	0.0796	0.0793	0.1178
\vdots	\vdots	\vdots	\vdots	\vdots
0.425	0.1959	0.1867	0.1813	0.2364
0.45	0.2283	0.2149	0.2068	0.2651
0.475	0.2647	0.2458	0.2340	0.2953
0.5	0.3056	0.2795	0.2628	0.3272

low-lying states, i.e. the states below the mass gap. In the case of Table III, we have ten such states. As we increase the temperature, more states from above the mass gap will contribute significantly. The results are shown in Fig. 14 for small coupling, $g = 0.1$, and in Fig. 15 for large coupling, $g = 4.0$. For $K = 16$ we find that, at a coupling value of $g = 0.1$, the free energy is quadratic in T . This is expected since for low values of temperature the quasi-massless modes dominate and their contribution should be similar to the massless states of the free theory. The fourth column of Table II, which considers contributions from states below the mass gap (e.g., $M^2 \in [0.0001, 0.00113]\kappa^2$), shows clearly that these states dominate.

At large coupling ($g = 4.0$) we obtain free energies that are about a hundred times smaller than the available weak-coupling free energies. This is less than the free energy $\tilde{\mathcal{F}}_0$ that would be contributed from a single massless state. However, it can be justified from the spectrum at these temperatures. Recall Figs. 9 and 10, especially those plots depicting the spectrum below the mass gap. For the data shown in Table III, the dominant symmetry sector is \mathcal{T} even, as expected since this is the sector which includes the lightest state. We denote the contribution of this lightest state by \mathcal{F}_1 . This single state accounts for almost all the contribution to the free energy for $0 \leq T \leq 0.5\kappa$. The latter result is close to the contribution $\tilde{\mathcal{F}}_0$, that would be made by a single, exactly massless mode, suggesting that this very light state may be approximated with a massless state. Finally, at this coupling and $K = 16$, we show in

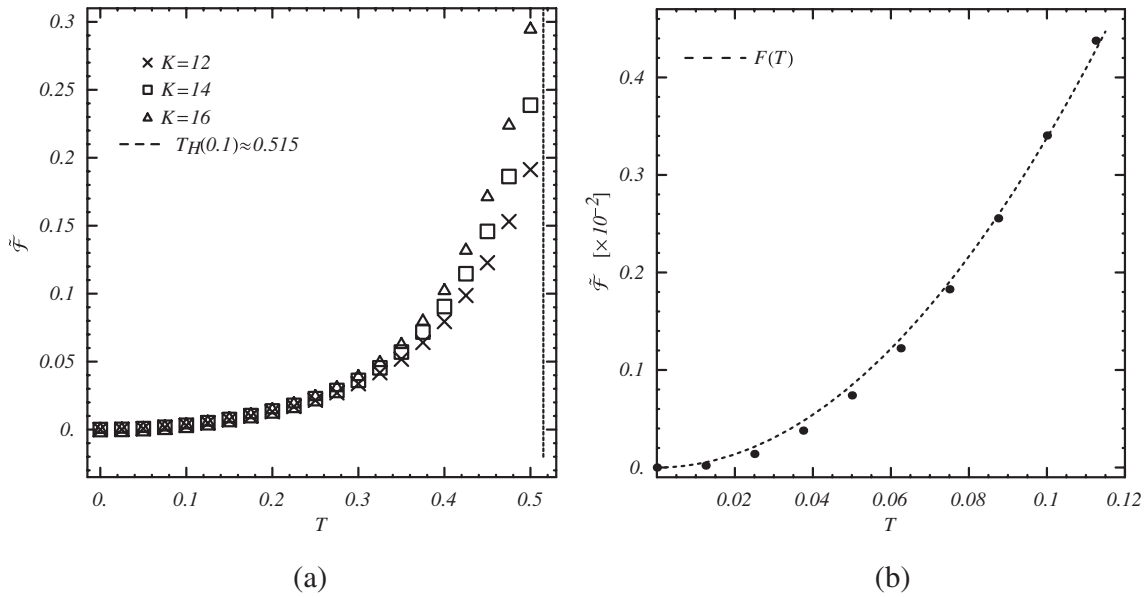


FIG. 14. Free energy $\tilde{\mathcal{F}}$ at weak coupling ($g = 0.1$) as a function of T at $K = 12$ (crosses), 14 (boxes), 16 (triangles) for (a) all temperatures $T < T_H$ (dashed vertical line) and (b) low temperatures with a quadratic fit $F(T) = \alpha_w T^2$.

Fig. 15(b) the behavior of the free energy at low T , which appears to be quartic.

2. Coupling dependence of the free energy

The behavior of the free energy as a function of the coupling is summarized in Figs. 16–18. For relatively low temperatures ($T \approx 0.1$) and for values of g on the order of 1 and above (see Fig. 16), the DoS fit misses the most important contribution, which is expected from the single lightest state in the \mathcal{T} -even sector. For instance, for the

resolution $K = 16$ at coupling $g = 4.0$, the lightest bound state has $M^2 = 0.0362\kappa^2$ and the next available state is at $M^2 = 4.86\kappa^2$. Although the fit in Fig. 9(c) seems to capture quite well the behavior of the states below the mass gap, states which are expected to dominate the thermodynamics at low temperatures, it yields $\tilde{\mathcal{F}}(T = 0.1) \approx 10^{-6}\kappa^2$. This is not what we expect from the CDF data. The free energy should be close to the contribution of a pair of massless supersymmetric partners, $\frac{\pi}{16(K-1)}T^2$. A way to improve the calculation of the free energy is to use the

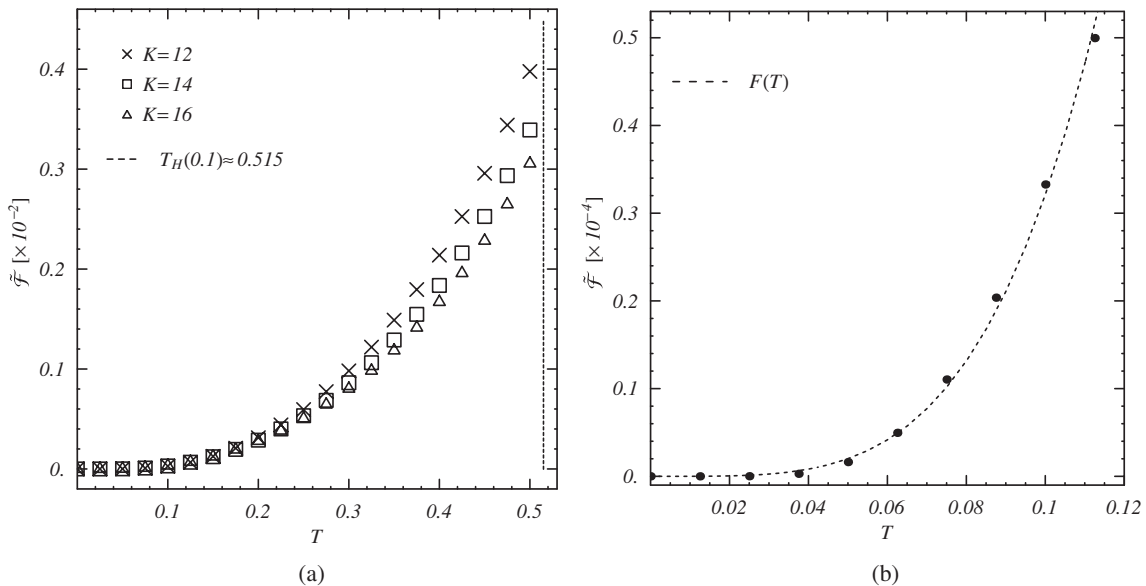


FIG. 15. Same as Fig. 14, but for strong coupling ($g = 4.0$). In (b) the low temperature behavior is described by a quartic fit $F(T) = \alpha_s T^4$.

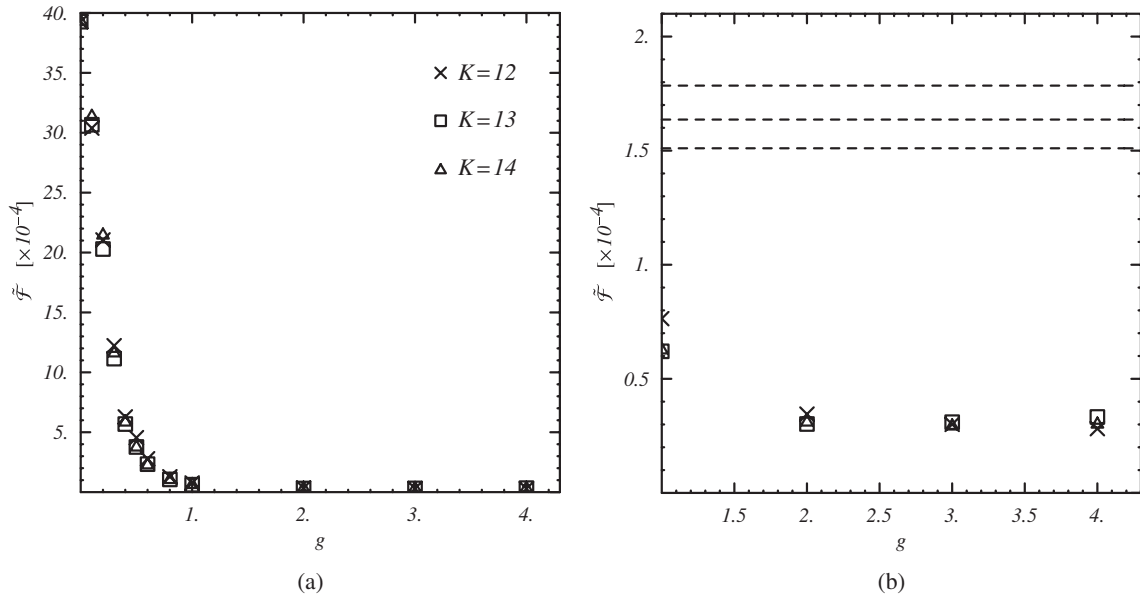


FIG. 16. The free energy as a function of the coupling g at temperature $T = 0.1\kappa$ (with $\kappa = 1$) and for resolutions $K = 12$ (crosses), 13 (boxes), 14 (triangles) with two different vertical scales. In (b) we see along with the data the contribution to the free energy that would be made by a pair of exactly massless superpartners. For fairly large values of g and at this temperature, the overall free energy is small compared to the contribution of a single pair of massless states. This is expected at this coupling region because the masses are very large and are suppressed by the modified Bessel function, $\mathbf{K}_1(x)$.

discrete spectrum and sum over the states instead of approximating this part of the spectrum with a fit function. By extracting the states' degeneracies from the CDF data and by utilizing Eq. (5.7), we obtain $\tilde{\mathcal{F}}(T = 0.1) \approx 3.33 \times 10^{-5} \kappa^2$, which matches the expectations much better. In fact, the value of $\tilde{\mathcal{F}}$ is the contribution of one supersymmetric \mathcal{T} -even state. The \mathcal{T} -odd sector does not contribute significantly to $\tilde{\mathcal{F}}$, since its lightest state

($M^2 = 3.651\kappa^2$) is heavily suppressed due to the Bessel factor $\mathbf{K}_1(M/T)$ at $T = 0.1$.

Although failing here, generally (at relatively weak couplings and small temperatures) the fit does a good job, mainly because the states below the mass gap are very light compared to those for large g , and therefore not suppressed by the Bessel function, $\mathbf{K}_1(M/T)$, of Eqs. (5.7) and (5.11). Therefore, for large values of the

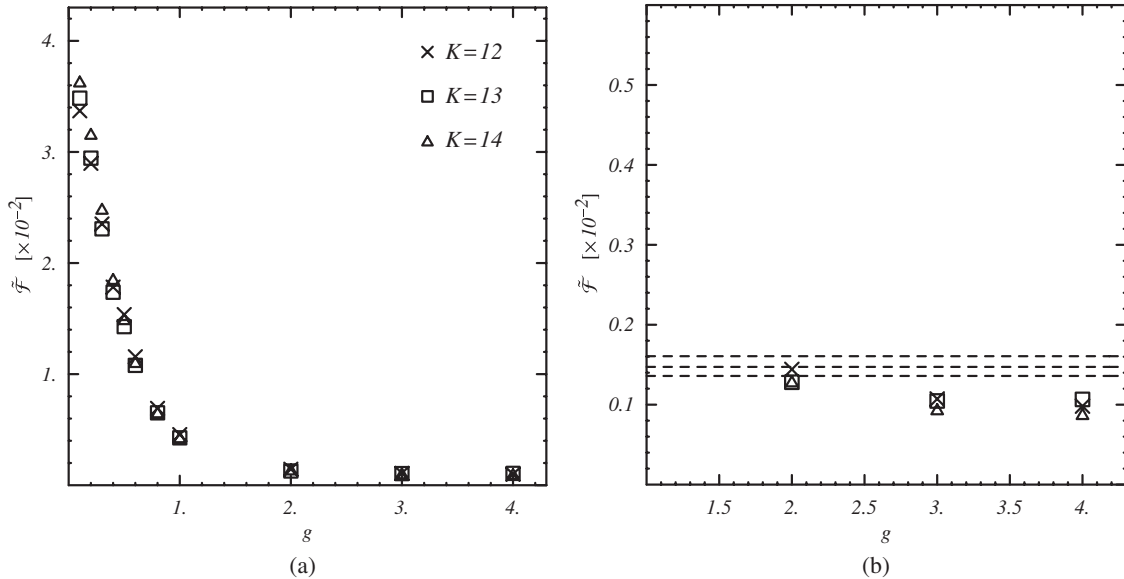


FIG. 17. Same as Fig. 16 but for $T = 0.3\kappa$. From (b) it is clear that, for values of $g > 1.0$ and at this temperature, the contribution to the free energy resembles the one from the nearly massless state in the \mathcal{T} -even sector.

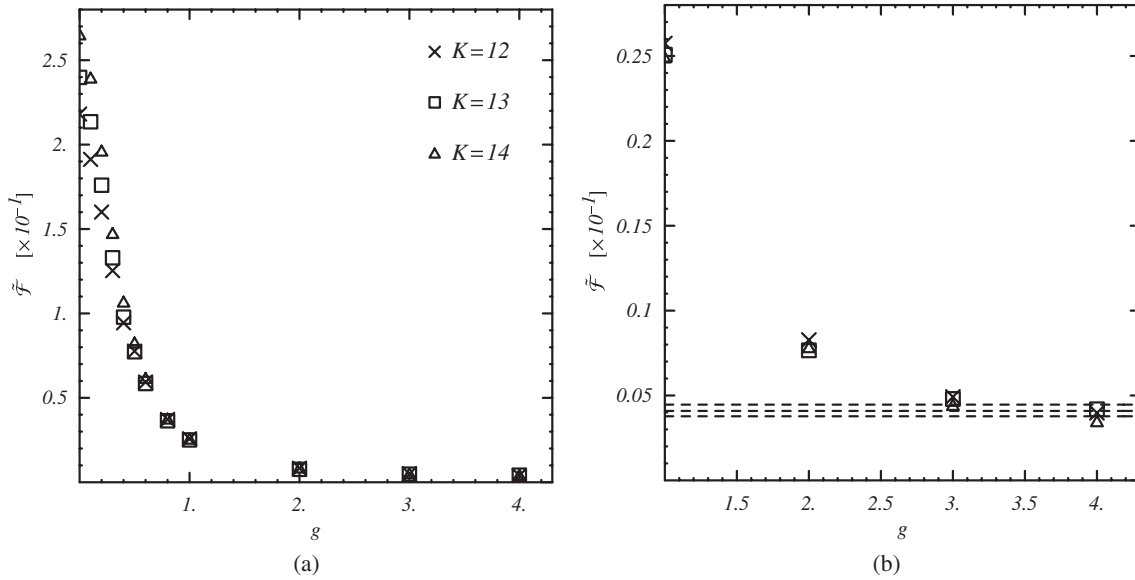


FIG. 18. Same as Fig. 16, but for $T = 0.5\kappa$. From (b) it is clear that more states contribute to the free energy at higher temperature, and at relatively large values of g .

coupling, we see that, as the temperature is gradually being increased, the contribution to the free energy becomes similar to the one coming from an exact massless mode; this contribution is included as dotted lines in Figs. 16–18. These results are also in accord with the results presented in Tables II and III.

As a check, we have compared results for the massive, strongly coupled theory (g large, $\kappa = 1$) and the massless theory ($g = 1$, $\kappa = 0$), where g is the only scale factor. We

expect the strongly coupled theory to have masses M related to the masses M_* of the massless theory by $M^2 = g^2 M_*^2$. At $g = 4.0$, we find $M^2 \approx 4.05^2 M_*^2$. The free energies are related by

$$\tilde{\mathcal{F}}(T, M^2, K) = g^2 \tilde{\mathcal{F}}_*(T/g, M_*^2, K).$$

We also calculated the free energy with the DoS method described earlier, and we found that it matches quite well the free energy of the theory with $g = 4.0$ and $\kappa = 1$. This

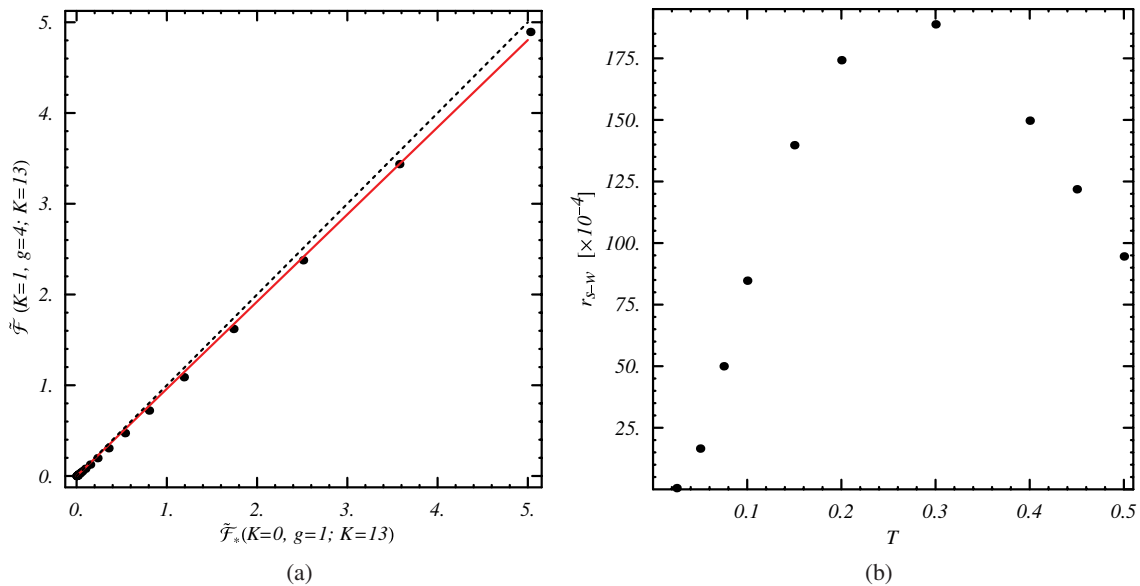


FIG. 19 (color online). (a) Comparison of the free energies $\tilde{\mathcal{F}}(g = 4, \kappa = 1; K = 13)$ and $F'(g = 1, \kappa = 0; K = 13) \equiv 4^2 \tilde{\mathcal{F}}_*$ at various temperatures between 0.015κ and 0.5κ with steps of $\Delta T = 0.015\kappa$, for $\kappa = 1$. The dashed line represents a perfect match, the solid line the best linear fit $\tilde{\mathcal{F}}(g = 4, \kappa = 1) = 0.97 F'(g = 1, \kappa = 0)$. (b) Strong/weak-coupling ratio r_{s-w} as a function of the temperature at $K = 16$.

is shown in Fig. 19(a). Therefore, by solving a numerically less challenging problem, i.e. the model with no CS term, we were able to determine the strong-coupling behavior of the theory with a CS term.

Having established that $g = 4.0$ is a relatively strong coupling, and by knowing the exact, weak-coupling ($g = 0$) free energy, let us calculate the strong/weak-coupling free-energy ratio r_{s-w} at $K = 16$, the highest available resolution in our calculations. At low temperature, $T = 0.1\kappa$, we get $r_{s-w}(K = 16) \approx 8.47 \times 10^{-3}$, and at $T = 0.5\kappa$ we obtain $r_{s-w}(K = 16) \approx 9.46 \times 10^{-3}$. Results for several temperatures are summarized in Fig. 19(b). A quartic fit to the strong-coupling data reveals that, at resolution $K = 16$, the ratio is

$$r_{s-w} = \frac{\tilde{\mathcal{F}}_s}{\tilde{\mathcal{F}}_w} \approx 0.95T^2.$$

This is consistent with the fact that at low temperatures and weak coupling, g , the massless states dominate and contribute to the free energy $\tilde{\mathcal{F}}_w$ proportional to T^2 . Our CDF data suggest that in the continuum limit, however, the lightest (nearly massless) state of the strongly coupled theory will become exactly massless, and also yield a contribution proportional to T^2 at low temperatures. On the other hand, we know that in the weak-coupling ($g \sim 0$) theory for finite K there are exactly $2(K - 1)$ massless pairs of bound states and the ratio r_{s-w} , for large K , becomes $r_{s-w}(K) = (2(K - 1))^{-1}$. Thus, we conclude that in the continuum limit

$$r_{s-w} \xrightarrow{K \rightarrow \infty} 0,$$

and the discrepancy between the strongly and weakly coupled theories becomes maximal. However, we cannot exclude the possibility that we may have more than one pair of massless states in the strongly coupled sector of the theory for large values of K , namely, a number of massless states proportional to K . Although from Fig. 2(a) which refers to the strongly coupled system (dual theory), one may try to argue in favor of the latter statement that the

mass of states at small K seems to follow a trend towards the massless limit for relatively large K . However, this is not a definitive result, at least from our data, because the highest resolution we have is only up to $K = 16$. Therefore, as far as thermodynamics is concerned, in this paper we will just assume that we only have one massless pair in the continuum limit of the strongly interacting sector. Further results for r_{s-w} for several values of T and K are presented in Fig. 19(b) and Table IV. It seems that r_{s-w} decreases with K starting at medium temperatures.

VI. DISCUSSION

We have studied the thermodynamics of $\mathcal{N} = (1, 1)$ super Yang-Mills theory in $1 + 1$ dimensions with fundamentals and a Chern-Simons term that gives mass to the adjoint partons. We used SDLCQ to solve the theory in the large- N_c approximation. The theory has two classes of bound states: glueballs, which form a closed string in color space, and mesonlike states, which form open strings. In the large- N_c approximation, these two sectors do not interact with each other, and make independent contributions to the thermodynamics. We previously calculated the contribution of the glueball sector but without a CS term. We found that the mesonlike sector dominates the glueball sector for combinatorial reasons, and, therefore, the results presented here represent the full thermodynamics of the theory. Adding a CS term to the theory introduces an additional parameter, and thus allows us to inquire about the coupling dependence of the theory.

We have been able to take the calculation up to resolution $K = 16$, which effectively means that we are diagonalizing matrices that are of order 7×10^6 by 7×10^6 in our approximation of the continuum field theory. We introduced a new Lanczos method, which is particularly valuable in our calculation of the Hagedorn temperature.

It is interesting that the spectrum for this theory has a mass gap, which we have discussed extensively. The states below the mass gap dominate the low temperature behavior of the theory, while the states above the mass gap and

TABLE IV. Data for strong- to weak-coupling ratio r_{s-w} for $K = 13, 14, 15, 16$ at various temperatures. We show $r_{s-w}(K = 16)$ as a function of T in Fig. 19(b).

$T (\kappa)$	$r_{s-w}(K = 13) (10^{-4})$	$r_{s-w}(K = 14) (10^{-4})$	$r_{s-w}(K = 15) (10^{-4})$	$r_{s-w}(K = 16) (10^{-4})$
0.025	0.12	0.19	0.29	0.50
0.05	10.44	11.32	12.89	16.56
0.075	42.95	41.24	42.71	49.97
0.1	84.74	76.62	75.77	84.72
0.15	161.82	138.05	130.57	139.74
0.2	217.32	180.08	166.48	174.25
0.3	262.02	208.95	186.77	188.81
0.4	235.75	180.94	154.97	149.70
0.45	207.06	155.59	129.71	121.84
0.5	174.62	128.31	103.76	94.57

below the point of inflection of the CDF determine the Hagedorn temperature. In fact, the very low temperature behavior is dominated by a few massless or nearly massless states in the theory.

The determination of the Hagedorn temperature from the states beyond the mass gap requires a detailed understanding of the SDLCQ spectrum and careful fitting techniques. We have checked our numerical methods by comparing the solutions of the free, massive theory obtained numerically to those extracted analytically.

We extrapolated the Hagedorn temperature at fixed coupling to the continuum limit. The process is repeated at various values of the coupling to determine the coupling dependence of the Hagedorn temperature. We find that it increases with the coupling from a value of about $T_H = \frac{1}{2}\kappa$ at $g = 0$ to a value of nearly 3.0κ at a coupling of $g = 4.0$.

We calculate the free energy of the theory as a function of both the temperature and the coupling. As the coupling vanishes, the bound-state spectrum can be obtained analytically; the analytic results agree with our SDLCQ calculations. The theory has $4(K - 1)$ massless fermionic bound states and an equal number of bosonic bound states. At low temperature and near-zero coupling, the free energy is simply given by the contribution of these massless states, which can be calculated analytically. As the temperature increases, the free energy grows quadratically and starts to diverge as the temperature approaches the Hagedorn temperature. As we discussed above, this point of divergence increases with the coupling.

At strong coupling and very low temperature, the nearly massless bound states dominate the free energy. We find one such fermionic and one such bosonic state. These states have very small masses at the highest resolution. Their masses appear to decrease with increasing resolution, suggesting that they will become massless in the continuum limit. Since the free-energy at low temperatures is proportional to the number of nearly massless states, the free energy at strong coupling is independent of the resolution and therefore has this fixed value in the continuum limit. On the other hand, at weak coupling the number of light states grows with the resolution and diverges in the

continuum limit, as does the free energy. We therefore find that at low temperatures the ratio of the free energies at strong and weak coupling goes to zero as we approach the continuum.

In summary, the relevance of the present work is three-fold. We verified that supersymmetric Yang-Mills theories can have very different thermodynamic properties in the weakly and strongly coupled regimes, as expected. We made progress on the way towards studying more “physical” models, i.e. models that share certain features with (supersymmetric) QCD. Third, we improved our methods to evaluate thermodynamic properties, in the sense that their calculation in higher dimensions is very similar to the one presented here, provided that we work in the large- N_c approximation. The obvious caveat is the condition that the mass spectrum be computable numerically within our SDLCQ framework. While issues of renormalization creeping in at higher dimensions are a roadblock for most generic theories, one can circumvent the problem by adding more supersymmetry. Judging from earlier work [31], a study of SYM with a Chern-Simons term in three dimensions seems feasible within our framework [32], while we will be hard pressed to overcome convergence problems in theories with too many degrees of freedom, such as extended supersymmetric theories or theories in even higher dimensions. Apart from tackling these more realistic theories, there is merit in studying the thermal properties of two-dimensional models, like super-QCD with extended supersymmetry. Interest in this theory stems from the fact that it is related to the theory of a D1-D5 brane system. The theory’s mesonic spectrum calculated with SDLCQ could be compared against the spectrum obtained from string theory calculations [33].

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- [1] S. S. Gubser, I. R. Klebanov, and A. W. Peet, *Phys. Rev. D* **54**, 3915 (1996).
 - [2] M. Li, *J. High Energy Phys.* 03 (1999) 004; A. A. Tseytlin and S. Yankielowicz, *Nucl. Phys.* **B541**, 145 (1999); A. Fotopoulos and T.R. Taylor, *Phys. Rev. D* **59**, 061701 (1999).
 - [3] A. Das and M. Kaku, *Phys. Rev. D* **18**, 4540 (1978).
 - [4] Y. Matsumura, N. Sakai, and T. Sakai, *Phys. Rev. D* **52**, 2446 (1995).
 - [5] O. Lunin and S. Pinsky, *AIP Conf. Proc.* **494**, 140 (1999).
 - [6] A. G. Cohen, D. B. Kaplan, E. Katz, and M. Unsal, *J. High Energy Phys.* 08 (2003) 024.
 - [7] J. R. Hiller, S. S. Pinsky, Y. Proestos, and N. Salwen, *Phys. Rev. D* **70**, 065012 (2004).
 - [8] S. J. Brodsky, *Nucl. Phys. B, Proc. Suppl.* **108**, 327 (2002).
 - [9] V. S. Alves, A. Das, and S. Perez, *Phys. Rev. D* **66**, 125008 (2002); A. Das and X. X. Zhou, *Phys. Rev. D* **68**, 065017 (2003); A. Das, arXiv:hep-th/0310247.
 - [10] A. N. Kvinikhidze and B. Blankleider, *Phys. Rev. D* **69**, 125005 (2004).

- [11] H. A. Weldon, Phys. Rev. D **26**, 1394 (1982).
- [12] M. Beyer, S. Mattiello, T. Frederico, and H.J. Weber, arXiv:hep-ph/0310222.
- [13] S. Elser and A.C. Kalloniatis, Phys. Lett. B **375**, 285 (1996).
- [14] H.-C. Pauli and S.J. Brodsky, Phys. Rev. D **32**, 1993 (1985); **32**, 2001 (1985).
- [15] S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Phys. Rep. **301**, 299 (1998).
- [16] F. Antonuccio, O. Lunin, and S.S. Pinsky, Phys. Lett. B **429**, 327 (1998).
- [17] F. Antonuccio, O. Lunin, and S. Pinsky, Phys. Rev. D **59**, 085001 (1999).
- [18] P. Haney, J.R. Hiller, O. Lunin, S. Pinsky, and U. Trittman, Phys. Rev. D **62**, 075002 (2000).
- [19] J.R. Hiller, S. Pinsky, and U. Trittman, Phys. Rev. D **64**, 105027 (2001).
- [20] R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965); Nuovo Cimento A **56**, 1027 (1968).
- [21] D. Kutasov, Nucl. Phys. **B414**, 33 (1994).
- [22] D.J. Gross, A. Hashimoto, and I.R. Klebanov, Phys. Rev. D **57**, 6420 (1998).
- [23] F. Antonuccio and S. Pinsky, Phys. Lett. B **439**, 142 (1998).
- [24] J.R. Hiller, S.S. Pinsky, and U. Trittman, Phys. Rev. Lett. **89**, 181602 (2002).
- [25] J.R. Hiller, M. Harada, S.S. Pinsky, N. Salwen, and U. Trittman, Phys. Rev. D **71**, 085008 (2005).
- [26] J.R. Hiller, S.S. Pinsky, and U. Trittman, Phys. Rev. D **67**, 115005 (2003).
- [27] J.R. Hiller, S.S. Pinsky, and U. Trittman, Nucl. Phys. **B661**, 99 (2003).
- [28] R. Alben, M. Blume, H. Krakauer, and L. Schwartz, Phys. Rev. B **12**, 4090 (1975); A. Hams and H. De Raedt, Phys. Rev. E **62**, 4365 (2000).
- [29] T. Iitaka and T. Ebisuzaki, Phys. Rev. E **69**, 057701 (2004).
- [30] J. Jaklič and P. Prelovšek, Phys. Rev. B **49**, 5065(R) (1994); M. Aichhorn, M. Daghofer, H.G. Evertz, and W. von der Linden, Phys. Rev. B **67**, 161103(R) (2003).
- [31] J.R. Hiller, S.S. Pinsky, and U. Trittman, Phys. Rev. D **66**, 125015 (2002).
- [32] J. Hiller, Y. Proestos, and U. Trittman, “Thermodynamics of Three-Dimensional Super Yang-Mills Theory with a Chern-Simons Term” (unpublished).
- [33] R.C. Myers and R.M. Thomson, J. High Energy Phys. **09** (2006) 066.