A Study on the Inhomogeneous Hypergeometric Differential Equation

Research Thesis

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by

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1. Introduction: The Hypergeometric Function and Linear Equations in the Complex Domain

The present section presents main mathematical facts needed throughout this paper, including a discussion on Frobenius theory in section 1.3 and an introduction to the hypergeometric equation in section 1.4. The problem studied and the main results are presented in in section 2. We discuss the integral formula in section 3, expansion in Jacobi polynomials in section 4, and solution for general parameters in section 5. We conclude our results in section 6 and include an Appendix discussing Jacobi series in section 7.

Consider a power series, which has the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + ... + a_n(x-x_0)^n + ..., \tag{1}$$

where x_0 and a_n are real coefficients and x is in the neighborhood of x_0 within the radius of convergence of the series. Results justifying convergence and divergence of series with real numbers are well known [1]. Now, if a_n, x_0 and z are complex, then these results also hold true if one replaces the absolute value of real numbers by the modulus of complex numbers.

Recall that for real numbers

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \tag{2}$$

and for complex numbers,

$$x = x_1 + ix_2, \tag{3}$$

where x_1 and x_2 are real, and the modulus is $|x| = \sqrt{x_1^2 + x_2^2}$. The results regarding power series, such as tests of convergence, also extend to the complex domain. For example, consider the hypergeometric series, whose sum is called the hypergeometric function,

$$F(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \tag{4}$$

where the parameters $a, b, c \in \mathbb{C}$, $c \neq 0$, and $(p)_n$ is the Pochhammer symbol defined as

$$(p)_n = \begin{cases} 1 & n = 0 \\ p(p+1)...(p+n-1) & n > 0 \end{cases}. \tag{5}$$

The series (4) converges for all $x \in \mathbb{C}$ such that $|x| < 1$, and it is symmetric in the sense that $F(a,b,c,x) = F(b,a,c,x)$.

Indeed, we can show that (4) converges absolutely via the ratio test with the absolute value being the complex modulus. The ratio test is stated as follows [2].

Theorem 1. Ratio Test *for the series* $\sum x_n$. *Assume that the following limit exists.*

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}. \tag{6}$$

If $L < 1$, the series converges absolutely and if $L > 1$, the series diverges.

For the hypergeometric series, we find that

$$L = \lim_{n\to\infty} \frac{|F_{n+1}|}{|F_n|} = \lim_{n\to\infty} \left|\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\right| / \left|\frac{(a)_n(b)_n}{(c)_n}\right| = \lim_{n\to\infty} \left|\frac{(a+n)(b+n)x}{(c+n)(n+1)}\right| = |x| \tag{7}$$

and hence the series converges for $|x| < 1$.

1.1. **Convergence.** The following result is well known for power series with real coefficients [3].

Theorem 2. Convergence *Consider a power series $S(x) = \sum a_n(x - x_0)^n$. If a_n, x_0, and x are real numbers, then one of the following holds:*

(i) $S(x)$ converges only for $x = x_0$ and the radius of convergence is $R = 0$.

(ii) $S(x)$ converges for all x and the radius of convergence is $R = \infty$.

(iii) There exists a radius of convergence $R > 0$ such that $S(x)$ converges for all real numbers x for $|x - x_0| < R$, and diverges for all real numbers x for $|x - x_0| > R$. We say that $S(x)$ is analytic at the point x_0.

Remark: This theorem also holds for complex numbers. However, the radius of convergence is now a disk of convergence centered at x_0 in the complex plane. Indeed, if $x = x_1 + ix_2$ then $|x| < R$ is the disk $x_1^2 + x_2^2 < R^2$. In fact, $f(x)$ is analytic at all points in this disk [15].

When $R = \infty$, $f(x)$ is called an entire function and it is analytic at all points in the complex plane. For example, consider the function

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{8}$$

We find its radius of convergence using the ratio test:

$$L = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$$

Since $L < 1$ for all x, then the series converges for all $x \in \mathbb{C}$. Hence, the radius of convergence is $R = \infty$ and $f(x) = e^x$ is an entire function. On the other hand, consider the function $f(x) = \ln(1-x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$. Again, we use the ratio test:

$$L = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)}\frac{n}{x^n}\right| = \lim_{n\to\infty} \left|x\frac{n}{n+1}\right| = |x|$$

Since this will converge absolutely for $L = |x| < 1$, the radius of convergence is $R = 1$ and $f(x) = \ln(1-x)$ is not an entire function. Note that $\ln(1-x)$ is not even defined at $x = 1$.

1.2. **Solutions to the differential equation.** The function $F(a,b,c,x)$ defined in (4) satisfies the linear differential equation

$$x(x-1)u''(x)+\big(-a+(a+b)x\big)u'(x)-cu(x)=0 \tag{9}$$

which therefore is called the hypergeometric differential equation. Indeed, we show that this is the case. We substitute the series into the differential equation letting

$$P(a,b,c,x)=\sum_{n=0}^{\infty}P_n x^n \tag{10}$$

so that

$$\sum_{n=2}^{\infty}P_n n(n-1)x^n-\sum_{n=2}^{\infty}P_n n(n-1)x^{n-1}-a\sum_{n=1}^{\infty}P_n n x^{n-1}+(a+b)\sum_{n=1}^{\infty}P_n n x^n-c\sum_{n=0}^{\infty}P_n x^n \tag{11}$$

We then get the above relations:

$$cP_0+aP_1=0 \tag{12}$$

$$[2P_2+2aP_2+(a+b)P_1+cP_1]\,x=0 \tag{13}$$

$$[P_n n(n-1)-P_{n+1}n(n-1)-aP_{n+1}(n+1)+(a+b)P_n n+cP_n]\,x^n=0 \tag{14}$$

and therefore,

$$P_{n+1}=\frac{(n+a)(n+b)}{n(n+c)}P_n \tag{15}$$

so that

$$P_n=\frac{(a)_n(b)_n}{(c)_n n!}. \tag{16}$$

Note that the coefficients of $u(x)$ and $u'(x)$ from (9) are not defined at $x=0$ and $x=1$. These are called singular points of the equation. The coefficients are analytic at all other points and these points ar called regular points of the equation.

1.3. **Singularities of Second Order Differential Equations with Analytic Coefficients: Frobenius Theory.** For linear differential equations in the complex plane, it is useful to investigate the nature of the point at ∞ by substituting $x=1/z$ in the differential equation. Now, we get that as $x\to\infty$, we have $z\to 0$. For example, this substitution, $u(x)=u(1/z)=v(x)$ in the hypergeometric differential equation (9) gives us

$$v''(z)+\left[az+\frac{bz}{1-z}\right]v'(z)+\frac{cz^2}{1-z}v(z)=0 \tag{17}$$

For each of the singular points, 0, 1 and ∞, there are two solutions, one that is analytic and another that is a power series multiplied by an analytic function. These solutions can be found using Frobenius theory. Note that it is essential to classify singularities since Frobenius theory requires regular singular points. For a reference to this subject, see, for example,

[16].

Consider the second order linear differential equation of the form

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0, \tag{18}$$

where $r(x)$ and $q(x)$ are analytic functions. We would like to solve this equation in a neighborhood around the point x_0, by which the solution has a power series expansion. However, this depends on the classification of the point x_0.

If $p(x)$ and $q(x)$ are analytic at the point x_0, then x_0 is called an ordinary point. The point x_0 is a regular singular point if $q(x)$ has a pole up to order 1 at $x = x_0$ and $r(x)$ has a pole of order up to 2 at $x = x_0$. Otherwise, a singular point x_0 is called an irregular singular point.

Here, we assume x_0 is a regular singular point. We denote

$$q_0 = \lim_{x \to x_0} (x - x_0)q(x) \tag{19}$$

and

$$r_0 = \lim_{x \to x_0} (x - x_0)^2 r(x). \tag{20}$$

Note that the values q_0 and r_0 correspond to the values of the indicial equation

$$r(r-1) + q_0 r + r_0 = 0. \tag{21}$$

There are three different cases to consider depending on the roots of the indicial equation (21). We will denote the two solutions of the indicial equation by r_1 and r_2.

Case 1: Two distinct roots that do not differ by an integer The two solutions of the indicial equation are such that $r_1 - r_2 \neq n$, where $n \in \mathbb{Z}$. Then there are two linearly independent solutions of the differential equation and they have the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n \tag{22}$$

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n(r_2)(x - x_0)^n \tag{23}$$

The values $a_n(r_1)$ and $b_n(r_2)$ are found by substituting the solution into the differential equation and determining the recurrence relation.

For example, consider the differential equation

$$9x^2y'' + 3x^2y' + 2y = 0. \tag{24}$$

The indicial equation can be read off immediately as $9r^2 - 9r + 2 = 0$. However, we will illustrate below how it arises naturally as we investigate the series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{25}$$

to find two linearly independent solutions, (22) and (23). From (25), we have

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad\qquad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \tag{26}$$

We plug (25) and (26) into the differential equation (24) to get

$$\sum_{n=0}^{\infty} 9(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0 \tag{27}$$

and simplify

$$(9r(r-1)+2)a_0 x^r + \sum_{n=1}^{\infty}[9(n+r)(n+r-1)a_n + 3(n+r-1)a_{n-1} + 2a_n]x^{n+r} = 0. \tag{28}$$

The coefficient of x^r must vanish, giving us the indicial equation is then $9r(r-1)+2=0$, with roots, $r_1 = 1/3$ and $r_2 = 2/3$. We can find the recurrence relation by requiring that the coefficient of x^{n+r} vanishes:

$$[9(n+r)(n+r-1)+2]a_n + 3(n+r-1)a_{n-1} = 0 \tag{29}$$

and therefore,

$$a_n = \frac{3-3n-3r}{9(n+r)(n+r-1)+2}a_{n-1}, \tag{30}$$

where $r = 1/3$ or $r = 2/3$. Hence, the two linearly independent solutions are

$$y_1 = \sum_{n=0}^{\infty} \frac{(-2-3n)}{9(n+5/3)(n+2/3)+2}a_n x^{n+2/3} \tag{31}$$

and

$$y_2 = \sum_{n=0}^{\infty} \frac{-1-3n}{9(n+4/3)(n+1/3)+2}a_n x^{n+1/3}. \tag{32}$$

Case 2: Roots differing by an integer

Assume now that the two roots differ by an integer: $r_1 - r_2 = n$, where $n \in \mathbb{N}, n \neq 0$. The larger root gives a solution of a similar form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n. \tag{33}$$

Another independent solution can be found in the form

$$y_2(x) = Ay_1(x)\log x + x^{r_2} \sum_{n=0}^{\infty} b_n(r_2)x^n. \tag{34}$$

The constant A may be zero. Here, we consider the example,

$$xy'' + y = 0 \tag{35}$$

Substituting in (34) into the differential equation (35) and rearranging, we get

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_k x^{r+n-1} + \sum_{n=0}^{\infty} a_k x^{n+r} = 0 \tag{36}$$

$$r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_k + a_{k-1}]x^{r+n-1} = 0 \tag{37}$$

The indicial equation is therefore $r(r-1) = 0$ with roots $r_1 = 1$ and $r_2 = 0$.
The recurrence relation is then

$$a_k = \frac{-a_{k-1}}{(n+r)(r+n-1)} \tag{38}$$

For $r_1 = 1$, the largest root, we get

$$a_k = \frac{(-1)^k}{(k+1)(k!)^2} \tag{39}$$

Hence,

$$y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^k}{(k+1)(k!)^2}x^k \tag{40}$$

A linear independent solution with $r_2 = 0$ has the form of (34).

Case 3: Repeated Roots
The two roots have the same value, $r_1 = r_2$. Then, two independent solutions are of the form

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \tag{41}$$

and

$$y_2(x) = y_1(x)\log x + x^{r_2}\sum_{n=0}^{\infty} b_n(r_1)x^n \tag{42}$$

As an example, consider the following differential equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0, \tag{43}$$

where $x = 0$ is a singular point. We substitute the general series solution, $\sum_{n=0}^{\infty} a_n x^{n+r}$, to the differential equation.

$$0 = 4x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}\right)$$
$$- 4x^2\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\right) + (1-2x)\left(\sum_{n=0}^{\infty}a_nx^{n+r}\right) \quad (44)$$

$$0 = \left(\sum_{n=0}^{\infty}4(n+r)(r+r-1)a_nx^{n+r}\right) - \left(\sum_{n=1}^{\infty}4(n+r-1)a_{n-1}x^{n+r}+\right)$$
$$+ \left(\sum_{n=0}^{\infty}a_nx^{n+r}\right) - \left(\sum_{n=0}^{\infty}2a_{k-1}x^{n+r}\right) \quad (45)$$

$$0 = (4r(r-1)+1)a_0x^r + \sum_{n=1}^{\infty}(4(n+r)(n+r-1)+1)a_n - (4(n+r-1)+2)a_{n-1})x^{n+r} \quad (46)$$

Assuming that $a_0 \neq 0$, the first expression is the indicial equation: $(4r(r-1)+1)a_0 = 0$, which has a double root, $r = 1/2$. With this in mind, we substitute our root into

$$(4(n+r)(n+r-1)+1)a_n - (4(n+r-1)+2)a_{n-1}) = 0 \quad (47)$$

to get

$$a_n = \frac{4(n-1/2)+2}{4(n+1/2)(n-1/2)+1}a_{n-1} = \frac{1}{k}a_{n-1}. \quad (48)$$

This is a recursive relation and assuming that $a_0 = 1$, then $a_n = \frac{1}{n!}$. Hence, the solutions are

$$y_1 = \sum_{n=0}^{\infty}a_nx^{n+r} = \sum_{n=0}^{\infty}\frac{1}{n!}x^{n+1/2} = x^{1/2}\sum_{n=0}^{\infty}\frac{1}{n!}x^n = x^{1/2}e^x \quad (49)$$

and a second, linearly independent solution is

$$y_2 = \sum_{n=0}^{\infty}b_nx^{n+r} + y_1\ln x = \sum_{n=0}^{\infty}b_nx^{n+1/2} + x^{1/2}e^x\ln x \quad (50)$$

1.4. **The Hypergeometric Equation.** As the example that is critical in this paper, the hypergeometric differential equation has three regular singular points at 0, 1, and ∞. The roots of the indicial equation for each singular point do not differ by an integer, so we get two independent solutions for each Fuschian point.

We look for the regularity of solutions of the inhomogenous second order differential equation. In other words, we investigate whether or not there are solutions that are analytic at the given Fuschian points. As an approach to this task, we will study the hypergeometric differential equation first by finding the solutions to the equation for each singular point as mentioned in Section 1.3.

We calculate these roots for the singular point $x = 0$. We again impose that the general solution is a series expansion of the form $\sum_{n=0}^{\infty} a_n x^{n+r}$ and we substitute this into the hypergeometric differential equation (9),

$$x(x-1)u''(x) + [-a + (a+b)x]u'(x) + cu(x) = 0 \tag{51}$$

$$x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} - x \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2} - \\ a \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} + (a+b)x \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} + c \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{52}$$

We simplify and rearrange to get

$$\sum_{n=0}^{\infty} a_n[(n+r)(n+r-1) - a(n+r)]x^{n+r-1} + \sum_{n=0}^{\infty} a_n[(n+r)(n+r-1) + (a+b)(n+r) + c]x^{n+r} = 0 \tag{53}$$

We then reindex to find that

$$\sum_{n=0}^{\infty} a_n[(n+r)(n+r-1) - a(n+r)]x^{n+r-1} + \\ \sum_{n=1}^{\infty} a_{n-1}[(n+r-1)(n+r-2) + (a+b)(n+r-1) + c]x^{n+r-1} = 0 \tag{54}$$

and therefore, the indicial equation, which comes from the $n = 0$ term,

$$a_0[r(r-1) + ar]x^{r-1} = 0, \tag{55}$$

is $r^2 + r(a-1) = 0$ and its solutions are $r_{11} = 0$ and $r_{21} = 1 - a$. A similar calculation can be done for the other two singularities. However, we will take a different approach and for $x = 1$, we can make the substitution $z = 1 - x$ such that the hypergeometric equation (9) becomes

$$z(z-1)u''(z) + [b + (a+b)z]u'(z) + cu'(x) = 0 \tag{56}$$

Hence the solution here for $z = 0$ is the solution for $x = 1$ except that the indicial equation differs slightly,

$$a_0[r(r-1) + br]x^{r-1} = 0 \tag{57}$$

giving us roots $r_{12} = 0$ and $r_{22} = 1 - b$. Therefore, there are two linearly independent solutions in the form of a power series expansion.

1.5. **Riemann Scheme.** We introduce the idea of a Riemann scheme for Fuchsian equations illistrated by the hypergeometric equation. The Riemann scheme is a table that displays the singularities and the roots of the indicial equation for each singularity [14]. For the hypergeometric differential equation, where there are three singularities, 0, 1 and ∞, the Riemann scheme is

$$R = \begin{bmatrix} x=0 & x=1 & x=\infty \\ r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix} = \begin{bmatrix} x=0 & x=1 & x=\infty \\ 0 & 0 & -\frac{a+b-1}{2} + \sqrt{\frac{(a+b-1)^2}{4} + c} \\ 1-a & 1-b & -\frac{a+b-1}{2} - \sqrt{\frac{(a+b-1)^2}{4} + c} \end{bmatrix}$$

as the calculations of these roots are shown in the previous section.

1.6. **Variation of Parameters.** In this section, we show the method of variation of parameters that we used in Section 1.3 to solve the following differential equation

$$y'' + q(x)y' + r(x)y = g(x). \tag{58}$$

We know that the general solution to the homogenous differential equation is a linear combination of two independent solutions $y_1(x)$ and $y_2(x)$ and has the form

$$y_h = c_1 y_1(x) + c_2 y_2(x), \tag{59}$$

where c_1 and c_2 are constants. We look for a particular solution in the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{60}$$

with $u_1(x)$ and $u_2(x)$ are two unknowns to be determined, using the method of variations of parameters. We also assume that

$$u_1' y_1 + u_2' y_2 = 0 \tag{61}$$

such that the derivatives of y_p are $y_p' = u_1 y_1' + u_2 y_2'$ and $y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$. Substituting this into the differential equation, we find that

$$\begin{aligned} (u_1'(x)y_1'(x) + u_2'(x)y_2'(x)) + u_1(x)\left[y''(x) + q(x)y_1'(x) + r(x)y_1(x)\right] \\ + u_2(x)\left[y_2''(x) + q(x)y_2'(x) + r(x)y_2(x)\right] = g(x). \end{aligned} \tag{62}$$

Simplifying and imposing (61) and the fact that the homogeneous solution is $y''(x) + q(x)y'(x) + r(x)y(x) = 0$, we get

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = g(x). \tag{63}$$

Hence our two equations are

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \qquad u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = g(x) \tag{64}$$

So we solve for $u_1'(x)$ and $u_2'(x)$:

$$u_2'(x) = \frac{y_1(x)g(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)} \qquad u_1'(x) = \frac{y_2(x)g(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)} \tag{65}$$

and integrate to find $u_1(x)$ and $u_2(x)$

$$u_2(x) = \int \frac{y_1(x)g(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}dx \qquad u_1(x) = \int \frac{y_2(x)g(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}dx \tag{66}$$

Note that the denominator is precisely the Wronskian, which is defined as

$$W = y_1(x)y_2'(x) - y_2(x)y_1'(x), \tag{67}$$

which does not equal zero, since y_1 and y_2 are linearly independent. Recall the particular solution has the form

$$Y_p = y_1u_1 + y_2u_2 \tag{68}$$

by which we substitute our values of u_1 and u_2,

$$Y_p = -y_1(x)\int \frac{y_2(t)g(t)}{W(t)}dt + y_2(x)\int \frac{y_1(t)g(t)}{W(t)}dt. \tag{69}$$

We can put the constants of integration to obtain the general solution

$$Y = Y_p + c_1y_1 + c_2y_2. \tag{70}$$

1.7. **The Gamma Function.** The gamma function is a meromorphic function defined as

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)^{-1}, \tag{71}$$

where γ is a constant such that $\Gamma(1) = 1$. Note that the gamma function has no zeroes but has poles at $z = 0, -1, -2....$ The recursive formula for Γ is given by

$$\Gamma(z+1) = z\Gamma(z). \tag{72}$$

We will utilize the gamma function in our solution to the hypergeometric differential equation.

2. The Problem Studied and Main Results

We study the regularity of solutions of inhomogeneous hypergeometric equations, more precisely, equations of the form

$$u''(x) + \left(\frac{a}{x} + \frac{b}{x-1}\right)u'(x) + \frac{c}{x(1-x)}u(x) = \frac{g(x)}{x(1-x)} \tag{73}$$

where $a, b,$ and c are constants and $g(x)$ is assumed to be analytic in a domain containing both singular points $x = 0$ and $x = 1$. In the case when $g(x) = 0$, the homogeneous equation is the well known hypergeometric equation (9),

$$w''(x) + \left(\frac{a}{x} + \frac{b}{x-1}\right)w'(x) + \frac{c}{x(1-x)}w(x) = 0, \tag{74}$$

which does not have solutions that are analytic at both $x = 0$ and $x = 1$, except for special values of the parameters $a, b,$ and c. However, we want to investigate whether the inhomogeneous equation has solutions that are analytic at both $x = 0$ and $x = 1$. We prove the following result:

Theorem 3. *Consider the following hypergeometric equation with an inhomogeneous term:*

$$u''(x) + \left(\frac{a}{x} + \frac{b}{x-1}\right) u'(x) + \frac{c}{x(1-x)} u(x) = \frac{g(x)}{x(1-x)} \tag{75}$$

where g is a function that is analytic in a domain containing both singular points $x = 0$ and $x = 1$.
Assume that:

$$a, b, a+b \in \mathbb{C} \setminus \{1, 0, -1, -2, -3, \ldots\} \tag{76}$$

and

$$c \neq n(n+a+b-1) \quad \text{for all } n = 0, 1, 2, 3, \ldots \tag{77}$$

Then equation (75) has a unique solution which is analytic at both $x = 0$ and $x = 1$, and therefore it is analytic on the whole domain where g is.
Moreover:
(i) If $\Re a, \Re b > 0$ this solution is given by the formula

$$\begin{aligned} u(x) = -\frac{1}{1-b} v_1(x) \int_1^x v_2(t)\, t^{a-1}(1-t)^{b-1}\, g(t)\, dt + \frac{1}{1-b} v_2(x) \int_1^x v_1(t)\, t^{a-1}(1-t)^{b-1}\, g(t)\, dt \\ - \frac{1}{1-b} v_1(x) \left[m_{a,b,c} \int_0^1 v_1(t) g(t) t^{a-1}(1-t)^{b-1} dt \right] \\ - \frac{1}{1-b} v_1(x) \left[\int_0^1 v_2(t) g(t) t^{a-1}(1-t)^{b-1} dt \right] \end{aligned} \tag{78}$$

where $v_1(x) = 1 + O(x-1)$, $v_2(x) = (1-x)^{1-b}[1 + O(x-1)]$ is a fundamental system of solutions of the homogeneous equation, and

$$m_{a,b,c} = \frac{\Gamma(-r_1)\Gamma(-r_2)\Gamma(1-b)}{\Gamma(1-b-r_1)\Gamma(1-b-r_2)\Gamma(b-1)}$$

where $r_{1,2}$ are the zeroes of $r^2 + (a+b-1)r - c = 0$.
(ii) For other values of a and b, the same formula above gives the analytic solution, only the integrals should be replaced by their Hadamard finite part.
(iii) Furthermore, if $g(x)$ is a polynomial, then $u(x)$ is a polynomial of the same degree.
(iv) Let p_n be the (nonnormalized) Jacobi polynomials on $[0, 1]$ given by the Rodrigues formula

$$p_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n} [q(x)^n w(x)] \quad \text{where } w(x) = x^{a-1}(1-x)^{b-1},\ q(x) = x(1-x) \tag{79}$$

Assume $g(x)$ is analytic on an ellipse $\mathcal{E}$ with foci at 0 and 1 and let $g(x) = \sum_{n=1}^{\infty} g_n p_n(x)$ be the Jacobi series expansion of g. Then the unique solution in (i) has the Jacobi expansion

$$u(x) = \sum_{n=0}^{\infty} \frac{g_n}{c - n(n+a+b-1)} p_n(x)$$

where the series converges uniformly on every compact set in $\mathcal{E}$.

The proof of Theorem 1 (i) and (ii) in the case $\Re a, \Re b > 0$ is found in §3. The proof of Theorem 1 (iv) and (v) in the case $\Re a, \Re b > 0$ is found in §4. The proof of (iii) and (v) for the remaining cases is found in §5.

3. Investigation by integral formula

We first establish the existence of a one parameter family of solutions which is analytic at $x = 0$ and then we investigate if any of these are also analytic at $x = 1$.

3.1. **Local study near each singularity.** Using Frobenius theory, we showed in section 1.4 that the *homogeneous equation* (74) has a fundamental system of solutions of the form

$$u_1(x) = \sum_{i=0}^{\infty} a_i x^i, \qquad u_2(x) = x^{1-a}\tilde{u}_2(x) \text{ where } \tilde{u}_2(x) = \sum_{i=0}^{\infty} a'_i x^i, \tag{80}$$

where $u_1(x)$ and $\tilde{u}_2(x)$ are analytic at $x = 0$ and the power series have radius of convergence of 1. Similarly, it has linearly independent solutions of the form

$$v_1(x) = \sum_{i=0}^{\infty} b_i(1-x)^i, \qquad v_2(x) = (1-x)^{1-b}\tilde{v}_2(x) \text{ where } \tilde{v}_2(x) = \sum_{i=0}^{\infty} b_i(1-x)^i, \tag{81}$$

where $v_1(x)$ and $\tilde{v}_2(x)$ are analytic at $x = 1$, and the series have radius of convergence of at least 1. Since any constant multiple of these solutions is again a solution, we fix these solutions by assuming

$$a_0 = 1, \; a'_0 = 1, \; b_0 = 1, \text{ and } b'_0 = 1. \tag{82}$$

We note that by the assumption (76), the solution u_2 is not analytic at $x = 0$ and the solution v_2 is not analytic at $x = 1$.

Lemma 1. *The Wronskian of the solutions u_1, u_2 is defined as $W[u_1, u_2] = u_1 u'_2 - u'_1 u_2$ and equals*

$$W_0(x) := W[u_1, u_2] = c_0 x^{-a}(1-x)^{-b} \text{ where } c_0 = 1 - a$$

Similarly, the Wronskian of the solutions v_1, v_2 is

$$W_1(x) := W[v_1, v_2] = c_1 x^{-a}(1-x)^{-b} \text{ where } c_1 = 1 - b$$

Proof.
It is well known that the Wronskian of two solutions of a second order equation satisfies $W'(x) = -P(x)W(x)$, where $P(x)$ is the coefficient of the first derivative, after dividing the equation by the coefficient of the second derivative; in our case, $P(x) = \frac{a}{x} + \frac{b}{x-1}$.
Separating the variables, $\frac{dW}{W} = -P(x)dx$, and integrating, we obtain $\ln(W) = -\int P(x)dx + C$ and therefore

$$W = Ce^{-\int P(x)dx}$$

which for our $P(x)$ gives

$$W = Cx^{-a}(1-x)^{-b}$$

To find the constant C for the Wronskian of u_1, u_2, we use (80) in the definition of the Wronskian, then using (82) we obtain:

$$W[u_1,u_2] = u_1(x)u_2'(x) - u_1'(x)u_2(x) = u_1(x)\left[x^{1-a}\tilde{u}_2(x)\right]' - u_1'(x)x^{1-a}\tilde{u}_2(x)$$
$$= u_1(x)\,(1-a)\,x^{-a}\,\tilde{u}_2(x) + u_1(x)\,x^{1-a}\,\tilde{u}_2'(x) - u_1'(x)\,x^{1-a}\,\tilde{u}_2(x) := x^{-a}\ f(x) \quad (83)$$

where $f(x) = u_1(x)\,(1-a)\,\tilde{u}_2(x) + u_1(x)\,x\,\tilde{u}_2'(x) - u_1'(x)\,x\,\tilde{u}_2(x)$ is an analytic function at $x = 0$ with its power series of the form $f(x) = (1-a)a_0a_0' + O(x)^1 = (1-a) + O(x)$. We obtained that $W[u_1,u_2] = x^{-a}[(1-a)+O(x)]$. On the other hand, it must equal $Cx^{-a}(1-x)^{-b} = Cx^{-a}(1+O(x))$ for some suitable C, so the constant is $C = 1-a$.

Similarly, we find the value of c_1 as follows. Again, we start with the definition of the Wronskian:

$$W[v_1,v_2] = v_1(x)v_2'(x) - v_1'(x)v_2(x) = v_1(x)\left[(1-x)^{1-b}\tilde{v}_2(x)\right]' - v_1'(x)(1-x)^{1-b}\tilde{v}_2(x)$$
$$= v_1(x)\,(1-b)\,(1-x)^{-b}\,\tilde{v}_2(x) + v_1(x)\,(1-x)^{1-b}\,\tilde{v}_2'(x) - v_1'(x)\,(1-x)^{1-b}\,\tilde{v}_2(x)$$
$$:= (1-x)^{-b}\ g(x), \quad (84)$$

where $g(x) = v_1(x)\,(1-b)\,\tilde{v}_2(x) + v_1(x)\,(1-x)\,\tilde{v}_2'(x) - v_1'(x)\,(1-x)\,\tilde{v}_2(x)$ is an analytic function at $x = 1$, which can be rewritten using its power series at $x = 1$ as $g(x) = (1-b)b_0b_0' + O(x-1) = (1-b) + O(x-1)$. Therefore, $W[v_1,v_2] = (1-x)^{-b}[(1-b)+O(x-1)] = Cx^{-a}(1-x)^{-b} = C(1-x)^{-b}(1+O(x-1))$. This means that the constant for $x = 1$ is $C = 1-b$. □

The general formula for solving linear, non-homogeneous, second order differential equations applied to (73) reads (see section 1.6):

$$u(x) = -u_1(x)\int_{x_0}^{x}\frac{u_2(t)g(t)}{W(t)t(1-t)}dt + u_2(x)\int_{x_1}^{x}\frac{u_1(t)g(t)}{W(t)t(1-t)}dt + C_1u_1(x) + C_2u_2(x), \quad (85)$$

where x_0, x_1 are arbitrary points, C_1, C_2 are arbitrary constants, u_1, u_2 are any two linearly independent solutions of the homogeneous equation and $W(t)$ is their wronskian.

3.2. **Solutions analytic at $x = 0$.** We first determine which solutions (85) are analytic at $x = 0$. It is convenient to fix the lower bounds of integration at 0; hence, we consider the general solution in the form:

$$u(x) = -u_1(x)\int_{0}^{x}\frac{u_2(t)g(t)}{W_0(t)t(1-t)}dt + u_2(x)\int_{0}^{x}\frac{u_1(t)g(t)}{W_0(t)t(1-t)}dt + C_1u_1(x) + C_2u_2(x) \quad (86)$$

More explicitly, using (80) and Lemma 1 we obtain:

$$u(x) = -u_1(x)\int_{0}^{x}\frac{\tilde{u}_2(t)g(t)}{c_0(1-t)^{1-b}}dt + \tilde{u}_2(x)x^{1-a}\int_{0}^{x}\frac{u_1(t)g(t)}{c_0t^{1-a}(1-t)^{1-b}}dt + C_1u_1(x) + C_2u_2(x) \quad (87)$$

[1]By $O(x)$ we denote a function of the form $xg(x)$ where g is analytic at $x = 0$.

Since the denominator in the second integrand in (87) may vanish at $t = 0$, we need to investigate the convergence of the integral.

The integral $\int_m^x \frac{1}{(t-m)^p} dt$ converges for all $p < 1$. In our case, we thus require that the second integral converges if $1 - a < 1$, thus $a > 0$. If a is a complex number, in a similar way we deduce that it must have positive real part. Similar restrictions will appear for b in §3.3.

In this section we assume $\Re a > 0$ and $\Re b > 0$.

Proposition 1. *If a, b satisfy $\Re a > 0$, $\Re b > 0$ and (76), then equation (73) has a one parameter family of solutions analytic at $x = 0$, namely*

$$u(x) = -u_1(x) \int_0^x \frac{u_2(t)g(t)}{c_0 t^{1-a}(1-t)^{1-b}} dt + u_2(x) \int_0^x \frac{u_1(t)g(t)}{c_0 t^{1-a}(1-t)^{1-b}} dt + C_1 u_1(x), \qquad (88)$$

where C_1 is an arbitrary constant.

Proof of Proposition 1.
The general solution of (73) is given by the formula (87). The first term in (87) is clearly analytic at $x = 0$. We need the following Lemma to investigate the second term.

Lemma 2. *(i) Let $f(x)$ be a function analytic at $x = 0$ and let $a \in \mathbb{C}$ with $\Re a > 0$. Then the function*

$$F(x) = x^{-a} \int_0^x t^{a-1} f(t)\, dt$$

is analytic on the same domain where f is analytic.
(ii) Let $g(x)$ be a function analytic at $x = 1$ and let $b \in \mathbb{C}$ with $\Re b > 0$. Then the function

$$G(x) = (1-x)^{-b} \int_1^x (1-t)^{b-1} g(t)\, dt$$

is analytic on the same domain where g is analytic.

Proof.
(i) Since f is analytic at the point of interest, $x = 0$, we can rewrite f in terms of a convergent power series: $f(x) = \sum_{i=0}^{\infty} f_n x^n$. This means

$$F(x) = x^{-a} \int_0^x t^{a-1} \sum_{i=0}^{\infty} f_n t^n\, dt = x^{-a} \sum_{i=0}^{\infty} \frac{f_n x^{a+n}}{a+n} = \sum_{i=0}^{\infty} \frac{f_n}{a+n} x^n$$

The series converges for all $a \neq 0, -1, -2, \ldots$ having the same radius of convergence as the series of f, and therefore, it is analytic at $x = 0$. Moreover, the integral is clearly analytic at all nonzero points where f is analytic.
(ii) Let $f(x) = g(1-x)$, that is, $g(1-x) = \sum_{i=0}^{\infty} f_n (1-x)^n$. Substituting this in, we have that

$$G(x) = (1-x)^{-b} \int_1^x (1-t)^{b-1} \sum_{i=0}^{\infty} f_n (1-t)^n\, dt = (1-x)^{-b} \sum_{i=0}^{\infty} \frac{f_n (1-x)^{b+n}}{b+n} = \sum_{i=0}^{\infty} \frac{f_n}{b+n} (1-x)^n$$

This integral converges for all b $\neq 0, -1, -2, \ldots$ and has the same radius of convergence as the power series of g. Therefore, it is an analytic function at $x = 1$ on the same domain of analyticity as the function g. $\square$
Using Lemma 2 (i), we see that the second term in (87) is also an analytic function at $x = 0$. We have shown that the first two terms in (87) are analytic. Finally, recall that $u_2(x)$ is a x^{1-a} multiple of an analytic function, and a is not an integer. Since we want an analytic solution in (87), we require that the coefficient, $C_2 = 0$ in front of $u_2(x)$ to be 0. Proposition 1 is now proved. $\square$

3.3. **Solutions Analytic at $x = 1$.** We now investigate which, if any, solutions (88) are analytic at $x = 1$.
As in the previous section, it is useful to know how the fundamental solutions behave at $x = 1$. It is therefore advantageous to express (88) in terms of the fundamental solutions v_1 and v_2, which we do as follows:
We write $u_1(x)$ and $u_2(x)$ as a linear combination of $v_1(x)$ and $v_2(x)$:

$$\begin{aligned} u_1(x) &= m_{11}v_1(x) + m_{21}v_2(x) \\ u_2(x) &= m_{12}v_1(x) + m_{22}v_2(x) \end{aligned} \tag{89}$$

or, in matrix notation:

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} M, \qquad \text{where } M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{90}$$

Plugging (89) into (88) and simplifing the results, we obtain:

$$u(x) = \int_0^x \left[-v_1(x)v_2(t)\det(M) + v_2(x)v_1(t)\det(M)\right] \frac{g(t)}{W_0(t)t(1-t)} dt + C_1\left[m_{11}v_1(x) + m_{21}v_2(x)\right] \tag{91}$$

where $\det M = m_{11}m_{22} - m_{12}m_{21}$ is the determinant of matrix M.

We rewrite W_0 in terms of W_1:

$$W_0 = u_1u_2' - u_1'u_2 = \det \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}$$

then using (89), we obtain

$$W_0 = \det \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = \det \left(\begin{bmatrix} v_1 & v_2 \\ v_1' & v_2' \end{bmatrix} M \right) = W_1 \det(M)$$

Substituting this back into (91):

$$u(x) = \int_0^x \Big[- v_1(x)v_2(t) + v_2(x)v_1(t)\Big] \frac{g(t)}{W_1(t)t(1-t)} dt + C_1\Big[m_{11}v_1(x) + m_{21}v_2(x)\Big] \tag{92}$$

In order to take advantage of Lemma 2 (ii) we change our lower limit of integration to 1. Relation (92) becomes:

$$u(x) = -v_1(x)\int_1^x \frac{v_2(t)g(t)}{W_1(t)t(1-t)}dt + v_2(x)\int_1^x \frac{v_1(t)g(t)}{W_1(t)t(1-t)}dt$$
$$+ v_1(x)\left[C_1 m_{11} - \int_0^1 \frac{v_2(t)g(t)}{W_1(t)t(1-t)}dt\right] + v_2(x)\left[C_1 m_{21} + \int_0^1 \frac{v_1(t)g(t)}{W_1(t)t(1-t)}dt\right] \quad (93)$$

We rewrite (93) using (81) in order to determine which the terms in the sum are analytic at $x = 1$:

$$u(x) = -v_1(x)\int_1^x \frac{\tilde{v}_2(t)g(t)}{C_1 t^{1-a}}dt + \tilde{v}_2(x)\,(1-x)^{1-b}\int_1^x \frac{v_1(t)g(t)}{C_1 t^{1-a}(1-t)^{1-b}}dt$$
$$+ v_1(x)\left[C_1 m_{11} - \int_0^1 \frac{v_2(t)g(t)}{W_1(t)t(1-t)}dt\right] + v_2(x)\left[C_1 m_{21} + \int_0^1 \frac{v_1(t)g(t)}{W_1(t)t(1-t)}dt\right] \quad (94)$$

Clearly, the first and third terms in (94) are functions analytic at $x = 1$. The second term is analytic at $x = 1$ by Lemma 2 (ii). Since the fourth term contains $v_2(x)$, which is not analytic at $x = 1$, its coefficient must be 0, and this will make $u_2(x)$ analytic at $x = 1$. Hence, our condition is as follows:

$$C_1 m_{21} + \int_0^1 \frac{v_1(t)g(t)}{W_1(t)t(1-t)}dt = 0 \quad (95)$$

If

$$m_{21} \neq 0 \quad (96)$$

then we have a unique value for C_1:

$$C_1 = -\frac{1}{m_{21}}\int_0^1 \frac{v_1(t)g(t)}{W_1(t)t(1-t)}dt, \quad (97)$$

To find the coefficient m_{21} in the linear combination $u_1 = m_{11}v_1 + m_{21}v_2$, we need to examine the matrix M.

3.4. **The constants m_{11} and m_{21}.** The matrix M in (90) is called a *monodromy matrix*: it shows how two fundamental solutions are related to each other. For the hypergeometric equations, the monodromy matrices are known, while for many other differential equations, finding them is still an open problem.

3.4.1. *The monodromy matrix M.* We use the formulas in [11] (which we checked against the ones found in [12], equation (9.7)). The Riemann scheme, which gives the roots of the indicial equation for each regular singularity, for our differential equation (74) is:

$$P\begin{bmatrix} x=0 & x=1 & x=\infty \\ r_{0,1}=0 & r_{1,1}=0 & r_{\infty,1}=r_1 \\ r_{0,2}=1-a & r_{1,2}=1-b & r_{\infty,2}=r_2 \end{bmatrix} \quad (98)$$

where $r_{1,2} = -\frac{a+b-1}{2} \pm \sqrt{\frac{(a+b-1)^2}{4} + c}$ are the characteristic indices of the hypergeometric equation (74) at $x = \infty$.

The paper [12] (see also [11]) gives the explicit relationship for the coefficients in the monodromy matrix in terms of the roots of the indicial equation, and we use this to calculate the coefficient m_{11} and m_{21}, More precisely,

$$u_1(x) = m_{11}v_1(x) + m_{21}v_2(x) \tag{99}$$

$$u_1(x) = \frac{\Gamma(r_{0,1} - r_{0,2} + 1)\Gamma(r_{1,2} - r_{1,1})}{\Gamma(r_{0,1} + r_{1,2} - r_{\infty,1})\Gamma(r_{0,1} + r_{1,2} - r_{\infty,2})}\, v_1(x) + \frac{\Gamma(r_{0,1} - r_{0,2} + 1)\Gamma(r_{1,1} - r_{1,2})}{\Gamma(r_{0,1} + r_{1,1} - r_{\infty,1})\Gamma(r_{0,1} + r_{1,1} - r_{\infty,2})}\, v_2(x), \tag{100}$$

or, more explicitly,

$$u_1(x) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1-b-r_1)\Gamma(1-b-r_2)} v_1(x) + \frac{\Gamma(a)\Gamma(b-1)}{\Gamma(-r_1)\Gamma(-r_2)} v_2(x),$$

and therefore,

$$m_{11} = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1-b-r_1)\Gamma(1-b-r_2)}, \quad m_{21} = \frac{\Gamma(a) \cdot \Gamma(b-1)}{\Gamma(-r_1) \cdot \Gamma(-r_2)} \tag{101}$$

3.4.2. *Making condition* (96) *explicit.* The Gamma function has no zeroes, but does have poles (points where it becomes unbounded), which may cause m_{21} in (101) to vanish. Since the Gamma function $\Gamma(x)$ has poles at $x = 0, -1, -2...$, then condition (96) becomes: $r_{1,2} \neq -n$ for all $n = 0, 1, 2, \ldots$, that is

$$-\frac{a+b-1}{2} \pm \sqrt{\frac{(a+b-1)^2}{4} + c} \neq -n,$$

which simplified gives (77).

This completes the proof of Theorem 3 (i) and (ii) in the case when $\Re a, \Re b > 0$.

4. Solution by expansion in Jacobi Polynomials

Let p_n be the Jacobi polynomials defined in Theorem 3 (iv).

Some facts regarding Jacobi polynomials that we use are contained in the Appendix §7. We note that, up to a multiplicative constant, they are the same as p_n in §7.2 with $\alpha = a-1$, $\beta = b-1$. Denote

$$\lambda_n = n(n + a + b - 1). \tag{102}$$

Then by (122) each polynomial p_n satisfies

$$p_n''(x) + \left(\frac{a}{x} + \frac{b}{x-1}\right) p_n'(x) = -\frac{\lambda_n}{x(1-x)}\, p_n(x) \tag{103}$$

which implies that

$$p_n''(x) + \left(\frac{a}{x} + \frac{b}{x-1}\right) p_n'(x) + \frac{c}{x(1-x)}\, p_n(x) = \frac{c-\lambda_n}{x(1-x)}\, p_n(x) \tag{104}$$

so that $p_n(x)$ is a solution of (73) for the case when $g(x) = (c - \lambda_n)p_n(x)$.
We are thus led to the idea of decomposing the nonhomogeneous term g as a linear combination of Jacobi polynomials.

4.1. **Solutions of** (73) **when** $g(x)$ **is a polynomial.** The Jacobi polynomials form a complete orthogonal set, so any polynomial $g(x)$ of, say, degree d, can be written as a linear combination of Jacobi polynomials; that is, there exist constants $g_1, \ldots, g_d$ so that

$$g(x) = \sum_{k=0}^{d} g_k\, p_k(x) \tag{105}$$

We look for a solution of (73) as a linear combination of Jacobi polynomials

$$u(x) = \sum_{k=0}^{N} u_k\, p_k(x) \tag{106}$$

which plugged in the equation gives

$$\sum_{k=0}^{N} u_k \left[p_k'' + \left(\frac{a}{x} + \frac{b}{x-1}\right) p_k' + \frac{c}{x(1-x)}\, p_k \right] = \sum_{k=0}^{d} \frac{g_k}{x(1-x)}\, p_k(x) \tag{107}$$

In view of (104) we must have $N \geq d$ and

$$\sum_{k=0}^{d} u_k(c-\lambda_k)\, p_k(x) + \sum_{k=d+1}^{N} u_k(c-\lambda_k)\, p_k(x) = \sum_{k=0}^{d} g_k\, p_k(x) \tag{108}$$

Since the Jacobi polynomials are linearly independent we must have

$$u_k(c-\lambda_k) = g_k \quad \text{for all } k = 0, 1, \ldots, d$$

and

$$u_k(c-\lambda_k) = 0 \;\; \text{for } k > d$$

Assuming

$$c - \lambda_k \neq 0, \quad \text{for all } k = 0, 1, 2, \ldots \tag{109}$$

we obtain

$$u_k = \frac{g_k}{c-\lambda_k}, \qquad k = 0, 1, \ldots, d \tag{110}$$

and $u_k = 0$ for $k > d$.
We thus found a unique solution of (73) as a polynomial of the same degree as $g(x)$:

$$u(x) = \sum_{k=0}^{d} \frac{g_k}{c-\lambda_k}\, p_k(x) \tag{111}$$

Note that in view of (102) the solvability conditions (109) are the same as (77).

4.2. **Solutions of** (73) **when** $g(x)$ **is an analytic function.** It is known that functions analytic on a large enough domain in the complex plane can be expanded in Jacobi series - see Theorem 4 in §7. Letting $z = 1 - 2x$, we obtain that if $g(x)$ is analytic in the interior of an ellipse $\mathcal{E}$ with foci at 0 and 1 then g has a unique expansion

$$g(x) = \sum_{k=0}^{\infty} g_k \, p_k(x), \tag{112}$$

and the series converges absolutely in the interior of the ellipse $\mathcal{E}$. The same calculations and justifications that we previously did for the case when g is a polynomial hold here, only now we find a solution of (73) as the series

$$u(x) = \sum_{k=0}^{\infty} \frac{g_k}{c - \lambda_k} \, p_k(x) \tag{113}$$

if c satisfies (109). The series converges absolutely on the same domain where the series (112) does since, using (102), we see that $|g_k|/|c - \lambda_k| < |g_k|$ for k large enough.
We thus proved Theorem 3 in the case when $a, b > 0$.

5. Solution for general parameters

The Hadamard finite part is a method for possibly diverging integrals and can be taken as the analytic continuation of a convergent integral. Analytic continuation is a technique used to extend the domain of a given analytic function in the complex domain. When an integral is divergent, we can proceed by dropping the divergent part and keeping the converging part.

Moreover, the Hadamard finite part of an integral can be manipulated the same way as an ordinary integral, such as with integration by parts and addition on the interval of integration.

For $a < 0$ or $b < 0$ some of the integrals in (78) are undefined. Section §7.4 shows that the integrals can be interpreted as Hadamard finite part for any α, $\beta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and then all the results used above, in the case when $\Re\alpha$, $\Re\beta > 0$ still hold.

6. Conclusion

The hypergeometric differential equation is a linear second order differential equation with two singularities in the complex plane and a third at infinity, all three singularities being of Fuchsian type. We have studied the regularity of solutions of the inhomogeneous hypergeometric equation where the inhomogeneous term, $g(x)$ is assumed to be analytic in a domain containing both singular points $x = 0$ and $x = 1$.

In the case when $g(x) = 0$, the homogeneous equation is the hypergeometric equation, which is known not to have solutions that are analytic at both $x = 0$ and $x = 1$, except for special values of the parameters a, b, and c. We investigate whether the inhomogeneous equation has solutions that are analytic at both $x = 0$ and $x = 1$.

We find that with an inhomogeneous term, there exists a unique solution that is analytic at both singularities, provided that the parameters of the equation satisfy explicit conditions.

For $a, b > 0$, we use integral formulas for solutions and the monodromy of the hypergeometric equation. For general parameters, when the integrals are divergent, we consider the Hadamard finite part of these integrals.

More precisely, we have established a unifying approach to representing analytic functions. We have reformulated the classical theory of expansion in terms of Jacobi polynomials with the idea of using the Hadamard finite part of the possible diverging integrals.

7. Appendix: Jacobi series expansions of analytic functions

Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ $(n = 0, 1, 2, \ldots)$ are classical orthogonal polynomials (for an exposition see, for example [6]) satisfy the Rodrigues formula:

$$P_n^{(\alpha,\beta)}(z) = C_n \frac{1}{W(z)} \frac{d^n}{dz^n} \left[Q(z)^n W(z)\right]$$

$$\text{where } W(z) = (1-z)^\alpha (1+z)^\beta,\ Q(z) = (1-z)(1+z),\quad C_n = \frac{(-1)^n}{2^n n!} \tag{114}$$

7.1. **Orthogonality of Jacobi polynomials for $\Re\alpha > -1$, $\Re\beta > -1$.** For this range of parameters the polynomials $P_n^{(\alpha,\beta)}(z)$ are mutually orthogonal on the interval $[-1, 1]$ in the sense that they satisfy

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(z)\, P_k^{(\alpha,\beta)}(z)\, W(z)\, dz = 0 \quad \text{if } n \neq k \tag{115}$$

It is well known that analytic functions can be expanded in Jacobi series ([7] p.245):

Theorem 4. *Let $\alpha > -1,\ \beta > -1$. Let f be analytic in the interior of an ellipse with foci at ± 1. Denote by $\mathcal{E}$ the greatest such ellipse.*
Then f has an expansion

$$f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{116}$$

which is convergent in the interior of $\mathcal{E}$ and divergent outside.

We note that the coefficients a_n in (116) can be calculated by multiplying (116) by $P_k^{(\alpha,\beta)}(z)(1-z)^\alpha(1+z)^\beta$, integrating on $[-1, 1]$, then interchanging the series with integration (possible because the series converges uniformly), and in view of (115) only the coefficient of a_k is not zero, yielding

$$\int_{-1}^{1} f(z)\, P_k^{(\alpha,\beta)}(z)(1-z)^\alpha(1+z)^\beta\, dz = a_k\, h_k^{(\alpha,\beta)},$$

$$\text{where } h_k^{(\alpha,\beta)} = \int_{-1}^{1} \left[P_k^{(\alpha,\beta)}(z)\right]^2 (1-z)^\alpha(1+z)^\beta\, dz \tag{117}$$

Finally, it is well known that the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ satisfy the differential equation:

$$(1-z^2)P'' + [\beta - \alpha - (\alpha+\beta+2)z])P' + n(n+\alpha+\beta+1)\, P = 0 \tag{118}$$

7.2. **Jacobi polynomials on** $[0,1]$**.** Substituting $p_n(x) = P_n^{(\alpha,\beta)}(\frac{1-z}{2})$ we obtain a sequence of polynomials orthogonal on $[0,1]$ (which we will also call Jacobi polynomials).
Formula (114) becomes

$$p_n(x) = c_n \frac{1}{w(x)} \frac{d^n}{dx^n} \left[q(x)^n w(x) \right],$$

$$\text{where } w(x) = x^\alpha (1-x)^\beta,\ q(x) = x(1-x),\quad c_n = \frac{(-1)^n 2^n}{n!} \tag{119}$$

the orthogonality relation (115) becomes

$$\int_0^1 p_n(x)\, p_k(x)\, w(x)\, dx = 0 \ \text{ if } n \neq k \tag{120}$$

By Theorem 4 it follows that a function $g(x)$ which is analytic in an elliptic domain with foci at 0 and 1 has a unique expansion

$$g(x) = \sum_{n=0}^{\infty} a_n\, p_n(x)$$

where the coefficients a_n are given by

$$\int_0^1 g(x)\, p_n(x)\, x^\alpha (1-x)^\beta\, dx = a_k \int_0^1 p_k^2(x)\, x^\alpha (1-x)^\beta\, dx \tag{121}$$

Equation (118) becomes the differential equation satisfied by p_n :

$$p'' + \left(\frac{\alpha+1}{x} + \frac{\beta+1}{x-1} \right) p' + \frac{n(n+\alpha+\beta+1)}{x(1-x)}\, p = 0 \tag{122}$$

which is a hypergeometric equation.

7.3. **Jacobi expansions for general parameters.** The Rodrigues formula (114) defines polynomials for all complex parameters α, β (though it should be noted that, while for $\alpha+\beta \neq -2,-3,-4,\ldots$ we have $\deg P_n^{(\alpha,\beta)} = n$, this is no longer true in the excepted cases). On the other hand the integral in (115) is convergent only for $\Re\alpha > -1$, $\Re\beta > -1$. But we still have completeness, and orthogonality in a generalized sense, as we illustrate below.

7.3.1. *Completeness.* Theorem 4 was extended in [8] to all α, β with $\alpha+\beta \neq -2,-3,-4,\ldots$. We reproduce below this result in a special case ($r=1$, $s=0$) which is of interest for the present problem.

Theorem 5. [8] *Let* Ω *be an open elliptic disk with foci* 1 *and* 0*, and let* h *be analytic on* Ω*. Let* α, β *complex numbers so that* $\alpha+\beta \neq -2,-3,-4,\ldots$*. Then, for every* $z \in \Omega$ $f(z)$ *has the expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\, c_n\, R_n(z) \tag{123}$$

where

$$c_n = F^{(n)}(1+\alpha+n, 1+\beta+n) = \frac{1}{B(n+\alpha+1, n+\beta+1)} \int_0^1 f^{(n)}(u) u^{n+\alpha} (1-u)^{n+\beta}\, du$$

and $R_n(z)$ are scalar multiples of $P_n^{(\alpha,\beta)}(\frac{1-z}{2})$, and are given by the formula $R_n(z) = R_n(-\alpha - n, -\beta - n, z-1, z)$ where

$$R_n(b, b', x, y) = \frac{1}{B(b, b')} \int_0^1 [ux + (1-u)y]^n \; u^{b-1}(1-u)^{b'-1} \, du \tag{124}$$

The series (123) *converges absolutely on Ω, uniformly on every compact set in Ω.*

As always, $B(b, b')$ denotes the beta function:

$$B(b, b') = \int_0^1 t^{b-1}(1-t)^{b'-1} \, dt = \frac{\Gamma(b)\,\Gamma(b')}{\Gamma(b+b')}$$

7.3.2. *Orthogonality.* By Favard's theorem the Jacobi polynomials must be mutually orthogonal with respect to a bilinear functional. It was recently shown [9] that this linear functional is still (115) if $\alpha, \beta \neq -1, -2, -3, \ldots$ where the divergent integrals should be understood as the *Hadamard finite part*.

The Hadamard finite part of a (possibly divergent) integral is constructed as follows. Consider $f(x)$ a function analytic at $x = 0$: $f(x) = \sum_{n=0}^{\infty} c_n x^n$ a series with nonzero radius of convergence R. The integral

$$x^{-\alpha} \int_0^x t^{\alpha-1} f(t) \, dt \tag{125}$$

converges for $\Re\alpha > 0$ and for $|x| < R$ we have

$$\begin{aligned} x^{-\alpha} \int_0^x t^{\alpha-1} f(t) \, dt = x^{-\alpha} \int_0^x t^{\alpha-1} \sum_{n=0}^{\infty} c_n t^n \, dt = x^{-\alpha} \sum_{n=0}^{\infty} c_n \int_0^x t^{n+\alpha-1} \, dt \\ = x^{-\alpha} \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} \, x^{n+\alpha} = \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} \, x^n \end{aligned} \tag{126}$$

and we see that the result is analytic at $x = 0$. For all other complex values of $\alpha \neq 0, -1, -2, \ldots$, the Hadamard finite part of the integral is defined by:

$$x^{-\alpha} H \int_0^x t^{\alpha-1} f(t) \, dt := \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} \, x^n \quad \text{for all } \alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$$

This definition represents the analytic continuation in α of the function (125) to α in the complex plane (when α is a negative integer the continuation has poles of order one).

7.4. **Jacobi series expansion for general parameters.** Reformulating Theorem 5 using Hadamard principal part we see that the coefficients in the Jacobi series (123) are given by (121) where the (possibly divergent) integrals should be interpreted as the Hadamard finite part:

Theorem 6. *Let $\alpha, \beta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.*
Consider the (non-normalized) Jacobi polynomials on $[0, 1]$ given by the Rodrigues formula:

$$p_n = \frac{1}{w} \frac{d^n}{dx^n} (q^n w) \quad \textit{where } w = x^{\alpha}(1-x)^{\beta}, \; q = x(1-x) \tag{127}$$

Let $\Omega \in \mathbb{C}$ be an open elliptic disk with foci 1 and 0, and let f be analytic on Ω. Then $f(x)$ has the expansion for all $x \in \Omega$ as

$$f(x) = \sum_{n=0}^{\infty} f_n \, p_n(x) \qquad \textit{where} \quad f_n = \frac{H \int_0^1 f p_n w}{H \int_0^1 p_n^2 w} \tag{128}$$

The expansion converges absolutely and uniformly on every compact set in Ω.

Proof.
Since the Hadamard principal part is the analytic continuation of the usual integral to the complex plane, and since the proof in [8] is also by analytic continuation (and the analytic continuation is unique) it is clear that the formulas should be the same.
A straightforward calculation can be also made to link (128) and (123), yielding

$$p_n = (-1)^n \, (\alpha + \beta + 2n)_n \, R_n(x) \quad \text{ and } \quad \frac{1}{n!} \, c_n \, (-1)^n \, (\alpha + \beta + 2n)_n^{-1} = \frac{H \int_0^1 f p_n w}{H \int_0^1 p_n^2 w} := f_n$$

where $R_n(x)$ are given by (124), $(a)_n$ denotes the raising factorial: $(a)_n = a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a)$. We illustrate below the main steps of the calculation.
In [8] R_n is shown to be a multiple of a Jacobi polynomial by using the binomial formula in (124), then integrating term by term (this is also correct if one uses the Hadamard principal part in case the integrals diverge). Then one compares with the expression of p_n obtained by using Leibniz rule to expand its Rodrigues formula.
Integration by parts also holds for Hadamard finite part, and p_n were proved to be an orthogonal set in [9]. It only remains to calculate the "norm" of p_n in this sense.
We first calculate the dominant term in p_n: it is found by retaining only the highest powers of x in its Rodrigues formula, yielding: $p_n = D_n x^n + ...$ where $D_n = (-1)^n(\alpha + \beta + 2n)_n$.
Then, using the fact that p_n is orthogonal to all the polynomials of degree less than n, and then using the Rodrigues formula and integrating by parts n times we obtain

$$\begin{aligned} H \int_0^1 p_n^2 w &= D_n \, H \int_0^1 p_n \, x^n \, w = D_n \, H \int_0^1 [Q^n w]^{(n)} \, x^n \\ &= D_n \, (-1)^n \, n! \, H \int_0^1 x^{n+\alpha}(1-x)^{n+\beta} \, dx = D_n \, (-1)^n \, n! \, B(n+\alpha+1, n+\beta+1) \end{aligned} \tag{129}$$

□

Proposition 2. *The Jacobi polynomials p_n defined by* (127) *for general α, β satisfy* (122).

Proof.
This follows from a more general result proved in [10] p.697. The notations (1)...(3) in [10] in our case are $\sigma = -1, \; \tau = 1, \; \delta = 0, \; qw^{-1}w' \equiv xL_1 + L_2 = -(\alpha+\beta)x + \alpha$; then formula (5) in [10] gives here $\mathcal{A}_1 = -(\alpha+\beta+2)x + \alpha + 1 + x(1-x)\partial_x$ and Proposition 4 gives (122).
□

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