On the Commutativity of the Boundary and Interior Operators in a Topological Space

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ON THE COMMUTATIVITY OF THE BOUNDARY AND INTERIOR OPERATORS IN A TOPOLOGICAL SPACE.—N. Levine [2] discovered that, in a topological space, the interior and closure operators will commute if and only if the set on which they operate is the symmetric difference of a set that is both open and closed and a set that is nowhere dense. The intent of this paper is to characterize those sets for which the interior and boundary operators will commute.

The following notation will be used:

- $cA$—closure of $A$
- $CA$—complement of $A$
- $Int A$—interior of $A$
- $BA$—boundary of $A$
- $A\Delta B$—symmetric difference of $A$ and $B$

**Lemma 1:** If $\{X, T\}$ is a topological space and $A$ and $E$ subsets of $X$, then $Int(A \cap E) = IntA \cap IntE$.

**Lemma 2:** If $\{X, T\}$ is a topological space, $A$ and $E$ are subsets of $X$, and $A$ is open and $E$ is dense, then $c(A \cap E) = cA \cap cE = cA$. This is an exercise on p. 57 in Kelley's book (1955).

**Lemma 3:** If $\{X, T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA \cap BIntA = \emptyset$.

**Proof:** $IntBA \cap BIntA = (cA \cap cCA) \cap cIntA = cIntA \cap IntcCA \cap cIntA = cIntA \cap IntcA \cap cIntA = \emptyset$.

**Lemma 4:** If $\{X, T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA = BIntA$ if and only if $IntcA = cIntA = IntA$.

**Proof:** Necessity. Suppose $IntBA = BIntA$. Then these sets must both be empty in order to be equal since by lemma 2 they have nothing in common. Thus

1. $IntBA = IntcA \cap cIntA = \emptyset$
2. $BIntA = cIntA \cap cCA = \emptyset$

Equations 1 and 2 imply $IntcA \subset cIntA$ and $cIntA \subset cCA = IntA$ respectively. Since $IntA \cap IntA$, it follows that $IntA = IntcA = cIntA$.

Sufficiency. Suppose $IntcA = cIntA = IntA$. This implies that $BIntA = \emptyset$ and $IntBA = \emptyset$, and thus the two sets are equal.

**Theorem:** If $\{X, T\}$ is a topological space and $A$ is a subset of $X$, then $IntBA = BIntA$ if and only if $A = E \cup P$, where $E$ is open and closed, $P$ is nowhere dense, and $E \cap P = \emptyset$.

**Proof:** Using lemma 4, the proof reduces to showing that $IntcA = cIntA = IntA$ if and only if $A = E \cup P$, where $E$ is open and closed, $P$ is nowhere dense, and $E \cap P = \emptyset$.

Necessity. Suppose $IntcA = cIntA = IntA$. By Levine's theorem (Levine, 1961), it follows that if $IntcA = cIntA$, then $A = E \cup P$ where $E$ is open and closed and $P$ is nowhere dense. Thus it is left to establish what further conditions the second equality places on $E$ and $P$. In Levine's proof, $E = cIntA$. Thus $E = IntA$ (i.e., $E = Int(E \cap P)$). $Int(E \cap P) = cCc[(E \cap CP) \cup (CE \cap P)] = cc[(CE \cap CP) \cup (P \cap E)] = cCc(CE \cap CP) \cup (P \cap E)$. By lemma 2 and the fact that $CP$ is dense it follows that $Cc(CE \cap CP) \cup (P \cap E) = cCcCE \cap cCc(P \cap E) = E \cap cC(P \cap E)$. Therefore, $E = E \cap cCc(P \cap E)$ which implies $P \cap E = \emptyset$.

Sufficiency. Suppose $A = E \cup P$, $E$ is open and closed, $P$ is nowhere dense, and $E \cap P = \emptyset$. By Levine's theorem, $cIntA = IntA$. Int$A = Int(E \cup P) = cCc(CE \cap CP) = cCcCE = E$. Thus $IntA$ is closed and it follows that $cIntA = IntcA = IntA$.

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REFERENCES

