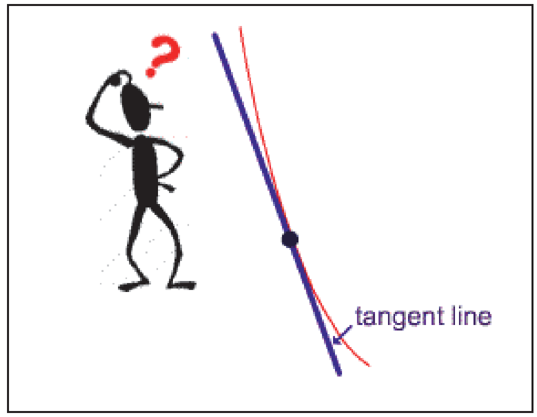
Derivatives at Several Points:
An Important Step between Derivative at a Point and Derivative Function

Alfinio Flores & Jungeun Park, University of Delaware
<alfinio@math.udel.edu>, < jungeun@udel.edu>
Milan Sherman, Drake University <milan.sherman@drake.edu>

Abstract

We present activities, with an interactive computer program, that serve to bridge two related but different concepts, derivative at a point and derivative function, and to help students understand better the relationships between the two. First, students work with tangent lines to the graph of the sine function at several points and then tabulate and graph the values of the slopes of these lines for the corresponding values of *x*. Then, students extend the function to an interval tracing the values of the slopes as the values of *x* change. Finally, students graph simultaneously the values of quotients of increments for several values of *x* to make more explicit the relation between the formal definitions of derivative at a point and derivative function.



Finding Derivatives at Several Points

In this article, we set up an intermediate step between the definitions of *the derivative at a point* and *the derivative of a function*. These are related, but very different kinds of mathematical objects; the *derivative at a point* is a real number, and the *derivative function* is a function. However, most books symbolize these two concepts with notation that is the same with the exception of one small difference in the use of literal symbols (x instead of x_0). Furthermore, instructors often use only the word "derivative" and do not clarify which concept they mean. Students may not realize that the slight difference in these notations implies a significant change in the underlying mathematics, and this oftentimes leads to the development of misconceptions (Park, 2011, 2013).

In the intermediate step, students look at the derivative at several points on the graph of the original function, to develop the concept of *derivative function* by first tabulating and plotting the values of the derivative at a point for several points. Students thus make an extension of *the derivative at a point* to *the derivatives at several points* and then an extension from *the derivatives at several points* to *the derivative at every point* which is *the derivative function*.

Prelude: Changes in Daylight Time

Many phenomena in nature are cyclical. Some depend on the seasons, which are repeated year after year. For example, the number of minutes of daylight per day for Columbus, OH for 2015 (U.S. Naval Observatory) are represented for every fifth day on Figure 1. We can expect a very similar graph for 2016 and the following years. Such cyclical phenomena are frequently modeled by using a sinusoidal function. Sometimes it is important not only to know the amount of minutes of daylight, but also how it is changing day to day, how fast is it changing, and whether it is decreasing or increasing. We can see that at the Solstices, the changes in the number of minutes of daylight are very small. At the Spring Equinox, the number of minutes of daylight increases quite fast while at the Fall Equinox, the number of minutes of daylight decrease equally fast.

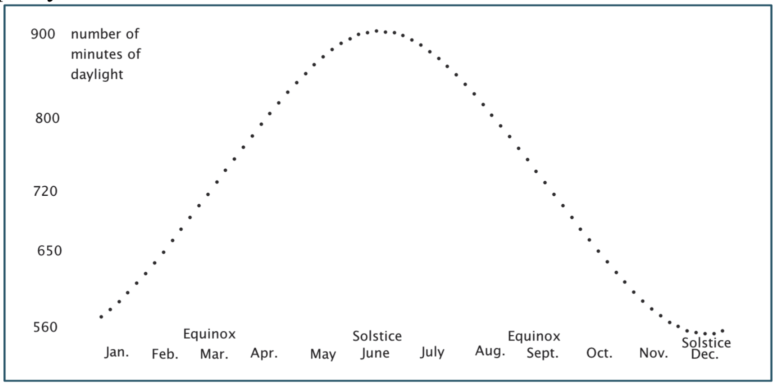


Figure 1. Duration of daylight in Columbus, OH (2015)

Overview of the Activities

We use $f(x) = \sin(x)$, with which students are familiar, to introduce students to the idea of the derivative at several points. Students use GeoGebra applets that are preset and available online for free. Students can interact with the applets by dragging line segments and/or points along the x-axis, or with a slider. No previous knowledge of GeoGebra is necessary. In the first activity, students find tangent lines to the graph by dragging preset red line segments (Figure 2) that form fixed angles with the x-axis that are easy to recognize, such as 0, $\pi/6$, $\pi/4$, $-\pi/6$ and $-\pi/4$ and for which students can compute the corresponding slopes, using their knowledge of trigonometric functions.

These activities can be used after the concept of *the derivative at a point* has been introduced from multiple perspectives: as the instantaneous rate of change, as the slope of the tangent line of a function at a point x_0, and as $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$. It is important to ensure that students understand what a *tangent line to a curve* means graphically, as well as accurately identify the point of tangency. Teachers may need to provide feedback as to whether the point identified is indeed the point of tangency or its approximate. The activities can also be conducted as a whole class demonstration where interactive figures are projected onto a screen and the instructor manipulates them. Students participate actively by estimating x values for points of tangency, filling in a table, and plotting their own graphs.

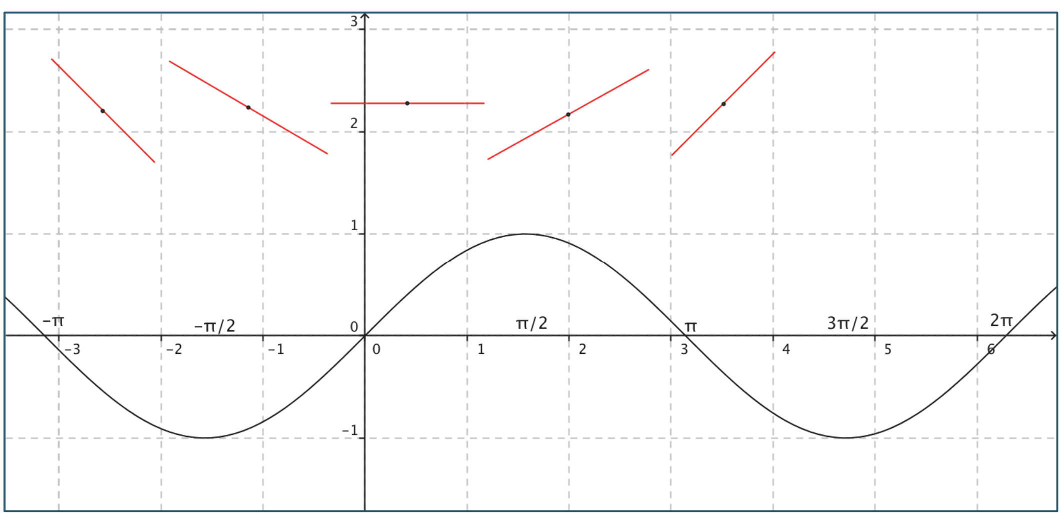


Figure 2. Movable segments with slope values of -1, $-\frac{\sqrt{3}}{3}$, 0, $\frac{\sqrt{3}}{3}$, 1.

From Derivative at a Point to Derivatives at Several Points

Finding x values where the slope of tangent line to the graph is known.

Students open the first GeoGebra applet (Flores, 2012a). The teacher can demonstrate placing the segment with a slope of $\sqrt{3}/3$ = 0.577… tangent to the graph (Figure 3) and find the corresponding x values, −1, 1, and 5.3 (approximate decimal values).

Students then put the horizontal segment tangent to the graph (Figure 3), to find the x values where the slope of the tangent line is 0, namely at $x = -\pi/2$, $\pi/2$, $3\pi/2$. Next, they find that the tangent line has a slope of 1 at $x = 0$ and $x = 2\pi$, and that the tangent line has a slope of −1, at $x = \pi$, $x = -\pi$, and $x = \pi$. Finally, students identify the x values where the slope is $-\sqrt{3}/3$ = − 0.577… namely x = −2.2, 2.2, 4.1 (approximate decimals).

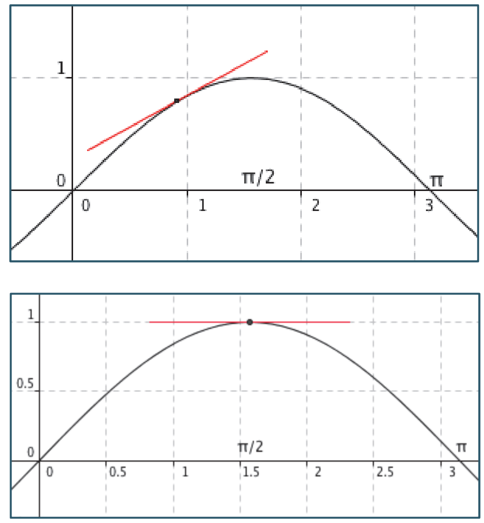


Figure 3. Using segments to find the slope of the tangent line at points on the graph

From Derivative at Several Points to Derivative Function

Making a table of known slopes.

Students summarize the information in Table 1. Students enter the x values they found from smallest to largest in the first column and the slopes of the corresponding tangent lines in the second column, that is, the value of the derivative at that point.

Table 1

Slope of tangent line for several x values

Approximate x-values	**Slope of the Tangent line (s)**
-π = −3.14	−1
−2.2	−0.577
-π/2 ≈ −1.6	0
−1	0.577
0	1
1	0.577
π/2 ≈ 1.6	0
2.2	−0.577
π ≈ 3.1	−1
4.1	−0.577
3π/2 = 4.7	0
5.3	0.577
2π ≈ 6.3	1

Graphing " x" vs. graphing slope.

Students then plot the points (x, s), where s is the slope, on grid paper or on a GeoGebra file (Figure 4). It is important that students verbalize that the values of the vertical coordinates of the points represent the slope of the tangent line to the original function at that value of x, that is, the derivative at each of the points for the original function, sine.

From discrete function to function on an interval.

Some students will join the plotted points with a continuous trace. The instructor should encourage these students to explain why this makes sense. Students can discuss what the new intermediate points on the continuous graph represent and articulate why the derivative is presumably defined at each point. Students can also use a GeoGebra applet to trace the slope values continuously by dragging a point representing values of the independent variable on the original function (Figure 5) (Flores, 2012b). This new function defined on the whole interval would be the derivative of the original function at each point in the interval. Students may guess that the cosine function is a good candidate to fit all the points on the graph.

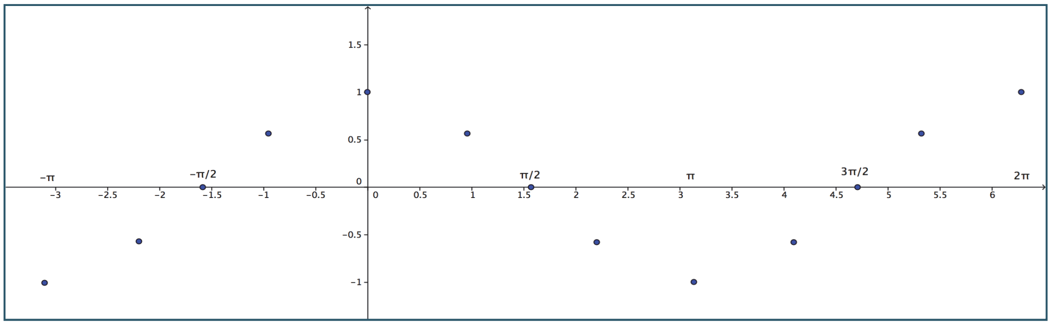


Figure 4. *x* vs. slope of tangent line at various points for sine function

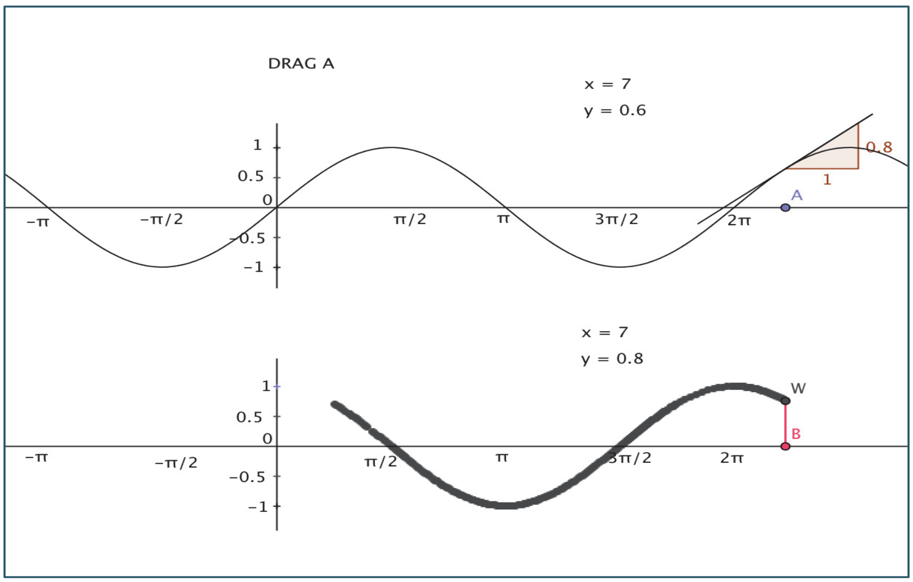


Figure 5. Tracing slope values

Derivative function on an interval.

Students may reinforce the connection between the tangents at a moving point and the derivative function of $f(x) = \sin(x)$ using an applet where the values of the independent variables of $f(x) = \sin(x)$ and its derivative function are the same (Figure 6) (Flores, 2012c).

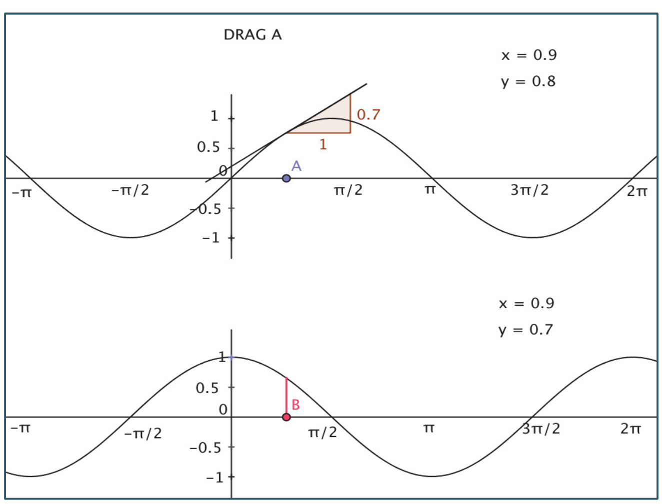


Figure 6. Dynamic tool showing the graphs of $f(x) = \sin(x)$ and its derivative function.

Simultaneous graphing of quotients of increments

To help students make explicit what x represents in the function, $\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$, defined on the interval, they consider the quotients of increments for the sine function simultaneously for several values of x, for a given value of h, and then for other values of h. Students use an interactive applet (Flores, 2012d) to observe how the values of the quotients of increments change as h approaches zero. The ordinates of the points in Figure 7 represent the values of the quotients $\frac{\sin(x+0.3) - \sin(x)}{0.3}$. The continuous curve is the graph of the function $f(x) = \cos(x)$. As students drag h on the slider, they can observe that as h gets closer to zero, for each of the x values shown, the quotient of increments $\frac{\sin(x+h) - \sin(x)}{h}$ gets closer to cos(x).

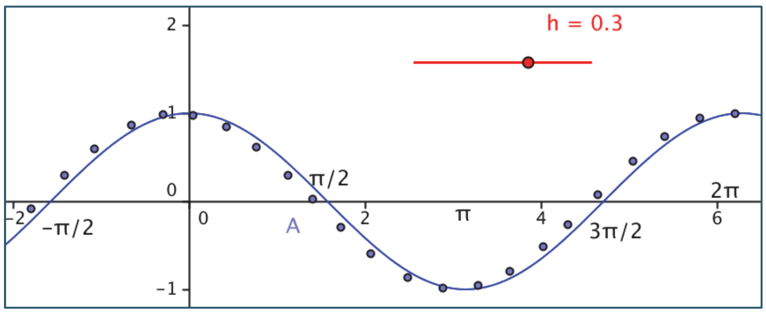


Figure 7. Values for quotients of increments $\frac{\sin(x+0.3) - \sin(x)}{0.3}$

Conclusion

Although at times it was not easy for students to verbalize the relationships illustrated, when we asked them to write down what they had learned, there were many positive responses:

- *I learned that the derivative at a point means the slope of tangent line to a curve at a point. I also learned that the derivative at a point could be the y-value of the derivative of the function.*
- *I learned that you can find the derivative of a function by finding the slopes of the tangent lines and graphing them with the same x-values.*

We also asked students how the lesson was different from their previous experience with the derivative. One student stated, “The visual aspect differs from my previous experience. It helps me picture the situation and figure out what is actually going on.” Another student said, “It was much more engaging and put the concepts into perspective.” We tried to get the students to see the correspondences and relationships for themselves, instead of being told. An “aha!” moment was described by a student: “as the lab progressed, it kind of clicked—cosine is a representation of $f'(x) = [\sin'(x)]$, here is how you plot the derivative, this is what it looks like, and here are a few ways of defining it.”

By considering the derivative at several points, students can develop a better understanding of the relationship between derivative at a point and the derivative function as a function. Students have an opportunity to appreciate a) how the derivative function is built up from the derivative at several points; b) why the ordinate of a point on the graph of the derivative of a function is the derivative at the corresponding x value of the original function; and c) why the definitions with the limit look “the same” except for “x_0” and “x”; and d) why one word, “derivative” is used, based on how the derivative function is built from the derivative at a point. Thus, instructors and students can communicate better about what “derivative” means.

References

Flores, A. (2012a). *Tangent lines to sine function*. Available http://www.geogebratube.org/student/m16534

Flores, A. (2012b). *Tracing the slopes of the tangent lines for sine function.* Available http://www.geogebratube.org/student/m7111

Flores, A. (2012c). *Slopes of tangent lines as y coordinates of a new function*. Available http://www.geogebratube.org/student/m7114

Flores, A. (2012d). *Quotients of increments for sine function for several values.* Available http://www.geogebratube.org/student/m23883

GeoGebra (2012) [Computer software]. Available from http://www.geogebra.org/cms/.

Park, J. (2011**).** *Calculus instructors' and students' discourses on the derivative.* Unpublished Doctoral Dissertation. Michigan State University.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematics Education in Science and Technology*, *44*(5), 624-640.

U. S. Naval Observatory (2012). Astronomical Applications Department. Duration of daylight for 2015, Columbus, Ohio. Available http://aa.usno.navy.mil/data/docs/Dur_OneYear.php

Alfinio Flores is professor of Mathematics Education at the University of Delaware. He enjoys teaching both mathematics methods and content courses *con ganas*. He uses technology, multiple approaches, and concrete materials to develop conceptual understanding of mathematical ideas.

Jungeun Park is an Assistant Professor of Mathematics Education at University of Delaware. Her major area of interest is undergraduate mathematics education, specifically teaching and learning the concept of derivative in Calculus in different languages.

Milan Sherman is an Assistant Professor of Mathematics Education at Drake University. His teaching and research interests include the use of technology for the teaching and learning of mathematics at the secondary and undergraduate level.

"In the domain of arithmetic, for instance, our brain might well compute 3 + 2 unconsciously, but not $(3 + 2)^2$, $(3 + 2) - 1$, or $-1/(3+2)$. Multistep calculations will always require a conscious effort."

Dehaene, S. (2014). *Consciousness and the brain: Deciphering how the brain codes our thoughts*, 107. New York, NY: Viking.