Weighted Dice: A Physical Application for Probability Studies

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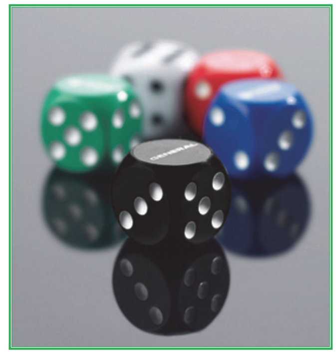
Abstract

Conventional dice were altered by the insertion of weights for the purpose of creating dice that do not behave according to the conventional rules of probability. The performance of these dice was then characterized by students in the classroom through several hundred experimental trials. Each trial involved one of the following: a one-die roll, a two-dice roll sum, or a three-dice roll sum. Results from these student trials were combined into one large database. These results with weighted dice were then compared against theoretical calculations with conventional dice for one-die rolls, two-dice roll sums, and three-dice roll sums.



The basic precepts of probability theory are shaped in many students long before they get into a classroom where a formal probability course is taught. In life, students often have experience with rolling dice, flipping coins, and playing cards, that allows them to already understand that the probability of rolling a 3 with a "balanced" die is $\frac{1}{6}$, the probability of flipping a "fair" coin and getting the result of a "tail" is $\frac{1}{2}$, and the probability of drawing a random card out of a deck of cards and having it be the "king of spades" is $\frac{1}{52}$. These basic foundations can then be refined to establish how to perform calculations for the probabilities for the sum of two dice being 6 or the probability of flipping four heads and two tails in six coin tosses, or the probability of getting a pair of kings out of a five-card poker hand.

However, in many advanced texts, the author moves from these familiar situations to a case where a coin is "unfair" or a die is "weighted" (Miller & Miller, 2004). The students are expected to accept on faith that a coin can be altered such that the probability of a "tail" changes from $\frac{1}{2}$ to $\frac{2}{5}$ or that a die can be loaded such that the probability of an odd value is twice the value of an even value. From a theoretical perspective, and with further development of the probability concepts, this is possible. From an experimental perspective, a question arises as to whether such unbalanced coins and weighted dice actually exist that can provide these altered probabilities. The subject of this paper is to undertake the production of weighted dice and then involve students in performing trials to compare this particular set of weighted dice to the expected values for conventional dice.

Procedure

Preparation. Conventional dice were obtained for the purpose of this experiment. These dice were intentionally chosen with a variety of colors to allow them to be distinguishable. The dice were measured to have an average mass of 4.57 ± 0.14 grams. Each die was altered by drilling a one-eighth inch hole through the die and then placing weights inside the hole. The location of the hole was chosen to be in the dot (or "pip") on the 4 side of the die nearest to the corner shared by the 5 and 6 faces, as shown in Figure 1. A one-eighth inch hole was drilled through the die at this location, such that the hole traveled completely through the die from the '4' to '3' side. The amount of material drilled out of each die was on average 0.35 grams. After the hole was created, two Water Gremlin™ round split-shot sinkers made

Figure 1. This picture shows two dice, one before and one after being weighted. As described in the text, the lead sinker was placed in a hole drilled in one of the pips in the 4 face of the die.

of lead and having a mass of 1.04 grams each, were inserted into the drilled hole, one in each end. The sinkers were pressed into the hole so as to fit tightly and not fall out. Fifteen weighted dice (10 blue and 5 green) were prepared in this manner (see Figure 2). The average mass of the fifteen altered dice was measured to be 6.30 ± 0.11 grams.

Figure 2. Fifteen dice, five green and ten blue, were weighted using this process.

Calculation. Students were asked to determine the sample spaces for single die rolls (S = {1, 2, 3, 4, 5, 6}), rolls for the sums of two dice, (S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}), and rolls for the sums of three dice (S = {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}). Students were then asked to calculate the probability for each of the results within these sample spaces for conventional dice. These calculations gave the students the opportunity to apply many of the fundamental probability concepts that had previously been introduced. For the single roll, the concept that when "n" outcomes are equally likely, the probability for any one particular outcome, "A", is given by $(A) = \frac{1}{n}$, quickly leads us to probabilities of $\frac{1}{6}$ for each possible outcome.

The two dice sum roll gives an opportunity for students to apply the concept that when the probability of a particular outcome depends on two independent events (such as two dice sum rolls), the probability of a particular outcome (e.g. $P(3,2)$) is the product of the probability of each individual outcome for the two dice ($P(3,2) = P(3) \cdot P(2) = \frac{1}{6} \cdot \frac{1}{6}$). This also gives the opportunity to see that, with a particular outcome consisting of n events, there are n! ways to arrange that outcome (e.g. $P(3,2)$ and $P(2,3)$). In addition, if the particular outcome is the sum of the dice (e.g. sum = 5), then the student has an opportunity to determine all the outcomes that lead to the specified sum of 5 ($P(3,2) + P(2,3) + P(1,4) + P(4,1) = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{4}{36}$), and thus establish the probability of achieving a sum of "5".

These concepts are further expanded in the rolling of three dice and summing the totals, where there are significantly different ways of achieving a particular sum. For instance some outcomes have three distinct numbers and hence $\binom{3}{1} = 3! = 6$ ways of achieving a sum (i.e. $P(2,3,4)$ for a sum of 9), other outcomes have two distinct numbers hence $\binom{3}{2} = 3$ ways of achieving a sum (i.e. $P(2,2,5)$ for a sum of 9), and other outcomes with only one distinct number and hence $\binom{3}{3} = 1$ way of achieving the sum (i.e. $P(3,3,3)$ for a sum of 9). The concept of the probability for any particular outcome of independent events is still applicable with three dice, so this concept is still applied while calculating the total probability. Thus students apply a wide variety of basic probability principles to these calculations.

Experiment. Using the weighted dice, students were asked to roll a pair of dice which were distinguishable (differently colored) 144 times each. They were to keep track of both the individual totals for each die, and of the sum. They repeated this experiment where they only recorded the sum of the two dice for an additional 144 data points each. This established 288 data points to compare to the single dice theoretical calculation and 288 data points for each student for the two dice sum theoretical calculation. Separately, the students were asked to roll three dice and record the sum of each roll 224 times each. There were five students so this resulted in n = 1440 for single die rolls, n = 1439 for two dice sum rolls (one data point was lost, as a student turned in results for one less roll than was requested), and n = 1120 for three-dice sum rolls.

Results. The results presented here represent the combined data of all students.
One die roll. For the one die roll results, three sets of data are displayed in Figure 3. The theoretically expected values and experimental values are clear, but the additional data are results based on 20,000 rolls of a single conventional die reported by Wolf (Hand, et al, 1994). These results are given for a comparison of the student data with the weighted die in order to illustrate the differences between the random variability of a conventional die and the purposeful variability of a weighted die. As the data shows, Wolf's data almost all fall within 10% of the theoretical values (represented by the error bars), whereas the probabilities for the outcomes 1 and 2 for the weighted die are more than 30% higher than the theoretical probabilities for those outcomes. Similarly, the probabilities for the outcomes 5 and 6 for the weighted die are roughly 30% lower than the theoretical probabilities for those outcomes. Wolf's data has been questioned as to whether the die was fair (Dunn, 2005), even with its comparatively modest variation in the neighborhood of 10% different than the theoretical outcomes.

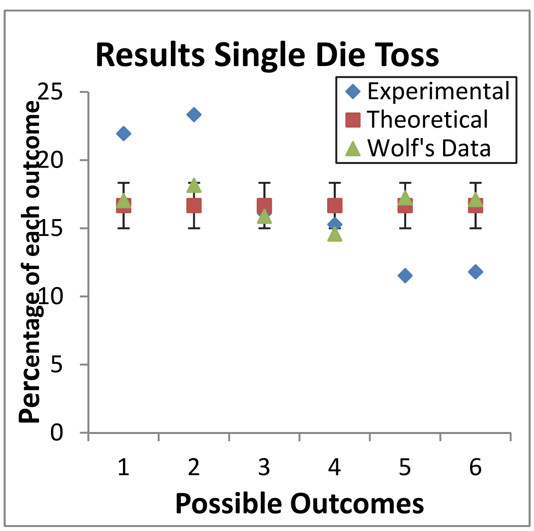


Figure 3. Results from single die roll experiment

Table 1. Theoretical vs. experimental results for single die roll

Result (n=1440)	Theoretical expectation with conventional die	Experimental result with weighted die	Percentage difference
1	240	316	31.7%
2	240	336	40.0%
3	240	232	-3.3%
4	240	220	-8.3%
5	240	166	-30.8%
6	240	170	-29.2%

Two dice sum rolls. For the two dice roll results (see Figure 4) only the theoretically expected outcomes for normal dice and the experimental outcomes for the weighted dice are presented. Based on the significant increase in 1's and 2's and the significant decrease in 5's and 6's in the single die roll results, one can reasonably assume that the results for rolling two dice at a time will show a marked increase in the frequency of low sums being rolled and a decrease in the frequency of higher sums being rolled. This is reflected in the data. The sums of 2, 3, 4, and 5, which have no possibilities of having either a 5 or a 6 as a result of either die, predictably had increased incidences of occurrence, ranging from 25.7% to 105.1% above the predicted values for conventional dice. Meanwhile, the sums of 9, 10, 11, and 12, which have no possibilities of having either a 1 or a 2 as a result of either die, predictably had decreased incidences of occurrence ranging from 30.0% to 42.5% below the predicted value for conventional dice.

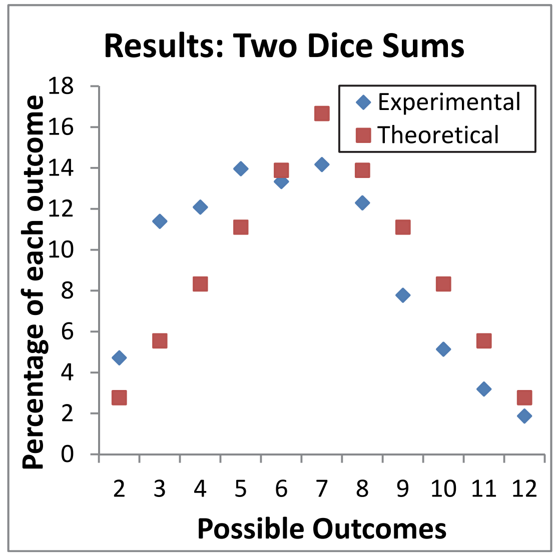


Figure 4. Results from two dice role experiment

The sum outcomes of 6, 7, and 8, which are all achieved by using combinations of the low occurring 5's and 6's of the weighted dice, as well as the high occurring 1's and 2's of the weighted dice, predictably had occurrences much closer to the theoretical predictions for normal dice, varying from 3.9% to 14.9% below predicted occurrence, as shown in Table 2.

Table 2.

Sum	Deviation	Sum	Deviation
2	70.1%	8	-11.4%
3	105.1%	9	-30.0%
4	45.1%	10	-38.3%
5	25.7%	11	-42.5%
6	-3.9%	12	-32.5%
7	-14.9%		

Three dice sum rolls. The data resulting from rolling three dice again includes just the theoretical expected outcomes for normal dice and the experimental outcomes for the weighted dice. Based on the significant increase in 1's and 2's and the significant decrease in 5's and 6's, one can reasonably assume that the results for rolling three dice at a time will show a marked increase in the frequency of low sums being rolled and a decrease in the frequency of higher sums being rolled. This is reflected in the data, as seen in Figure 5. The sums of 4, 5, and 6, which have no possibilities of having either a 5 or a 6 as a result of any of the dice, had increased incidences of occurrence ranging from 31.1% to 176.4% higher than predicted. The sum of 3 had a slightly lower incidence than predicted. This result occurred with a very limited number predicted (only 5 occurred, 5.18 predicted), so this is not statistically meaningful.

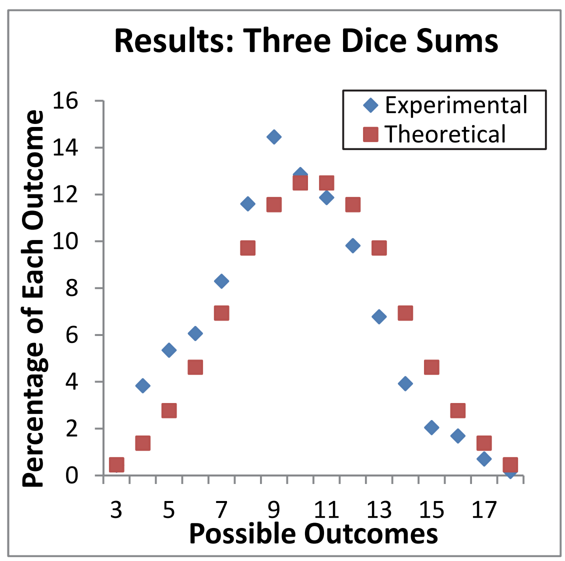


Figure 5. Results from three dice roll experiment

Meanwhile, the sums of 15, 16, 17, and 18, which have no possibilities of having either a 1 or a 2 as a result of any of the dice, predictably had decreased incidences of occurrence ranging from 38.9% below expectation to 61.4% lower incidence than predicted. The sum outcomes of 7, 8, and 9, which are all achieved by using combinations that have a preponderance of the highly occurring 1's and 2's and few incidences of the low occurring 5's and 6's of the weighted dice, showed increased incidences ranging from 19.4% to 25.0%. Similarly, the sum outcomes of 12, 13, and 14, which utilize more of the low occurring 5's and 6's and fewer of the high occurring 1's and 2's, showed decreased incidences ranging from 15.1% to 43.4%. The outcomes of 10 and 11, which are reached with a mostly balanced number of 5's and 6's and 1's and 2's were both within 5% of the predicted value, as indicated in Table 3.

Table 3.

Sum	Deviation	Sum	Deviation
3	-3.6%	11	-5.0%
4	176.4%	12	-15.1%
5	92.9%	13	-30.2%
6	31.1%	14	-43.4%
7	19.6%	15	-55.6%
8	19.4%	16	-38.9%
9	25.0%	17	-48.6%
10	2.9%	18	-61.4%

Conclusion

The purpose of this exercise was to use a fabricated a set of weighted dice that would not perform in a conventional way and thus allow students to evaluate this different behavior and extend their understanding of probability theory as a result. Texts frequently refer to weighted dice or coins, but then proceed with probabilities for these outcomes that appear purely theoretical, and are not based in any realistic weighting approach (i.e. "a die is loaded in such a way that each odd number is twice as likely to occur as an even number," (Miller & Miller, 2004, p.32)). In this exercise, weighted dice have been established for which one can make a reality based statement such as, "these dice are weighted in such a way that the occurrence of a 1 and a 2 each have a probability of occurrence of $\frac{21}{96}$", the occurrence of a 3 and a 4 each have the probability of occurrence of $\frac{1}{6}$ and the probability of occurrence of a 5 and a 6 each have the probability of occurrence of $\frac{11}{96}$. The extension of the dice rolling to include both two and three dice sums then allows the students to apply classical probability theory and observe how much of an impact the weighting of the dice has on the experimental outcomes in comparison to the expected outcomes with ordinary dice.

Further exploration can include the students using these experimentally established probabilities to establish probabilistic models to predict the results of two and three dice sum rolls and comparing them to the existing data. In addition, different weighting approaches, such as inserting a single mass in one corner, or inserting a mass into the center of one of the faces of a die rather than the approach used here, could be investigated experimentally in an attempt to establish weighted dice with different results.

References

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