Making a Mathematical Symphony: Emphasis on Relational Thinking and Connections

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Abstract

Relational thinking is a necessary and vital component for true conceptual understanding of mathematical thinking and application. Teachers and administrators who realize and nurture this pedagogical component through the study of vertical knowledge, collaboration, and ongoing professional development are solidifying a strong foundational mathematical journey for their students.

Imagine listening to an orchestra playing a symphony concert where each section plays independently ignoring the conductor's guidance. Sounds are achieved but the entire culminating piece is never fully realized. Rather, the performance stands as a discordant combination of unrelated notes and beats. In contrast, when each section of an orchestra relates their sounds with other sections via the conductor's guidance, they become interconnected, interdependent and intertwined resulting in rich and beautifully coordinated music. The same can be said for mathematics instruction. Areas of mathematics taught in isolation, rather than in an interconnected manner, inhibit the cognitive connections for building a stronger conceptual understanding.

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In the Connections Process Standard (Grades K-2), the National Council of Teachers of Mathematics (NCTM, 2000) describes such relational thinking as connections "between one mathematical concept and another, between different mathematics topics, between mathematics and other fields of knowledge, and between mathematics and everyday life. … [which] are supported by the link between the students' informal experiences and more-formal mathematics." Similarly, the Standards for Mathematical Practice (SMPs) embedded within the Common Core State Standards for Mathematics also address the importance of mathematical connections (CCSSI, 2010). For example, SMP 7: *Look for and make use of structure*, describes how students may leverage earlier observations about mathematical structures to explore new areas of the discipline. Indeed, fostering a relational vision of mathematics is an essential aspect of helping students develop a deep understanding of the discipline. Although this claim applies across all mathematical domains, the development of a strong foundation regarding numbers and arithmetic operations, or numeracy, is of particular interest. As a mathematics intervention specialist, I worked with children who struggled with whole-number and arithmetic concepts (or numeracy), and I witnessed, first-hand, how such deficiencies could reverberate throughout an individual's mathematical career. Thus, I find it useful to focus these remarks and examples on aspects of teaching practice related to numeracy.

Relational Thinking: Examples from Teaching Practice

I recall, on one particular day, Shawna, a third-grade student, entered the math lab and very happily told me, "We learned multiplication today! Three times ten equals thirty!" When I asked her what this statement meant, she paused for a moment and replied, "I don't know." It occurred to me that Shawna had likely constructed this information as an isolated fact rather than relating to prior experiences with skip counting or constructing equal groups of items. Perhaps those experiences (skip counting, making equal groups) were constructed in isolation as well, and her present mathematical landscape consisted of many small islands dotting a vast ocean. After Shawna's response, I remember placing three bowls on the table and asking her to place 10 items in each bowl. I can still recall Shawna's face and her verbal response, "Oh!" when it became apparent what the formerly memorized fact truly meant! Likewise, I might have enacted other appropriate instructional decisions at that moment. For example, I might have attempted to draw a connection to the verbal aspect of number (Thomas, Tabor, & Wright, 2010) – specifically, her forward number word sequence (e.g., "If you were going to count to find the answer to 3 × 10, what would that count sound like?"). Alternately, I could have presented Shawna with a full 10-frame and asked her to imagine how many dots there would be if I presented two additional frames. With each of these scenarios, though, my aim would be to help the student develop a relationship between new and existing mathematical knowledge.

Relational Thinking in a Whole-Class Setting

Another opportunity to develop relational thinking occurred in a first-grade classroom. I noticed that the teacher was desperately attempting to help her students understand the concept of "doubles plus one" as a means for structuring certain quantities (e.g., 13 = 6 + 6 + 1). I heard her pleading with her students, "Thirteen, it's six plus seven! It's doubles plus one more!" Was the strategy of doubles plus one being characterized as accessing a collection of facts with no relational thinking? Were students connecting this experience with some knowledge quantity? If the aim is to foster relational thinking, the teacher, at this point, might have opted to connect this experience to a quantitative model. Here, mathematical tools such as linking cubes (see Figure 1) could be used to model the extent to which a particular quantity (e.g., 13) may be considered as doubles plus one.

Figure 1*:* Linking Cubes Displaying
13 as Doubles (6+6) plus One

Additionally, this tool could be flashed (briefly presented and then concealed) in different "doubles plus one" configurations to help develop children's quantitative mental imagery (Thomas & Tabor, 2012). Lastly, such experiences with linking cubes (or other mathematical tools) would provide the occasion for a natural transition into five-plus and ten-plus quantitative structures (see Figure 2). Note, the color structure of the arithmetic rack allows for useful connections between doubles and structuring quantities via five and ten (e.g., 5 + 5 + 3 = 13) (Wright, Martland, & Stafford, 2006).

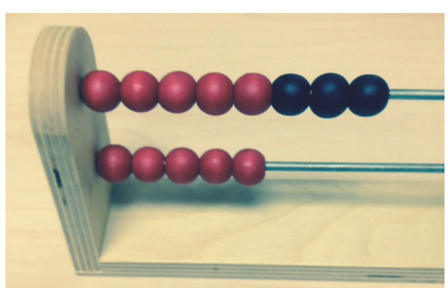

Figure 2. Arithmetic Rack Displaying
13 as Ten (5+5) plus Three

When relational thinking across aspects of number or arithmetic strategies is not engendered, children's mathematical knowledge exists as memorized facts with no ability to reason and connect to future concepts likely resulting in more superficial forms of mathematical fluency (Thomas, 2012).

Elaboration on Relational Thinking

When considering the development of relational thinking, it is useful to examine the manner in which school experiences may be structured. With respect to curriculum knowledge, "[t]here are two additional aspects, *lateral* and *vertical curriculum knowledge*. *Lateral knowledge* includes knowledge of what is being studied in the students' other subjects; this gives the teacher the ability to relate mathematics to student knowledge developed in other subjects." *Vertical knowledge* denotes the "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 2004, p. 204).

One example of vertical curriculum knowledge is the deliberate connection of students' experiences with skip-counting verbal sequences and sorting physical items into equal groups. Just as musicians make the shift from single/multiple note blending to working on stanzas, teachers must work to engender a cognitive shift away from seeing individual unit items and towards seeing composite units within composite units (e.g., groups of groups) to gain true multiplicative understanding (Wright, Ellemor-Collins, & Tabor, 2012). Arguably, this cognitive shift is most likely to occur within an interconnected, relational mathematical landscape. In this instance, the act of skip-counting is not abandoned by the child, but rather it takes on a new and deeper meaning as it becomes connected to quantitative experiences involving equal sharing and/or the repeated construction of equal groups.

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Given the importance of vertical connections within mathematics, no longer is it enough for teachers to focus solely on the standards for their particular grade level. Rather, teachers must be cognizant of students' entire mathematical journey (past, present, and future) in order to assure the development of conceptual understanding. Necessarily, this means that teachers must become willing to explore content beyond the purview of their own grade level and consider mathematical growth from the perspective of learning trajectories. One example of such a perspective is the Common Core State Standards for Mathematics *hexagon trajectories map* (www.turnonccmath.net) (Confrey, et al., 2012). Indeed, mathematical concepts must be revisited, experienced and incorporated into new concepts to create a robust landscape of understanding. On this point, the Connections Standard (P-2) in NCTM Principles & Standards for School Mathematics (2000) states: "It is the responsibility of the teacher to help students see and experience the interrelation of mathematical topics, the relationships between mathematics and other subjects, and the way that mathematics is embedded in the students' world." Here, it is important to note the tremendously significant role that teachers play with respect to developing a relational understanding of mathematics.

Recommendations to Foster Relational Thinking

There are several over-arching principles that teachers may draw from the ideas above;

(1) **Guiding students by revisiting and connecting prior concepts to present understanding is important in several ways.** Through conversation teachers are able to listen to and observe the thinking of their students and also discover and correct any possible misconceptions. With this dialogue

comes a great opportunity for students to share their thinking, explain their reasoning and draw upon their experiences to continue to make sense of and bridge mathematical domains. Returning to the Connections Standard (P-2):

> [Teachers] should plan tasks in new contexts that revisit topics previously taught, enabling students to forge new links between previously learned mathematical concepts and procedures and new applications, always with an eye on their mathematics goals. When teachers help students make explicit connections - mathematics to other mathematics and mathematics to other content areas - they are helping students learn to think mathematically.

As teachers, we may use visual supports to foster students' relational thinking. When Shawna made the connection of her original notion of $3 \times 10 = 30$ with skip counting and attending to groups, she was able to make connections to previously taught concepts. This experience provided her the support needed to connect her existing conceptual understanding to new mathematical ideas. By celebrating and valuing her response, I encouraged her to apply this support to additional multiplication problems.

(2) **Administrative support is also important.** Such support provides ongoing, thought-provoking and reflective professional development focused on how mathematical knowledge develops among children over time (e.g., learning trajectories). It is important to note the deep complexity of such development and teachers must be given considerable time and opportunity to construct their own knowledge and skills. Opportunities for interpreting children's mathematics, planning, and reflection must be a sustained experience for teachers rather than an isolated event.

One example of statewide, sustained professional development organized around relational thinking is the *Kentucky Numeracy Project* (KCM, 2013). The centerpiece of this initiative is sustained, collaborative professional learning experiences centered on refinement of teaching practices with respect to students' conceptions of number and operations. While large-scale programs may be quite powerful, in absence of such initiatives, teachers may organize professional learning communities in more local contexts (e.g., school level) to productive ends.

Whatever the format, though, such professional development aimed at relational thinking should emphasize the following ideas: First, these experiences should focus on the manner in which mathematical understanding develops among children over time. Here, any number of learning trajectories may be leveraged to help teachers better understand mathematical growth in longitudinal terms; Second, these experiences should examine connections between different mathematical subdomains. For example, teachers could delve into the manner in which the *Operations and Algebraic Thinking* standards relate to the *Number and Operations in Base 10* standards, and how these two subdomains relate to *Measurement and Data* standards (CCSSI, 2010). The aim is to recast instructional design as an opportunity to make mathematical connections, rather than as an isolated focus on a single idea or standard.

Revisiting and reflecting upon the verticality of children's mathematical development will help ensure that relational thinking and connections are being achieved, potential standard gaps identified, and bridges of instructional support are set in place so continuity of learning is not interrupted. Given the complexities of mathematical development (Clements, 2004; Steffe, 1992), making effective instructional decisions requires considerable expertise; moreover, the development of teaching expertise is almost always the result of meaningful and sustained professional development. Let us endeavor to become skilled and knowledgeable conductors so that our students may reap the rewards of a rich mathematical symphony.

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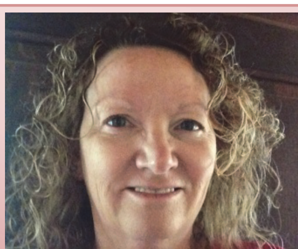

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"Alas, we are not manufactured, in our current edition of the human race, to understand abstract matters-we need context."

Taleb, N. N. (2010). *The black swan: The impact of the highly improbable*, 132. New York, NY: Random House.