Discovery: A Squaring Pattern for Two-Digit Numbers and Beyond

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Delving into and expanding the thinking of students is an important challenge for mathematics teachers. This article describes the thinking of one student, Sam, about perfect square numbers. His thinking and the probing of his teacher helped Sam develop his own pattern for mentally calculating perfect squares. Sam writes his pattern as a formula involving digits of the original number to be squared. The article describes his thinking process and the algebra used to verify his ideas.

Introduction

In a high school geometry class discussion related to the Pythagorean Theorem, students were asked to look for patterns to help them remember common perfect square numbers. In a lively discussion students agreed they could learn perfect squares up to 25^2 (See Figure 1) and generated many helpful patterns.

The class discussed that square numbers appear frequently in algebra and geometry classes and would be very useful to know. They also reluctantly agreed that, while they could simply square numbers on their calculator, practicing certain perfect squares until they were committed to memory would speed their calculations. The class agreed that finding patterns among the square numbers would ultimately help them to learn because they would understand the relationship between these numbers.

A Standards-based Task

Frequently standards documents suggest that teachers provide opportunities for students to learn and use perfect squares. The current Ohio Academic Content Standards for Mathematics Benchmark H for grades eight through ten in Number, Number Sense, and Operations state that students should, "Find the square root of perfect squares" (p. 162). The new Common Core State Standards for Mathematics include in grade 8 under the domain of expressions and equations,

n	n^2	n	n^2	n	n^2
1	1	11	121	21	441
2	4	12	144	22	484
3	9	13	169	23	529
4	16	14	196	24	576
5	25	15	225	25	625
6	36	16	256		
7	49	17	289		
8	64	18	324		
9	81	19	361		
10	100	20	400		

Fig 1 Whole numbers, n, ($1 \leq n \leq 25$) and corresponding perfect squares

standard 2 that students should "Evaluate square roots of small perfect squares" (CC, 54). Additionally in the Common Core Standards, number eight of the Standards for Mathematical Practice, it states that students should, "Look for and express regularity in repeated reasoning" (CC, 8).

Patterns in Squares

Some patterns that a current group of high school geometry students recognized are as follows:

- $12^2 = 144$ and $21^2 = 441$ (Reversal of all digits);
- $13^2 = 169$ and $14^2 = 196$ (Reversal of the last two digits);
- If n has a 5 in the units digit, n^2 ends in 25;
- If n has a 0 in the units digit,n^2 ends in 00;
- When squaring two different whole numbers with the same units digit, the squared number will also have the same units digit (e.g., 2^2, 12^2, 22^2 all have a units digit of 4);
- There is a pattern of adding increasing odd numbers as illustrated below. Specifically, $n^2 = (n-1)^2 + n + (n-1)$, e.g., $7^2 = 62 + 7 + 6$;

n^2	1	4	9	16	25	36	49	64	81
$n^2 - (n-1)^2$	--	3	5	7	9	11	13	15	17

- The square of a number can be found by multiplying the number by the previous number and adding the original number. (e.g., $21^2 = 21(20) + 21$). Specifically $n^2 = n(n-1) + n$. One student, Sam, went beyond the typical classroom discussion. He accepted the quest for perfect square patterns as a challenge and below is what he concluded.

The class agreed that finding patterns among the square numbers would ultimately help them to learn because they would understand the relationship between these numbers.

Sam's Perfect Squares Formula

$(10x + y) = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$

Sam's Thinking (In His Own Words)

Let $10x + y$ represent a 2-digit number where x represents the tens digit and y represents the ones digit.

One-digit number example: If 7 is $10x + y$ then $x = 0$ and $y = 7$.

$7^2 = 49$

$(10x + y)^2 = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$

$(10(0) + 7)^2 = (10(0) + 7)(10(0) + 10) + (10(0)(7) - 100(0) - 10(7)) + 7^2$

$7^2 = 7(10) + (0 - 0 - 70) + 49$

$7^2 = 49$

Two-digit number example: If 19 is $10x + y$ then $x = 1$ and $y = 9$.

$19^2 = 361$

$(10x + y)^2 = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$

$(10 + 9)^2 = (10(1) + 9)(10(1) + 10) + (10(1)(9) - 100 - 9(10)) + 9^2$

$19^2 = 19(20) + (90 - 100 - 90) + 81$

$19^2 = 380 - 100 + 81$

$19^2 = 361$

For numbers with 3 or more digits, let x represent the tens place and everything to the left. Let y represent the ones digit. Use the same formula as for two-digit numbers.

<u>Three-digit number example:</u> If 783 is $10x + y$ then $x = 78$ and $y = 3$.
$783^2 = 613,089$
$(10x + y)^2 = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$
$(10(78) + 3)^2 = (10(78) + 3)(10(78) + 10) + (10(78)(3) - 100(78) - 10(3)) + 3^2$
$(780 + 3)^2 = (780 + 3)(780 + 10) + (2340 - 7800 - 30) + 9$
$783^2 = 613,089$

<u>Four-digit number example:</u> If 1637 is $10x + y$ then $x = 163$ and $y = 7$.
$1673^2 = 2,679,769$
$(10x + y)^2 = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$
$(10(163)+7)^2=(10(163)+7)(10(163)+10)+(10(163)(7)-100(163)-10(7))+7^2$
$(1630 + 7)^2 = (1637)(1640) + (11410 - 16300 - 70) + 49$
$1673^2 = 2,679,769$

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The way that I came up with my formula was a very step-by-step process. And it all started one day in math class last Tuesday, when I was looking over some things to get ready for the perfect square quiz the next day. I was messing around with different number patterns and that's when I realized that for the squares of the numbers 11-19, you could multiply the original number by 20, then subtract 100, then add the units digit squared, and you would get the whole number squared. $13^2 = 13 \times 20 - 100 + 3^2$ and $14^2 = 14 \times 20 - 100 - 100 + 4^2$. Making the formula for that, $(10x + y)^2 = (10x + y) \times 20 - 100 + y^2$, and that is when my teacher challenged me to find the formula for the 20's, and 30's, to see if they were related and if you could turn that into one formula.

Once I got to the 20s it was a little bit more difficult. The only thing that stayed pretty steady in the formula for the 20s was that you multiplied $10x + y$ by 30, and you always added y^2. But between the 10s and 20s, you subtracted 90 more. So instead of subtracting 100 like you did in the 10s, you subtracted 190 on 21^2. But between each number in the 20s, when you would go up one on the exponents, for example, 22^2 would be next, you subtracted 10 less. So, the formula for 21^2 is: $(10x + y)^2 = (10x + y) \times 30 - 190 + y^2$. But for 22^2, the formula is: $(10x + y)^2 = (10x + y) \times 30 - 180 + y^2$, because you're subtracting 10 less than you did before. And following that pattern, the formula for 23^2 is: $(10x + y)^2 = (10x + y) \times 30 - 170 + y^2$. And that trend follows through all the 20s.

In the 30s, it gets a little bit more complicated. First you subtract by 90 more on 31^2 than you did on 21^2, and you are multiplying by 10 more. So the formula for 31^2 is: $(10x + y)^2 = (10x + y) \times 40 - 280 + y^2 = (10x + y)^2$ (Editor's note: confirm by setting $x = 3$ and $y = 1$). Another thing that changed between the 20s and the 30s is instead of the intervals of subtracting being 10, they're now 20. So the formula for 32^2 is: $(10x + y)^2 = (10x + y) \times 40 - 260 + y^2$. And then by following that same pattern, the formula for 33^2 is: $(10x + y)^2 = (10x + y) \times 40 - 240 + y^2$, and so on through the 30s.

That same pattern is followed from then on, for 41^2 you would subtract by 370 and have intervals of 30. And for 512 you would subtract by 460 and have intervals of 40 all the way through the 50s. But that is when my teacher suggested, that what we had, was one formula for the 10s, but beyond that, all we had was a pattern. So we had to find out the formula. All that we knew was $(10x + y)^2 = (10x + y) \times (10x + 10) + ? + y^2$. So we algebraically found out that the space we were unsure of was $10xy - 100x - 10y$ like this.

$$(10x + y)^2 = (10x + y) \times (10x + 10) + ? + y^2$$
$$100x^2 + 20xy + y^2 = 100x^2 + 100x + 10xy + 10y + ? + y^2$$
$$20xy = 100x + 10xy + 10y + ?$$
$$10xy = 100x - 10y = ?$$

So putting all of these observations together we obtained the following formula: $(10x + y)^2 = (10x + y) \times (10x + 10) + (10xy - 100x - 10y) + y^2$.

The beauty of Sam's formula is... its originality. His work is one example of the ability of a student to think deeply about a mathematical concept

Analysis of Sam's Method

Algebraically you can see that Sam's formula is equivalent to $(10x + y)^2$.

$$(10x + y)^2 = (10x + y)(10x + 10) + (10xy - 100x - 10y) + y^2$$
$$= 100x^2 + 100x + 10xy + 10y + 10xy - 100x - 10y + y^2$$
$$= 100x^2 + 20xy + y^2$$

And $(10x + y)(10x + y) = 100x^2 + 20xy + y^2$

The beauty of Sam's formula is certainly not its complexity but its originality. His work is one example of the ability of a student to think deeply about a mathematical concept in perhaps a way different from his peers or even his teacher. He admits that he would probably not use this formula to calculate squares of 3- and 4- digit numbers mentally, but rather he developed the formula as an extension of his thinking about his original 2-digit formula that is easy to use mentally. I applaud students for their original mathematical ideas that are a reflection of their ability to think about mathematics and an extension to our classroom lesson. Sam's formula serves as both as a tool for calculating squares mentally and an inspiration to the rest of the class.

Works Cited

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SAM BIEDERMAN is a high school student in Diane Kahle's geometry class. His interests are playing drums, wrestling, camping, and computers. His favorite subjects are mathematics, science and, woodshop. He is currently working on his Eagle Scout Project and is planning to go on a 10-day backpacking trip this coming summer at Philmont Scout Ranch in New Mexico.

The Self-Organizing Mind

"One of the ingenious properties of this associative architecture is that it is self- organizing: information is categorized, grouped, and stored in a way that reflects the world in which we live."

Buonomano, D. (2011). *Brain bugs: How the brain's flaws shape our lives, 33.* W. W. Norton & Company, NY.