WHY CERTAIN GEOMETRIC RATIOS ARE CALLED "HARMONIC"

Kenneth Cummins
Kent State University
Kent, Ohio

In school geometry there are various instances of the ratio of interior division ratio to exterior division ratio--or cross-ratio. One of these arises in the study of the bisectors of the interior and exterior angles at a vertex of a triangle. It can be shown that if $\overrightarrow{CD}$ and $\overrightarrow{CE}$ are interior and exterior angle bisectors at C then

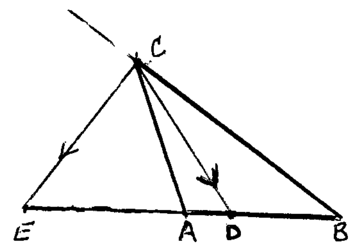


Figure 1

$$\frac{AD}{DB} = \frac{CA}{CB} \text{ and } \frac{AE}{-EB} = \frac{CA}{CB} \text{ whence } \frac{AD}{DB} = \frac{-AE}{EB}$$

or that AD:DB = -AE:EB which says that the interior division ratio of AB formed by D equals the negative of the exterior division ratio of AB formed by E, or that the ratio of the two ratios (internal and external) above equals -1.

Another case arises when from an external point E two tangents are drawn to a circle as shown in the accompanying figure. Regarding points D and E as interior and exterior points of division of segment AB, it can be shown by high school methods that

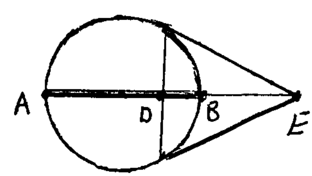


Figure 2

$$\frac{AD}{DB} = \frac{-AE}{EB} \text{ or that } \frac{AD}{DB} \Big/ \frac{AE}{EB} = -1$$

Although not traditionally found in high school geometry, one can prove by using the Theorems of Menelaus and Ceva that in this complete quadrilateral points D and E separate the segment AB such that the interior ratio/exterior ratio = -1.

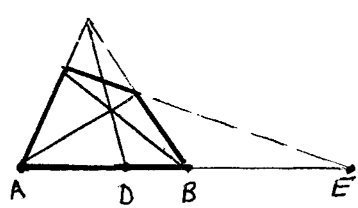


Figure 3

(Indeed, this as well as those above are good sources of exercises in measurement in informal geometry and in making conjectures.)

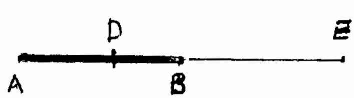
There are many other examples of such a relation and in all the cases above we note the common property that the quotient or ratio of the interior division ratio by the exterior division ratio is always -1. Such division is called harmonic division and the points A, D, B, E are said to form a harmonic set. This can be mentioned in our classes and then we can raise the question, "Why do we use the term 'harmonic'--is it related to music?"

We now attempt to investigate the question and we recall from the physical theory of music that overtones or "harmonics" come from strings or air columns vibrating in parts. The fundamental tone comes from the body's vibrating as a whole, the octave when it vibrates in halves, the next harmonic when it vibrates in thirds, etc.

Figure 4

This is shown in Figure 4. Now the Pythagoreans perceived the relation between lengths of vibrating segments and tones. The Greek word harmonikos meant "agreement in sound" and the early scholars called the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ "harmonic" according to what is seen on the musical staff above. The reciprocals of the terms above form an arithmetic sequence and this is how a harmonic sequence is defined.

Let us now look again at the relation $\frac{AD}{DB} = \frac{AE}{-EB}$.



In successive steps (without reasons) we have

$$\frac{AD}{DB} = \frac{AE}{-EB} \Rightarrow \frac{DB}{AD} = \frac{BE}{AE} \Rightarrow \frac{AB-AD}{AD} = \frac{AE-AB}{AE} \Rightarrow \frac{AB}{AD} - 1 = 1 - \frac{AB}{AE}$$

and $\frac{AB}{AD} + \frac{AB}{AE} = 2 \Rightarrow \frac{1}{AD} + \frac{1}{AE} = \frac{2}{AB} \Rightarrow \frac{1}{AD} + \frac{1}{AE} = \frac{1}{AB} + \frac{1}{AB}$

and $\frac{1}{AB} - \frac{1}{AD} = \frac{1}{AE} - \frac{1}{AB}$ (*) or $\frac{1}{AD}, \frac{1}{AB}, \frac{1}{AE}$

are in arithmetic sequence, since the two differences are equal as shown in (*). Therefore, the three reciprocals AD, AB, and AE are in harmonic sequence and hence the statement by early geometers that a cross-ratio equal to -1 gives rise to a harmonic sequence and is consequently called "harmonic" has been confirmed.

Yes, there is a relation between cross-ratio equal to -1 and music ! --- and in elementary geometry a relation between the separation of a third side of a triangle by the interior and exterior angle bisectors of the opposite angle and music !

REFERENCE

Taylor, E. H. and Bartoo, G. C. An Introduction to College Geometry. New York: The Macmillan Company, 1949, p. 67.

CAN (SHOULD) HIGH SCHOOL TEACHERS OF MATHEMATICS BE MATHEMATICIANS!

Ray Heitger
Ottawa Hills High School
Toledo, Ohio

If your high school program in mathematics is to be the type of program which will provide the students with the benefits they should get from it, you and your colleagues must keep up with the mathematics being used today and with possible applications of that mathematics. Some ideas about how you can accomplish this, and which have worked well for me, are the following.

- Read
    * Books about mathematics
    * Books about something else (e.g., physics)
- Lunch time seminars
- Math days at universities
- Talk to other teachers.
    * E.g.: science, social studies, business