Developing Children's Proportional Reasoning: Instructional Strategies That Go the Distance

Katie Larson, Lake Middle School - Millbury, Ohio

Abstract: *Proportional reasoning is one of four key areas of instruction for sixth and seventh grade mathematics as established by The Common Core State Standards for School Mathematics (CCSSM, 2010). This article describes the importance of proportional reasoning and provides educators with strategies that can be implemented in the classroom to help students develop their proportional reasoning.*

Keywords. *proportional reasoning, elementary mathematics*

1 Introduction

One of the most important mathematical competencies for middle grades students to develop is the ability to reason proportionally. Proportional reasoning is difficult to develop because it requires students to make significant shifts in thinking. Students must transition from using mostly additive strategies, which are emphasized throughout elementary school, to strategies that are multiplicative. Multiplicative relationships between two ratios allow students to understand that a recipe for punch that calls for four cups of juice and ten cups of water has the same concentration of juice as a different recipe calling for six cups of juice and fifteen cups of water, because both recipes have a two to five relationship. If a photograph was originally 6″ x 8″ but enlarged so that the width changed from 8″ to 12″, a student using an additive strategy might incorrectly add 4″ to each dimension and determine that the new height is ten inches. A student that correctly utilizes a multiplicative strategy would respond that the new height is nine inches because the edges of the photograph are enlarged to 1.5 times its original size. Helping students approach problems proportionally and use multiplicative strategies is a challenge.

There are two contributing factors that explain why students prefer additive strategies over those that are multiplicative (Resnick & Singer, 1993). First, multiplicative relations develop much slower than additive ones. Secondly, students' knowledge about additive strategies is much more developed. As a result, students who become frustrated when trying to solve proportion problems often incorrectly resort to additive strategies. Throughout the elementary years, questions are phrased, "how many more/fewer or how much all together?" In the middle grades the questions change to, "how many times bigger or what part of?" Not only is the phrasing of the problems different, but the new types of questions require more abstract thinking.

Proportional reasoning is challenging for students to learn, but helping students purposefully use ratios and proportions is imperative for middle grades students. Proportional thinking is the basis for a wide variety of mathematical concepts. Misconceptions about proportional thinking need to be addressed early on, as students can experience difficulty from their earlier lack of understanding that extends into later years (Fujimura, 2001). Proportional reasoning can be applied to many areas of mathematics including similar figures, probability, slope, scaling, and percents. Beyond the middle grades, being able to recognize multiplicative comparisons, similarity, and relationships amongst quantities is central to the idea of algebra and other upper level mathematics courses. Equally important is the role of proportionality in solving real-life problems such as adapting recipes, comparing costs, mixing oil and gas using specified ratios, and evaluating sports statistics.

Because of the importance of proportions, the Common Core State Standards for Mathematics (CCSSM, 2010) has established expectations regarding rational numbers, ratios, and proportions throughout the middle grades. The CCSSM has made proportional relationships a critical focus of the sixth and seventh grade mathematics curriculum, naming it as one of the four key areas to which instructional time should be devoted. More specifically, students need to compute with unit rates, recognize situations where two quantities are proportional, represent proportional relationships, and solve multi-step ratio and percent problems.

2 Characteristics of a Proportional Thinker

Although there are varying levels of proportional reasoning, a student that is thinking proportionally at a formal level should be able to exhibit the following characteristics (Langrall, 2000).

2.1 Recognize the difference between additive and multiplicative change

Consider the following task from Lo and Watanabe (1997), "Yesterday I bought 8 candies with 12 quarters. Today, if I go to the same store with 9 quarters, how many candies can I buy?" A student that inappropriately views the relationship between candies and quarters as additive would say the answer is 5 because 8 is four less than 12, and 5 is four less than 9. A student that correctly interprets this as a multiplicative scenario acknowledges the answer is 6 candies because 8 candies to 12 quarters and 6 candies to 9 quarters are both 2 to 3 relationships.

2.2 Recognize situations where a ratio is reasonable and appropriate

A proportional reasoner can distinguish whether or not it makes any sense to utilize ratios in order to solve a problem (Langrall, 2000). It does not make sense to say that if one boy has two dogs, then three boys have six dogs. The number of dogs is not associated with the number of boys. Conversely, if two roses cost ten dollars, then four roses cost twenty dollars. In this situation, comparing ratios does make sense. Even though the quantities being compared have changed, together the relationship between them remains constant.

2.3 Ability to unitize a situation

Unitizing is the ability to use a reference unit and then reinterpret a situation based upon that unit (Lamon, 2005). The flexibility of unitizing 24 popsicles means there could be one 24 pack, two 12 packs, four 6 packs, and 24 individual popsicles. Since unitizing is a flexible strategy, it allows students to unitize differently depending on what makes sense in a particular problem and leads to multiple solution strategies which may evoke different levels of sophistication in proportional thinking.

2.4 Ability to analyze situations relatively rather than in absolute terms

Consider two different recipes for punch. One recipe calls for four cups of juice and ten cups of water. The other recipe calls for six cups of juice and fifteen cups of water. A student viewing these recipes in absolute terms will likely say that the recipe that calls for six cups of juice is the more concentrated recipe because it has more juice, without consideration for the quantity of water. Whereas, a student viewing the juice problem using a relative perspective will note that both juice recipes have the same concentration because the ratio of juice to water in both recipes is equivalent. The ability to compare the relationship between two quantities, rather than viewing each as independent of the other, is an important step toward proportional thinking (Lamon, 2005).

3 Developing Proportional Reasoning: Five Instructional Strategies

Helping students develop the ability to reason proportionally requires diligence and patience. It is important to note that students do not develop proportional reasoning in one or two lessons, a chapter, unit, or even a year. It is a process that begins in late elementary school and continues into early high school. As a result, instruction should nurture and provide numerous opportunities for this type of thinking slowly and overtime. Expecting students to develop a solid understanding of proportionality too quickly is counterproductive. There are several research-based instructional strategies that can help students develop proportional reasoning.

3.1 Use a variety of proportion-type problems and sequence accordingly

Langrall (2000) notes that there are four different proportion problem types - Rate, Part-part-whole, Associated Sets, and Growth. Rate problems involve well-known measurements such as speed or cost per item. Part-part-whole problems involve a subset of a whole as it is compared to its complement, such as boys with girls or the number of boys as compared to the number of students in the whole class. Associated set problems pertain to quantities specific to a given situation such as pencils and students or people and candy. Growth problems, otherwise known as "stretcher" and "shrinker" problems, require scaling up or scaling down and involve a relationship between two linear quantities such as height, length, or width. Scaling up and scaling down also cause changes in the area of plane figures, or volume of solid figures when the dimensions are changed.

Instruction should include a balance of all four semantic types. Textbooks are often weighted heavily in favor of only one or two problem types instead of a balance of all four; consequently, supplemental problems may need to be used. Early in their development of proportional reasoning, students perform best on associated set problems because they can use pre-existing knowledge of patterns, counting, and matching techniques (Lamon, 1993). Students usually find growth problems the most difficult because of a tendency to apply additive strategies rather than multiplicative ones. As a result, sequencing instruction should begin with associated sets or part-part-whole type of questions before moving on to growth problems.

3.2 Choose tasks that have multiple solution strategies and a variety of contexts

Using a variety of contexts gives students exposure to the variety of situations and the types of scenarios that apply to multiplicative relationships (Kent, Arnosky, & McMonagle, 2002). While students may not be able to recognize a multiplicative relationship in one situation, they might be able to in a different situation. Using a variety of tasks also elicits different solution strategies. Students can respond to the questions in different ways depending on their level of understanding.

3.3 Build upon students' intuitive knowledge

Through experience in daily life and exposure to ideas through their school work, students possess prior knowledge that can be used while solving proportion problems. Learning that allows students to use their prior knowledge and intuition is important because it allows for the development of personal sense-making strategies. Students need time to develop strategies on their own. High-level cognitive tasks require time for students to grapple and explore their ideas. When teachers jump in too early and begin providing assistance by offering shortcuts or procedures, the complexity of the task is greatly reduced (Stein & Smith, 1998). Students have remarkable metacognitive abilities to monitor and judge the reliability of their thinking without direct instruction; therefore, instruction should be designed to take advantage of students' invented strategies (Lamon, 1993). When rules and procedures are not learned with connections and meaning, students forget or do not understand when or why to use them (Lamon, 2001).

3.4 Utilize multiple representations to develop fluency in proportional reasoning

Manipulatives, pictures, and diagrams are important tools that help represent proportional situations. The availability of manipulatives, especially early on, helps with sense-making and encourages informal problem solving strategies (Lo & Watanabe, 1997). Cubes assist with regrouping quantities or in unitizing, especially in associated-sets and part-part-whole problems. Ratio tables are a record-keeping tool that help display the building up or scaling down of quantities in proportional situations. Building ratio tables provide students with opportunities to discuss and present construction strategies. Early on, problems should be situational and solved using objects and pictures. Gradually instruction can build up to more complex problems and methods of solving (Langrall, 2000).

3.5 Informal strategies before cross-multiplication procedures

Many traditional mathematics curricula focus on a limited number of proportion-type problems and use the cross-products procedure for solving proportions, without ever helping students develop a reason for why the strategy works. Instruction should not promote specific strategies (Lamon, 1993). Students who are encouraged to use their own strategy rather than a single algorithm are generally more successful in developing proportional reasoning (Ben-Chaim et al., 1998). As a result, symbolic algebra and the cross-products method should only be introduced after students have had an opportunity to develop their informal strategies.

4 Conclusions

It is essential that students develop strong proportional reasoning skills in their middle school years. It is a process that cannot be learned in one unit or chapter, rather it develops slowly through the use of many different problem types and contexts. Some students struggle to become full-fledged proportional thinkers because of the difficulty transitioning to multiplicative thinking. The difficulty that students have with proportional reasoning, if not improved, can be a hindrance to their success in future math courses in high school and beyond (Che, 2009).

Proportional reasoning instruction should build from students' intuitive knowledge. Before any type of algorithm or symbolic representations is presented, students need to experience a myriad of problem types and be provided with time to solve and make sense of problems using their own informal strategies. Students have many real world experiences and strategies at their disposal by the time they reach the middle grades. Additionally, they develop strategies independent of instruction and keep these strategies as part of their informal knowledge system. Students can use this knowledge to solve problems without the need for direct instruction from a teacher (Ben-Chaim et al., 1998). Teachers should allow students time to construct meaning and use informal techniques before introducing conventional symbolism used with proportions. By using these ideas, students will become proportional thinkers who are able to solve real world problems.

References

Ben-Chaim, D., Fey, J.T., Fitzgerald, W.M., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics*, *36*, 247-273.

Che, S.M. (2009). Giant pencils: developing proportional reasoning. *Mathematics Teaching in the Middle School*, *14*, 404-408.

Fujimura, N. (2001). Facilitating children's proportional reasoning: a model of reasoning processes and effects of intervention on strategy change. *Journal of Educational Psychology, 93*, 589-603.

Kent, L. B., Arnosky, J., & McMonagle, J. (2002). Using representational contexts to support multiplicative reasoning. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook*. Reston, VA: National Council of Teachers of Mathematics, 145-152.

Lamon, S.J. (1993). Ratio and proportion: Children's cognitive and metacognitive processes. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research*. Hillsdale, NJ: Lawrence Erlbaum Associates, 131-156.

Lamon, S.J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F.R. Curcio (Eds.), *The roles of representation in school mathematics: 2001 yearbook*. Reston, VA: National Council of Teachers of Mathematics, 146-165.

Lamon, S.J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. New York, NY: Routledge.

Langrall, C. (2000). Three balloons for two dollars: Developing proportional reasoning. *Mathematics Teaching in the Middle School, 6*(**4**), 254-261.

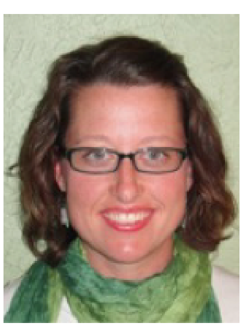
Lo, J.J., & Watanabe T. (1997). Developing ratio and proportion schemes: a story of a fifth grader. *Journal for Research in Mathematics Education, 28*, 216-236.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

CCSSM (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from Common Core State Standards website: `http://www.corestandards.org/assets/CCSSI_Math\%20Standards.pdf`.

Resnick, L.B., & Singer, J.A. (1993). Protoquantitative origins of ratio reasoning. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research*. Hillsdale, NJ: Lawrence Erlbaum Associates, 107-130.

Stein, M.K., & Smith, M.S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*(**4**), 268-275.



Katie Larson, `klarson@lakeschools.org`, is a seventh grade math teacher at Lake Middle School in Millbury, Ohio. She earned a Bachelor's degree from the University of Missouri and a Master's degree in Middle Childhood Education from the University of Toledo.