Maximizing the Area of a Sector With Fixed Perimeter

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Historically, many maxima and minima were found long before Newton and Leibniz developed calculus. Ivan Niven's (1981) classic Maxima and Minima Without Calculus *provides a systematic and thorough account of solving extreme-value problems using elementary algebra, geometry, and trigonometry. Niven devotes a chapter to isoperimetric problems: problems that ask "for the region of largest area in a given class of regions . . . of a specified perimeter" (p. 77). We use technology as a tool to solve the isoperimetric problem for the sector of a circle—an investigation inspired by a project in Farrell and Boyd (2007).*

Introduction

The thoughtful use of technology can enhance the mathematical understanding of advanced concepts and big ideas in school mathematics. According to the National Council of Teachers of Mathematics (NCTM, 2000), "Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise" (p. 17). Moreover, "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, p. 24). This article is about using technology to explore isoperimetric sectors of circles and explains some connections that may help foster students' mathematical understanding.

Polygons, Regular Polygons, and Circles

Before considering the areas of sectors of circles, we review some related theorems about polygons, regular polygons, and circles. Each of these could be turned into an exploration all its own. Background information and proofs of these theorems can be found in Niven (1981).

1. Regular polygons enclose larger areas than the corresponding irregular polygons with the same perimeter. For example, when we consider quadrilaterals with a fixed perimeter of 20 ft, the square with 5-ft side lengths will have the maximum area. In general, for n-sided polygons with a fixed perimeter, the regular n-gon encloses the maximum area.
2. For any two isoperimetric regular polygons with n and n+1 sides, the (n+1)-sided polygon encloses a larger area. For example, as shown in Figure 1, a square with a 12-cm perimeter encloses a larger area than an equilateral triangle with a 12-cm perimeter.
3. A circle of a given perimeter (circumference) encloses a greater area than any polygon with the same perimeter. Because a circle can be interpreted as a polygon with infinitely many sides, this can be seen as an extension of Theorem 2.

As a combined illustration of Theorems 2 and 3, Figure 1 shows an equilateral triangle, a square, and a circle—all with the same perimeter. Notice that their areas differ substantially.

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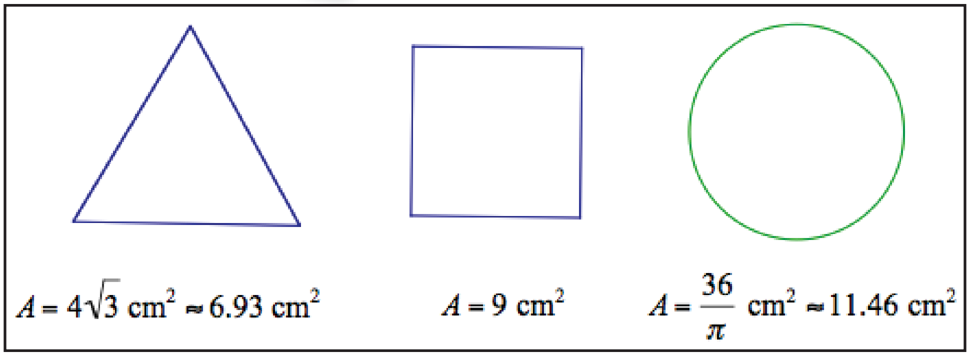

Fig 1 Three plane figures with a common perimeter of 12 cm.

Technology provides a compelling alternative available to students without knowledge of calculus, even to students in the middle grades.

Sector of Circles

The isoperimetric problem for the sector of a circle is especially intriguing because it involves two variables: the radius of the circle and the measure of the central angle. In Appendix A, we give a standard calculus solution to this problem. But technology provides a compelling alternative available to students without knowledge of calculus, even to students in the middle grades.

We can explore a sector for any fixed perimeter. Suppose we have a circular sector with perimeter p = 100 ft. We wish to determine the greatest area that the sector can enclose and discuss the mathematics associated with this process. We use the *Geometer's Sketchpad*, *TI-nspire CAS*, and *Microsoft Excel* as tools for the investigation, but the methods shown can be adapted to many other tools. Readers who wish to implement these ideas should use the tools they know and have available to them.

First, students should be introduced to, or reminded of, the ideas of an arc of a circle (both minor and major) and the associated sectors. A quick and easy *Geometer's Sketchpad* construction can illustrate the two possible sectors associated with a central angle, such as BOD in Figure 2. At this stage, the students might suspect that a major sector will yield the maximum area for a sector with fixed perimeter.

Students can be reminded that the perimeter of a circle is its circumference, and they can be reminded of, or guided through activities to determine, the value of π, the formula for the circumference, or both. Using prior or newly developed knowledge, they should be able to express the perimeter of sector BOD as the sum of the lengths of the radii OB and OD and the arc length BD (minor or major).

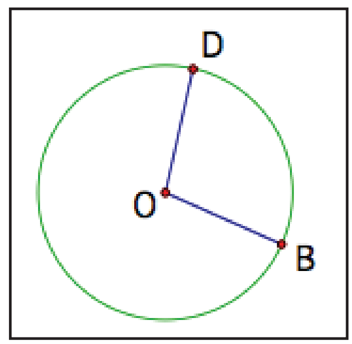

Fig 2 The central angle of a circle determines two arcs and two associated sectors.

Dynamic Construction of a Sector With Fixed Perimeter

The construction in Figure 3 uses a horizontal line segment of length 100 ft to represent the fixed perimeter. We use the control point T on this line segment to distribute lengths to the two radii and the arc length of a sector that is constructed using the Circle and Measurement transfer features of a *TI-nspire* Geometry page. Because the *TI-nspire* Geometry software does not measure reflex angles, the central angle θ is calculated from the arc length s and radius r as shown on the screen shots in Figure 3. The values of θ and A are automatically updated as the user moves the position of T along the horizontal segment.

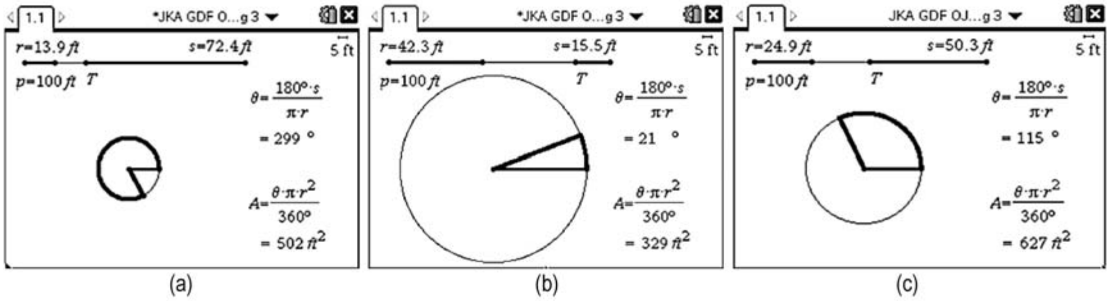

Fig 3 Three possible circular sectors with a perimeter of 100 ft.

Students can observe that, in the PacMan-like major sector (Fig. 3a), the large amount of perimeter used by the arc length portion reduces the radius of the circle and yields a relatively small sector area A. Further exploration reveals that an obtuse angle between 110° and 120° is maximal (Fig. 3c).

Figure 3c suggests that the maximum area of a sector with a perimeter of 100 ft is about 627 ft^2, but this is a fairly rough approximation. Although the *TI-nspire* computations are done with great precision, the scaling and measurements on a Geometry page are approximations that are limited by the number of pixels on the screen. Moreover, all of the displayed values are rounded. Making students aware of these limitations can be used to motivate the need for a more accurate solution.

An Algebraically Driven Numerical Exploration

To obtain further precision, algebraic and numerical representations can be used in combination with a spreadsheet. To begin this process, as in Figure 3, let r be the radius of the (dynamic) circle, and let θ be the measure of the related central angle BOD. If s is the associated arc length, then it can be expressed as a fraction of the circumference:

$$s = 2\pi r \cdot \frac{\theta}{360^\circ} = \frac{\pi r \theta}{180^\circ}$$

Thus, the perimeter of the sector $p = 2r + s$ is

$$2r + \frac{\pi r \theta}{180^\circ} = 100 ft.$$

Suppressing the units (feet and degrees) and solving for r in this equation yields

$$r = \frac{18000}{\pi\theta + 360}.$$

Lastly, we observe that the area A of the sector is a fraction of the area πr^2 enclosed by the corresponding circle:

$$A = \pi r^2 \cdot \frac{\theta}{360} = \frac{\pi r^2 \theta}{360}$$

In Table 1, these formulas are entered into a spreadsheet. This allows us to combine the problem-solving strategy of guess and check with using a systematic list. The sector areas vary substantially even though they all have the same perimeter—100 ft. Note that we are obtaining values for the same variables investigated in Figure 3, but without any possible measurement errors. Again, we are led to explore θ values in the neighborhood 120° to seek the maximum possible area for the sector.

After several steps of systematic numerical investigation, we could obtain values in our spreadsheet as shown in Table 2. In this case, all of the sector areas appear to be the same for all six θ values in the table because the differences in the areas are beyond the decimal accuracy shown. Actually, the greatest area occurs for $\theta \approx 114.591559°$, shown in boldface type in the table. Notice in this case that $r \approx 25$ ft $= p / 4$ and $s \approx 50$ ft $= p / 2$. Moreover, this result implies that the *maximum area of the sector occurs when the arc length is twice the radius*. This supports and refines what we observed in Figure 3c.

To obtain further precision, algebraic and numerical representations can be used in combination with a spreadsheet.

Table 1 The Radius, Arc Length, and Sector Area as Functions of the Central Angle

Angle Measure (degrees) θ	Radius r (ft) $18000 \div (\pi\theta + 360)$	Arc Length s (ft) $\pi r\theta \div 180$	Sector Area A (ft^2) $\pi r^2\theta \div 360$
30	39.62595044	20.74809913	411.0815739
60	32.81703871	34.36592258	563.8939058
90	28.00495768	43.99008465	615.9702294
120	24.42363218	51.15273563	624.6678001
150	21.65442455	56.69115090	613.8071249
180	19.44922648	61.10154704	594.1889134
210	17.65165419	64.69669163	571.0018138
240	16.15824688	67.68350625	546.8234017
270	14.89782554	70.20434892	522.9460712
300	13.81981332	72.36037335	500.0034259
330	12.88728460	74.22543080	478.2821256

Table 2 The Radius, Arc Length, and Sector Area as Functions of the Central Angle

Angle Measure (degrees) θ	Radius r (ft) $18000 \div (\pi\theta + 360)$	Arc Length s (ft) $\pi r\theta \div 180$	Sector Area A (ft^2) $\pi r^2\theta \div 360$
114.591556	25.00000033	49.99999934	625.00000000
114.591557	25.00000022	49.99999956	625.00000000
114.591558	25.00000011	49.99999978	625.00000000
114.591559	**25.00000000**	**49.99999999**	**625.00000000**
114.591560	24.99999989	50.00000021	625.00000000
114.591561	24.99999978	50.00000043	625.00000000

Compelling numerical and visual evidence should give students confidence that the maximum area of a sector occurs when the arc length is twice the radius.

A Revealing Visual Representation

Figure 4 depicts the isoperimetric sector of maximum area, partitioned into numerous congruent subsectors. Figure 5 rearranges these subsectors to form a region that is nearly a quadrilateral in shape. Indeed, Figure 5 is nearly a square, the quadrilateral with maximum area for a given perimeter.

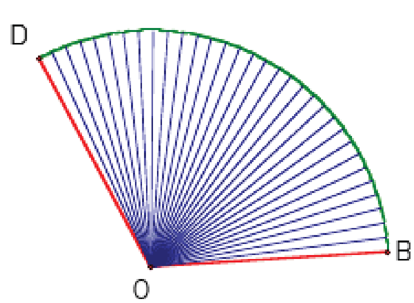

Fig 4 Maximal sector BOD divided into subsectors.

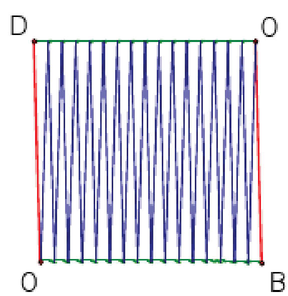

Fig 5 The subsectors of sector BOD rearranged to form a figure approximating a square.

A Proof Using Elementary Algebra

The compelling numerical and visual evidence should give students confidence that the maximum area of a sector occurs when the arc length is twice the radius. Yet, this is still merely a conjecture. To complete the mathematical reasoning process, a proof is called for, and one is well within reach of high school students.

To this point, we have expressed A as a function of θ. Now ask the students to express A as a function of r: $A(r) = \frac{pr}{2} - r^2$, or in our particular case, $A(r) = 50r - r^2$.

Then ask: What kind of function is this? Does it fit the data in Table 1? What are the properties of this function? Does it have a maximum value? If so, what is the maximum, and for what value of r does it occur? Figure 6 shows a scatter plot of an extension of Table 1 with a graph of $A(r)= 50r - r$ overlaid.

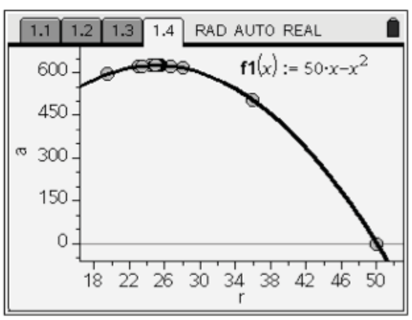

Fig 6 The graph of $f(x)=50x-x^2$ appears to fit the (r,A) data pairs from the spreadsheet in Table 1.

From earlier work with quadratic functions of the form $f(x) = ax^2 + bx + c$, students should have established that

$$x = \frac{-b}{2a}$$

produces an extreme value for $f(x)$. In this case, that means

$$r = \frac{-50}{2(-1)} = 25ft.$$

produces the sector with maximum area, which is exactly what we wanted to prove.

This can readily be generalized to the case

$$A(r) = \frac{pr}{2} - r^2.$$

Concluding Remarks

The NCTM's (2009) *Focus in High School Mathematics: Reasoning and Sense Making* calls for "all students in every high school mathematics classroom [to be held] accountable for personally engaging in reasoning and sense making" (p. 6). The recently released *Common Core State Standards for Mathematics* (2010) includes reasoning and sense making as standards for mathematical practice and asks students to "construct viable arguments," "model with mathematics," and "attend to precision" (pp. 6, 7).

Our approach allows students to engage in these mathematical practices and to explore deep connections among several representations of a rich and classic problem—without the need for calculus. The process ultimately leads to an elementary proof of a surprising result: The maximum area of a sector equals the area of a square with the same perimeter. This investigation, which is based on multiple uses of technology, is intended to develop the students' mathematical proficiency, that is, the blending and interweaving of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001). The innovative use of technology can provide learners the mathematical power to confront and make sense of the big ideas of mathematics in practical and meaningful ways. Ω

The innovative use of technology can provide learners the power to confront and make sense of the big ideas of mathematics.

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