What is the Whole in Cornhole? Introducing and Capitalizing upon Disequilibrium with Fraction Operations

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Abstract: *In this article, we describe a lesson study process aimed at examining and refining division of fractions experiences with pre-service teachers. Through a deliberate, research-based design process, the authors constructed a lesson focused on explicating the nature and mechanics of the traditional fraction division algorithm; however, implementation revealed unexpected yet powerful mathematical experiences that existed aside from the primary lesson goals. Specifically, pre-service teachers experienced significant disequilibrium regarding the shifting nature of "the whole" when working on a particular subset of lesson tasks. The authors describe their design process, implementation, and present several conclusions gleaned from this experience.*

Keywords. *lesson study, elementary, preservice teachers, fractions*

1 Introduction

As a majority of states move into the era of Common Core State Standards (CCSSI, 2010), mathematics teachers from across the country are working to help students develop a deeper understanding of the discipline through more focused and rigorous experiences. Indeed, these standards were written in response to existing state goals that were often perceived as a 'mile wide and inch deep'; however, helping children construct a rich understanding of mathematics proves all but impossible if teachers themselves have yet to construct such understanding. Certainly, many mathematics teachers have robust disciplinary foundations, but there is considerable documentation of mathematical fragility among teachers including (and perhaps especially) among preservice teachers (PSTs) (Ball, 1990; Ma, 1999; Stylianides, Stylianides, & Philippou, 2007). In particular, work with fractions often proves particularly challenging to teachers. When posed with a division task involving fractions, Ma (1999) found that only 43% of U.S. teachers could arrive at a correct answer. Here, it is important to note that this measure only gauged computational proficiency. Measures of conceptual understanding produced far more disquieting outcomes. Only a very small portion (5%) of U.S. teachers were able to construct a mathematically sound context to describe the fraction division task. More recently, Newton (2008) also examined PSTs' knowledge of fractions and found the following:

> Their success rate on problems that assessed computational skill, knowledge of basic fraction concepts, and ability to solve fraction word problems ranged from 70% to 78%. More important, their errors were not simply minor or random. They misapplied fraction algorithms and attended to superficial conditions when choosing a solution method. They also demonstrated little flexibility in solving problems (p.1104).

1.1 Our Journey

In the fall of 2008, a group which consisted of a faculty member and a number of graduate students from the University of Cincinnati's Department of Curriculum and Instruction convened to explore lesson study as well as the difficulties involved in the teaching and learning of fractions. As members of the post-secondary education community, we are particularly interested in the mathematical development of PSTs. Indeed, we discovered that potential teachers also exhibited considerable limitations in their conceptual understanding of fractions (Ball, 1990; Simon, 1993). Our journey began with weekly conversations where we examined work by Stigler and Hiebert (1999) and Ma (1997), to name a few. We read the book *Young Mathematicians at Work - Constructing Fractions, Decimals, and Percent* (Fosnot & Dolk, 2002) and discussed the stories the authors wrote about their approaches to teaching and learning. As a result of these conversations, we decided to enact the process of Japanese Lesson Study (JLS). Our initial aim was to develop a research lesson that addressed the following goals for PSTs:

1. Understand the relationship of dividing by a fraction and multiplying by its reciprocal.

2. Create a visual representation of what it means to multiply by the reciprocal and how that can be demonstrated in the visual.

3. Develop problem solving strategies and methods of representation.

Despite our intention to create tasks that would help student meet the goals listed above, in each implementation of the lesson, PSTs seemed to get caught up in the mathematical question "What is the whole?" This paper describes the tasks and the conversations among PSTs along with what we learned about our students' understanding of the concept of the whole.

Although we will focus on the mathematical work of PSTs, we contend that the disequilibrium and constructed understanding of this lesson experience are not confined to aspiring teachers. Rather, the words, thoughts, and actions we describe simply reflect individuals who are coming to understand fractions more deeply, and these exchanges could just as easily have taken place in the elementary or middle grades classroom.

2 Painting the Boards

After considering a number of different contexts for exploring fractions (i.e., subdividing land, paving roads, etc.), we devised a context that involved painting cornhole boards. Cornhole involves two boards (see Fig. 1) that are placed approximately 30 feet apart. The goal is to toss beanbags into the hole. Because points are awarded for 'near misses' (landing on the board but not going in the hole), we imagined a scenario of larger boards for novice players and smaller boards for more advanced players. This context was chosen because we were confident that the PSTs would know about the game of cornhole: a) it is a very popular game in our area; b) the cornhole boards could be represented in an iconic manner; and c) the context allowed for us to alter the size of the boards for different tasks.

Fig. 1: *Cornhole Board*

Our lesson plan was created so that students would draw representations of the following four tasks and in the order listed:

- **Task 1:** How much paint is needed to cover 4 medium-sized cornhole boards if a quart of paint covers $\frac{1}{2}$ of a board?
- **Task 2:** How much paint is needed to cover 4 large-sized (novice level) cornhole boards if a quart of paint covers $\frac{2}{7}$ of a board?
- **Task 3:** How much paint is needed to cover 4 small-sized (advanced level) cornhole boards if a quart of pint covers $\frac{7}{8}$ of a board?
- **Task 4:** How much paint is needed to cover 4 extremely small-sized (expert level) cornhole boards if a quart of paint covers $1\frac{1}{3}$ of a board?

In each implementation of this lesson, we had PSTs work in groups. The first two tasks were completed without any apparent difficulty. The cornhole board representations created by the different groups consisted of rectangles divided into strips denoting the portion of the board covered by paint.

The groups of PSTs approached Task 3 in the same manner, with representations of cornhole boards divided into strips. It was this task that brought an interesting (and nearly identical) debate with each implementation of the lesson. PST groups were divided as to whether the correct solution was $4\frac{1}{2}$ or $4\frac{4}{7}$. PSTs who came to the conclusion that $4\frac{1}{2}$ quarts of paint was correct typically justified their solutions by dividing each board into eight equal-sized pieces with a total of four unpainted pieces remaining after 4 quarts of paint were used. Because 4 is half of 8, they concluded that the solution was $4\frac{1}{2}$ quarts of paint (see Fig. 2).

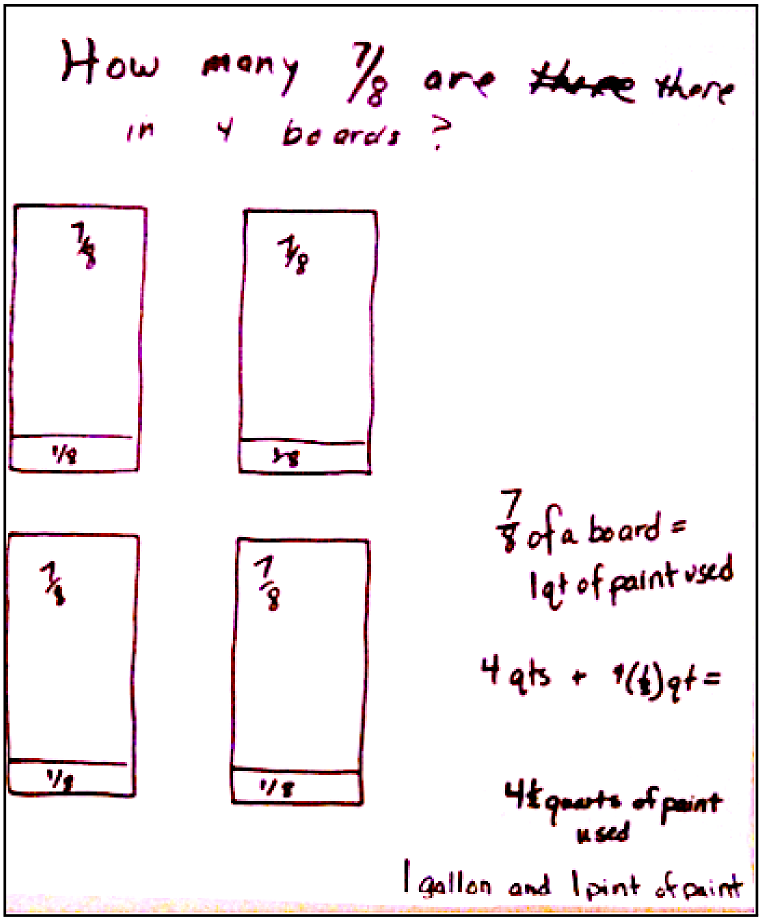

Fig. 2: *Student work associated with Task 3*

PSTs who believed the answer was $4\frac{4}{7}$ also began by dividing the boards into eight equal-sized pieces. However, they explained to the class that because a quart of paint covered $\frac{7}{8}$ of a board, the paint could be visually divided into seven segments with each segment covering one of the eight pieces of the board. After using four quarts of paint, four pieces on one of the boards remained unpainted and each piece would require one of the seven segments of a quart of paint, therefore requiring $\frac{4}{7}$ quarts of additional paint (see Fig. 3).

After several implementations of the lesson we became convinced that Task 1 and Task 2 set the stage for a debate between $\frac{4}{8}$ and $\frac{4}{7}$ on Task 3. It appeared to us that PSTs were having difficulty keeping track of what was the whole - the quart of paint or the board. Task 3 brought about rich, authentic mathematical conversations both within groups and between groups. While PSTs worked on this task, we circulated around the room and asked

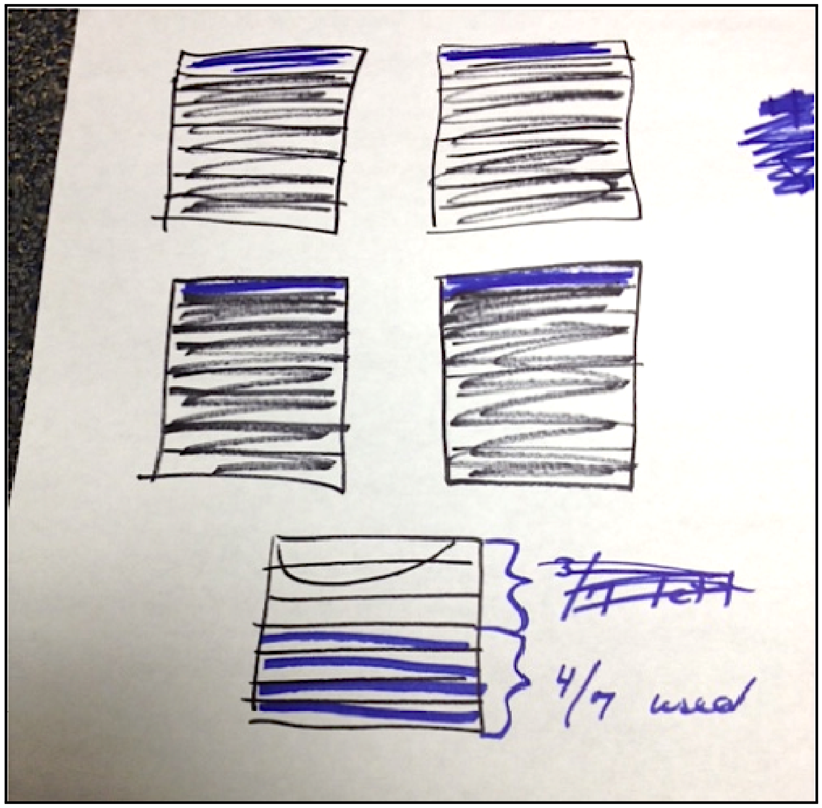

Fig. 3: *Alternative student work associated with Task 3*

groups to explain to us how they arrived at their solutions. If a group arrived at a solution of $4\frac{1}{2}$, we encouraged them to question their solution through dialogue. The following conversation arose from such a context. It is important to note how a very strategic question (hypothetically equating half of a quart with half of a board) prompts additional thinking and dialogue.

(T1): So I get that you have half a board left to paint . . . and that means a half a quart?

(PST 1): Well no it wouldn't mean half, though, because it's seven . . . $\frac{7}{8}$.

(PST 2): $\frac{7}{8}$ is a whole.

(PST 1): So it wouldn't be half.

(PST 2): Yeah, but you need $\frac{8}{8}$ to make a whole.

(PST 3): Whatever.

(PST 1): Would it be a little less than h —or a little more than half?

(PST 2): So four and half, minus a little . . . give or take some.

(PST 3): Just a little more than half . . . wouldn't it?

This conversation continued for some time as the PSTs came to believe that it would, indeed, take a little more than half of a quart, but struggled to determine exactly how much paint it would take.

To convince their peers, a different PST group was quite creative their construction of a pictorial representation to support their response of 4 *and* $\frac{7}{8}$. This group depicted the quart of paint divided into 7 pieces. Using chart paper, they were able to draw the quart of paint and boards in such a way that $\frac{1}{8}$ of the board was equal in size to $\frac{1}{7}$ of the quart of paint. They then moved the unpainted piece from each of the four boards and overlaid them on the picture representing the quart of paint, demonstrating that it would take $\frac{4}{7}$ of a quart of paint to cover the remaining $\frac{4}{8}$ of a board. However, when they presented their solution to the class, they still expressed some uncertainty about their solution (see Fig. 4).

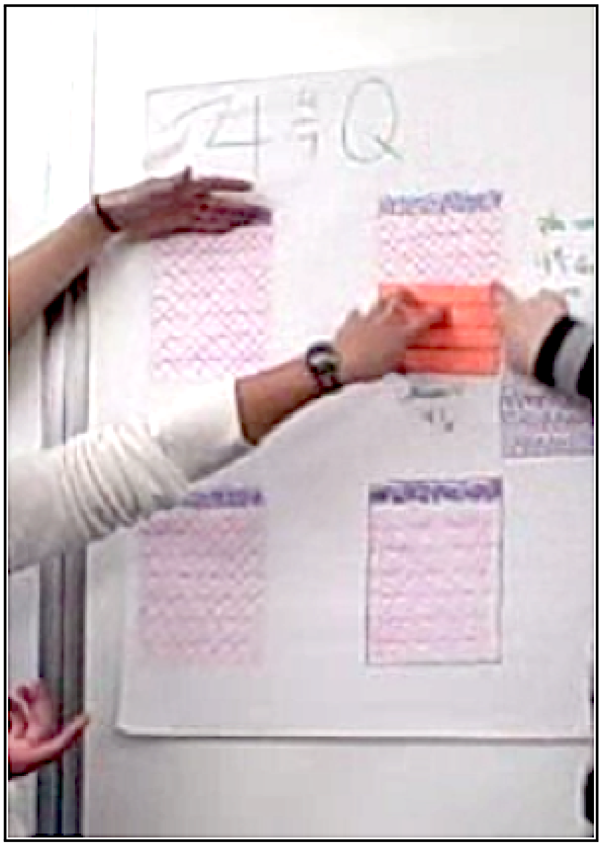

Fig. 4: *A pictorial representation intended to convince peers*

Soon after this group presented their representation of the problem, the issue of 'what was the whole' (boards or quarts) came to a head. Several groups stuck by their assertion that the solution was 4 *and* $\frac{1}{2}$. despite the presentation by the group described above. Many of the students who had become convinced that the fractional part of the solution was not $\frac{1}{2}$ still had difficulty determining the exact quantity of paint that would be needed. One attempt to clarify the issue involved the creation of a chart comparing the portion of the board covered by the paint. For example, students reasoned that 1 quart covered $\frac{7}{8}$ of a board, so $\frac{1}{2}$ of a quart should cover $\frac{7}{16}$ of a board. During this discussion some students began omitting the words "paint" and "board" from their presentations which brought forth the following conversation.

(T1): Okay but is $\frac{1}{2}$ equal to $\frac{7}{16}$?

(PST 1): No.

(T1): So, what's going on then? So were maybe the people who were saying $\frac{1}{7}$ equals $\frac{1}{8}$, correct?

(PST 1): No.

(PST 1): I was going to say, we're not comparing it ... OK, when we're looking at it, we're not comparing these fractions to a whole. We're comparing it to the whole which is a fraction. Like the whole quart isn't covering a whole board. So we have to be taking the whole as not 1, but the whole as in $\frac{7}{8}$ would be a whole.

(T1): OK.

(PST 1): Does that make ... ?

(T1): $\frac{7}{8}$ is a whole of what?

(PST 1): A whole quart.

(T2): $\frac{7}{8}$ is a whole quart.

(PST 1): So $\frac{7}{8}$ is equal to 1.

3 Conclusions

Based on our experiences with PSTs, there are a few key, generalizable ideas that may be extracted from this lesson experience.

3.1 The Power of Representations

Constructing representations of fractions and fraction-operations is a powerful platform to wrestle with challenging ideas. When students (including PSTs) learn the algorithm for dividing by fractions, many do not have a solid understanding of why it works. Constructing robust foundational understanding of fractions and wholes is essential for meaningful understanding of operations. The representations created by the PSTs shed light on this by allowing them to "see" the answer. The representations helped some PSTs see why $\frac{1}{8}$ of a board was equivalent to $\frac{1}{7}$ of a quart of paint as they physically pointed to the $\frac{1}{8}$ and the $\frac{1}{7}$. Using these representations, PSTs discussed the issue of what was the whole (board versus quart). Indeed, by emphasizing the depiction of fractional quantities, we elicited deeper interactions with mathematical ideas.

3.2 Task Design is Key

Thoughtful selection of context, task construction, and task ordering are essential to maximizing mathematical thinking. The first two tasks were designed to have whole number solutions. Through these two tasks PSTs became comfortable with their representations and did not struggle with the mathematics. Task 3 was specifically written to have a non-whole-number solution. Because the PSTs had already established a representational modality (drawing boxes, for most groups), they could focus primarily on the mathematics in their representations rather than having their focus divided between the actual creation of the representation and the mathematics of progressively more challenging tasks.

3.3 Questioning as a Catalyst for Discourse

Key questions, rather than explanations, are essential to maximize intragroup and intergroup discussions. We wanted PSTs to create their own understanding of the relationship between dividing by a fraction and multiplying by the reciprocal rather than just accepting our explanation. To this end we gave the groups time to explore the tasks with minimum interference. When we did engage the groups, we asked them to explain their representations. If a group was convinced that the solution to Task 3 was $4\frac{1}{2}$, we tried to guide them to rethink their solution through questions and encouraged them to discuss their representations with other groups. Questions that we have found to be particularly powerful include:

- So, I see from your representation that there is $\frac{1}{2}$ of a board left to paint, and you state that it will take $\frac{1}{2}$ of a quart to finish the job. *Does that mean that $\frac{1}{2}$ of a quart is equal to $\frac{1}{2}$ of a board*?
- How much paint would it take to cover just one of those $\frac{1}{8}$ strips on your representation?

We do recognize, however, that not all teacher talk can be delivered in interrogative terms. Indeed, there may be moments when we, as teachers, elect to offer up a conjecture or call attention to mathematical features that have, perhaps, been overlooked. In these instances, we found it quite beneficial to deliver such information in a very tentative manner. That is, tempering the tone and strength of our remarks allowed us to adopt the posture of mathematical co-constructor alongside the students rather than some external 'keeper of right and wrong.'

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