A Pythagorean-Like Theorem

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"The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. Oh, joy, oh, rapture. I've got a brain!"

The Scarecrow in The Wizard of Oz

Introduction

The Pythagorean Theorem is widely considered to be the "most proved" theorem in all of mathematics. Eli Maor (2007) attributes the Pythagorean Theorem's appeal in part to the large number of proofs of this proposition. In the Pythagorean Proposition, Elisha Scott Loomis, an Ohio native, collected over 350 different proofs of this Theorem. Maor (2007) claims this number is now well over 400. Eighty-one proofs of the Pythagorean Theorem and associated java applets can be found on *Cut the Knot* at http://www.cut-the-knot.org/pythagoras/. The Theorem, Proposition 47 in Book I of Euclid's Elements, states "In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle" (Joyce, 1998). Perhaps it is the Theorem's simplicity that makes it so appealing to a number of would-be provers.

One of the proofs that illustrate the wide appeal of the Pythagorean Theorem is the following proof attributed to President James A. Garfield. Begin with the trapezoid shown in Figure 1.

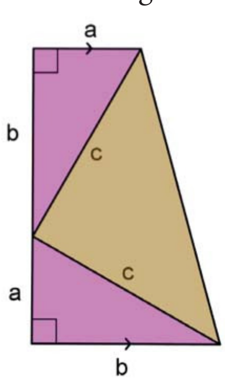

Fig 1 Garfield's Proof

The area of the trapezoid is $\frac{1}{2}(base_1 + base_2)(height)$. Making the appropriate substitutions and simplifying, we have:

$$\frac{1}{2}(base_1 + base_2)(height)$$
$$= \frac{1}{2}(a+b)(a+b)$$
$$= \frac{1}{2}(a^2 + 2ab + b^2)$$
$$= \frac{1}{2}a^2 + ab + \frac{1}{2}b^2$$

Notice, however, the trapezoid consists of two congruent right triangles, each with legs of length *a* and *b*, giving an area of $\frac{1}{2}ab$ and an isosceles right triangle with legs of length *c* and an area of $\frac{1}{2}c^2$. Therefore, $\frac{1}{2}(a^2 + 2ab + b^2) = 2\left(\frac{1}{2}ab\right) + \frac{1}{2}c^2$. Simplifying this, we have $\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = ab + \frac{1}{2}c^2$ which simplifies to $a^2 + b^2 = c^2$, the familiar version of the Pythagorean Theorem

A Theorem about Areas

President Garfield's proof captures an essential feature of the Pythagorean Theorem. First and foremost, the Pythagorean Theorem is a geometric statement about areas; in particular, areas related to the sides of a right triangle. Though Garfield's proof is carried out in an algebraic manner, at its heart is an area argument. In its most common form, a high school student may memorize $a^2 + b^2 = c^2$ while missing the main points of the theorem: namely that it applies *only* to a right triangle, and the total area of the squares constructed on the legs of a right triangle is equal to the area of the square constructed on the hypotenuse.

Generally speaking, the Pythagorean Theorem states that two squares constructed on the legs of a right triangle can be cut up and rearranged into one larger square constructed on the hypotenuse of the right triangle. Though the Theorem does not give a method for this dissection, numerous dissection proofs abound, often in the form of a "proof without words" (Nelson, 2001). One dissection method is shown in Figures 2 and 3.

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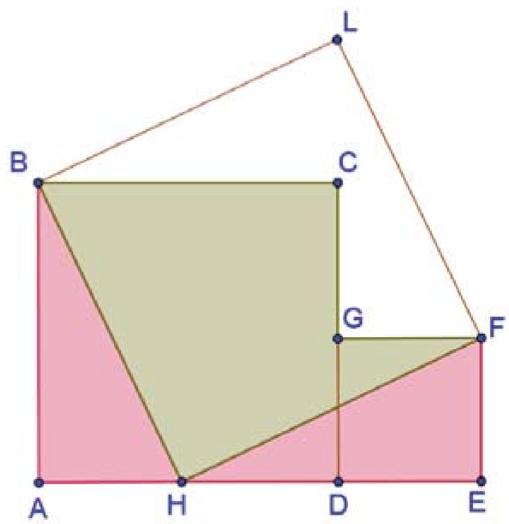

Fig 2 Dissection method

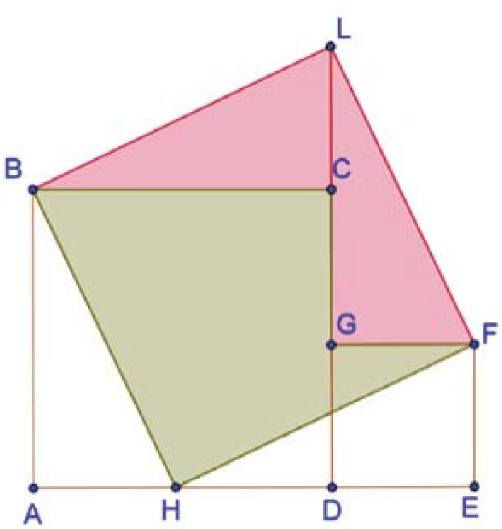

Fig 3 Dissection method

Squares ABCD and DEFG are placed side-by-side as shown in Figure 2. Point H is marked on side AD such that the distance from A to H is equal to the distance from D to E. Dissect the two squares by cutting along BH and HF. The two right triangles BAH and HEF can easily be shown to be congruent by Side-Angle-Side. Moreover, the square ABCD is on a leg of right triangle HEF, and square ABCD is on the other corresponding leg of right triangle DEFG.

Triangles BAH and HEF can be placed in the white space of square BHFL, completing square BHFL as shown in Figure 3. This square BHFL is constructed on the hypotenuse of both right triangles BAH and HEF. This dissection demonstration of how to cut up two squares and reassemble the parts into a larger square on the hypotenuse of a right triangle is a proof of the Pythagorean Theorem.

A Theorem about Areas of Similar Figures

We can use the Pythagorean Theorem to establish a more general geometric statement about the areas of similar figures. Proposition 31 in Book VI of Euclid's Elements states: "In right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle." In a nutshell, as long as the figures constructed on the sides of a right triangle are similar to each other, the sum of the areas of the figures constructed on the legs is equal to the area of the figure constructed on the hypotenuse. The Pythagorean Theorem could be viewed as a special case of this proposition. Indeed, the Pythagorean Theorem and its converse are Propositions 47 and 48, respectively, in Book I of the Elements. Thus, Proposition 31 is a generalization of Proposition 47.

The "figure" referred to in Proposition 31 that is usually encountered in a high school geometry class is a square, but according to this proposition, any similar shapes can be used. Consider semicircles I, II, and III drawn on the sides of right triangle ABC as shown in Figure 4.

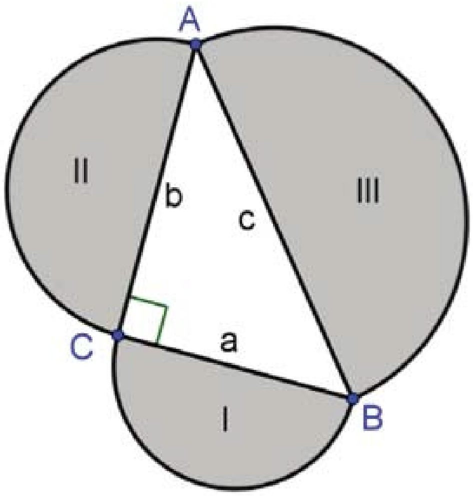

Fig 4 Euclid's Proposition 31 with semi-circles

The radii of the semicircles are respectively $\frac{a}{2}$, $\frac{b}{2}$, and $\frac{c}{2}$. Their respective areas are $\frac{a^2\pi}{4}$, $\frac{b^2\pi}{4}$, and $\frac{c^2\pi}{4}$. Summing the areas of I and II, we have $\frac{a^2\pi}{4}+\frac{b^2\pi}{4}$. Factoring, we have $\frac{\pi}{4}(a^2+b^2)$.

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Applying the Proposition 31, our result is $\frac{\pi}{4}c^2$, which is the area of III. Of course, any similar figures will do. Consider the regular septagons constructed on the sides of the right triangle shown in Figure 5.

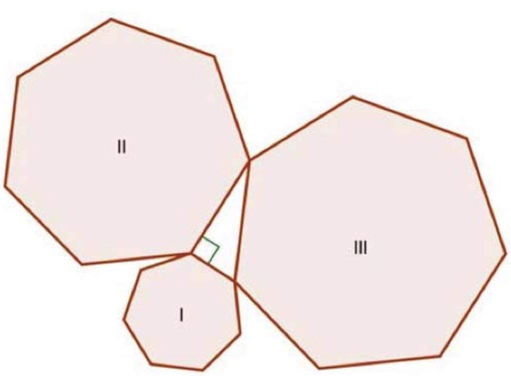

Fig 5 Proposition 31 with regular septagons

Because the polygons are similar, their areas are proportional to the square of the length of one of their sides, hence, I=Ka^2, II=Kb^2, and III=Kc^2 for some constant value K (which happens to be $\frac{7}{4}\cot\left(\frac{180^\circ}{7}\right) \approx 3.6339$). Summing areas I and II we have Ka^2+Kb^2. As before, this leads to $K(a^2+b^2)$ which by Proposition 31 leads to Kc^2, the area of III.

Is it possible that the Pythagorean Theorem, or that its generalization found in Proposition 31, could hold for triangles other than a right triangle? The following theorem, with its unmistakable Pythagorean flavor, certainly seems to suggest that this might be the case. Whereas the Pythagorean Theorem concerns the sums of areas of squares constructed on the sides of a triangle with a 90º angle, this theorem concerns the sums and differences of areas of equilateral triangles constructed on the sides of a triangle with a 60º angle.

Begin with any triangle ABC containing a 60º angle, and construct equilateral triangles on the sides as shown in Figure 6. The sum of areas of the equilateral triangles on the sides containing the 60º angle, diminished by the area of the equilateral triangle constructed on the side opposite the 60º angle, is always equal to the area of the given triangle. Referring to Figure 6, I + II – III = IV (Gutierrez, 2009).

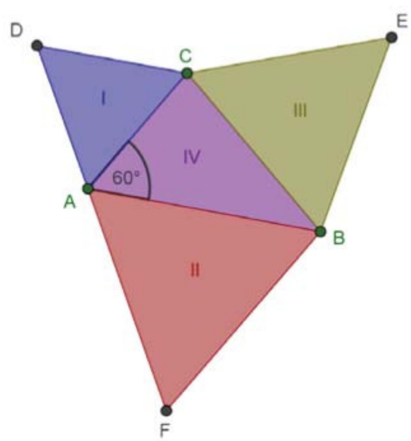

Fig 6 Equilateral triangles on sides of arbitrary traingle ABC

Consider this rewording of the Pythagorean Theorem: *Begin with any triangle ABC containing a 90º angle, and construct squares on the sides. The sum of areas of the squares on the sides containing the 90º angle, diminished by the area of the square constructed on the side opposite the 90º angle, is always zero.*

When stated in this manner, both the Pythagorean Theorem and this Pythagorean-Like Theorem are making geometric statements about areas and about invariants. One Theorem involves right angles and squares; the other involves 60º angles and equilateral triangles. It appears this theorem has more in common with the Pythagorean Theorem than one would initially suspect.

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Some Special Cases

Before diving into a proof of this theorem, it may be useful to establish the validity of the theorem for some special cases. The two cases we will consider are: (1) ABC is an equilateral triangle, and (2) ABC is a right triangle.

Case 1. When ABC is equilateral (see Fig 7), the four equilateral triangles are congruent to each other and all the areas are equal. Therefore, I + II – III = IV.

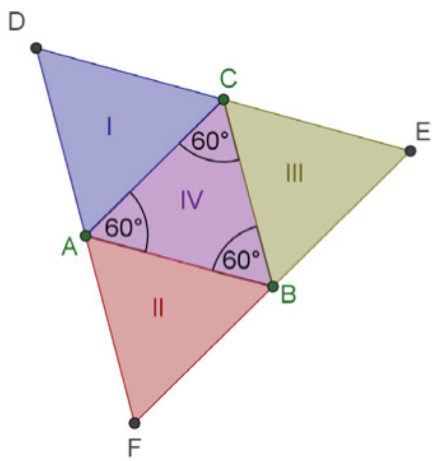

Fig 7 Equilateral triangles on sides of equilateral triangle ABC

Case 2. Without loss of generality, let angle C be 90º. This is shown in Figure 8. Recalling how the sides of a 30-60-90 triangle are in the proportion $1{:}\sqrt{3}{:}2$, the areas of the equilateral triangles must be in the proportion 1:3:4. In other words, II is four times as large as I, and III is three times as large as I. Therefore, I + II – III = I + 4I – 3I = 2I. Notice by Case 1 and by Case 2, II is twice as large as IV, which makes IV twice as large as I, thus IV = 2I. Therefore, I + II – III = IV.

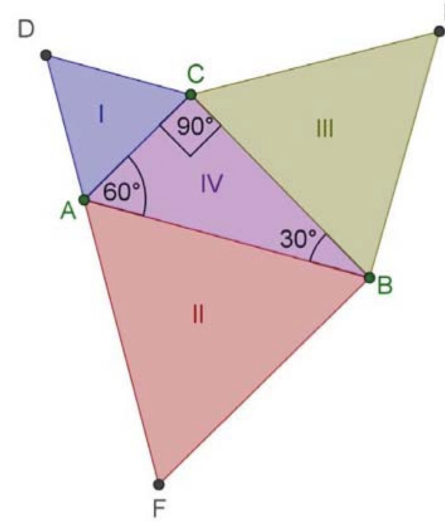

Fig 8 Equilateral triangles on sides of special right triangle ABC

Why Does This Theorem Work?

The proof of this theorem involves the Law of Cosines, which is a generalization of the Pythagorean Theorem. Recall the Law of Cosines to be $c^2 = a^2 + b^2 - 2ab\cos(C)$, where C is the included angle of sides a and b. We will also need to recall that the area of an equilateral triangle is given by the formula $\frac{s^2\sqrt{3}}{4}$ and that the area of any triangle given two sides a and b and the included angle C is $\frac{1}{2}ab\sin(C)$. Referring to Figure 9, we will begin by summing areas I and II, the subtracting off area III, which is I+II–III=$\frac{b^2\sqrt{3}}{4}+\frac{c^2\sqrt{3}}{4}-\frac{a^2\sqrt{3}}{4}$.

Applying the Law of Cosines, we can express c^2 in terms of a and b and the 60º angle. This leads to I + II – III = $\frac{b^2\sqrt{3}}{4}$ +

$\frac{c^2\sqrt{3}}{4} - \frac{(b^2+c^2-2bc\cos(60^\circ))\sqrt{3}}{4}$.
Applying the distributive property, collecting like terms, and evaluating the cos(60°), we have I + II – III = $\frac{bc\sqrt{3}}{4}$. The area of IV is given by $\frac{1}{2}bc\sin(60^\circ)$, which simplifies to $\frac{bc\sqrt{3}}{4}$, as well. Hence, we have shown that I + II – III = IV.

Conclusion

The Pythagorean Theorem may be the most widely known theorem in all of mathematics (Maor, 2007). However, the meaning of the Theorem can get lost in the memorization of $a^2 + b^2 = c^2$ without context. The Pythagorean Theorem is a theorem about areas related to right triangles. When viewed as a geometric statement about areas, there are new possibilities for students to explore. This Pythagorean-Like Theorem is just one example. Is there any other way to prove this theorem? What if the given triangle had a 120° angle instead of a 60° angle? What if the figures constructed on the sides of the triangle were squares or other regular polygons? What should the 60° angle be changed to in order for I + II – III = 0? All of the proofs presented in this article, as well as proofs for the questions above, are well within the reach of high school students in Geometry or Algebra 2. The algebraic proofs involve little more than careful application of the distributive property, factoring out a greatest common monomial, and the Law of Cosines. Any student can follow the reasoning, and perhaps feel as the Scarecrow feels: "Oh, joy, oh, rapture. I've got a brain!" Ω

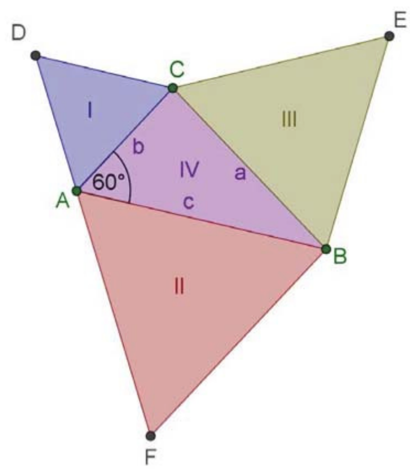

Fig 9 Equilateral triangles on sides of triangle ABC with measure angle A = 60°.

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