On the Commutativity of the Boundary and Interior Operators in a Topological Space

Staley, David H.
ON THE COMMUTATIVITY OF THE BOUNDARY AND INTERIOR OPERATORS IN A TOPOLOGICAL SPACE.—N. Levine [2] discovered that, in a topological space, the interior and closure operators will commute if and only if the set on which they operate is the symmetric difference of a set that is both open and closed and a set that is nowhere dense. The intent of this paper is to characterize those sets for which the interior and boundary operators will commute.

The following notation will be used:

- $cA$—closure of $A$
- $CA$—complement of $A$
- $Int A$—interior of $A$
- $BA$—boundary of $A$
- $A\mathcal{V}B$—symmetric difference of $A$ and $B$

**Lemma 1:** If $\{X,T\}$ is a topological space and $A$ and $E$ subsets of $X$, then $Int(A\cap E) = IntA \cap IntE$.

**Lemma 2:** If $\{X,T\}$ is a topological space, $A$ and $E$ are subsets of $X$, and $A$ is open and $E$ is dense, then $c(A \cap E) = cA \cap cE = cA$. This is an exercise on p. 57 in Kelley's book (1955).

**Lemma 3:** If $\{X,T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA \cap BIntA = \emptyset$.

**Proof:**

$$IntBA \cap BIntA = Int(cA \cap CA) \cap cIntA = IntcA \cap IntcCA \cap cIntA \cap cIntA = IntcA \cap cIntA \cap cIntA = \emptyset.$$ 

**Lemma 4:** If $\{X,T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA = BIntA$ if and only if $IntcA = cIntA = IntA$.

**Proof:** *Necessity.* Suppose $IntBA = BIntA$. Then these sets must both be empty in order to be equal since by lemma 2 they have nothing in common. Thus

1. $IntBA = IntcA \cap cIntA = \emptyset$
2. $BIntA = cIntA \cap cCA = \emptyset$

Equations 1 and 2 imply $IntcA \subseteq cIntA$ and $cIntA \subseteq cCA = IntA$ respectively. Since $IntA \subseteq IntA$, it follows that $IntA = IntcA = cIntA$.

**Sufficiency.** Suppose $IntcA = cIntA = IntA$. This implies that $BIntA = \emptyset$ and $IntBA = \emptyset$, and thus the two sets are equal.

**Theorem:** If $\{X,T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA = BIntA$ if and only if $A = E \cup \mathcal{P}$, where $E$ is open and closed, $\mathcal{P}$ is nowhere dense, and $E \cap \mathcal{P} = \emptyset$.

**Proof:** Using lemma 4, the proof reduces to showing that $IntcA = cIntA = IntA$ if and only if $A = E \cup \mathcal{P}$, where $E$ is open and closed, and $E \cap \mathcal{P} = \emptyset$.

**Necessity.** Suppose $IntcA = cIntA = IntA$. By Levine's theorem (Levine, 1961), it follows that if $IntcA = cIntA$, then $A = E \mathcal{V} P$ where $E$ is open and closed and $P$ is nowhere dense. Thus it is left to establish what further conditions the second equality places on $E$ and $P$. In Levine's proof, $E = cIntA$. Thus $E = Int(cE \cap \mathcal{P})$. $Int(cE \cap \mathcal{P}) = c[(cE \cap \mathcal{P}) \cup (CE \cap P)] = c[(CE \cap CP) \cup (P \cap E)] = c[(CE \cap CP) \cup (P \cap E)]$. By lemma 2 and the fact that $CP$ is dense it follows that $C[(CE \cap CP) \cup (P \cap E)] = CCE \cap CE \cap (P \cap E) = CE \cap CP$. Therefore, $E = E \cap CC(P \cap E)$ which implies $P \cap E = \emptyset$.

**Sufficiency.** Suppose $A = E \cap \mathcal{P}$, $E$ is open and closed, $P$ is nowhere dense, and $E \cap \mathcal{P} = \emptyset$. By Levine's theorem, $cIntA = IntA$. Int $A = Int(E \cup \mathcal{P}) = CC(CE \cap CP) = CCE = E$. Thus $IntA$ is closed and it follows that $cIntA = IntA = IntA$.

DAVID H. STALEY, Ohio Wesleyan University, Delaware, Ohio.

REFERENCES

