Uncertainty in PDV
For Accelerating Objects

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Background

- Trying to justify window size in expanding ring experiment
- Was I cheating by increasing the window size?
- What else is happening to the distribution?
- Derived “optimal window” for velocity
- Looking for “optimal window” for acceleration
- Stumbled onto Cohen's work
Interferometry

• **Fields of a light wave**
  - The E field amplitude of a wave traveling in the z direction
    \[ E(t) = E_1 \sin(\omega_1 t) \mathbf{i} + E_1 \cos(\omega_1 t) \mathbf{j} \]
  - Orthogonal B field
    \[ B(t) = -B_1 \cos(\omega_1 t) \mathbf{i} + B_1 \sin(\omega_1 t) \mathbf{j} \]

• **Fields add**
  \[ E(t) = E_1 \sin(\omega_1 t) \mathbf{i} + E_2 \sin(\omega_2 t) \mathbf{i} + E_1 \cos(\omega_1 t) \mathbf{j} + E_2 \cos(\omega_2 t) \mathbf{j} \]

• **Light intensity is a square law**
  - From Poynting and Maxwell
    \[ S(t) = \frac{1}{\mu_0 c} \left( E_1^2(t) + E_2^2(t) + 2 E_1 E_2 \cos(\omega_1 t - \omega_2 t) \right) \]
  - This equation is proportional to intensity
  - The third term is related to the frequency difference in two signals
Basic Assumptions

• The signal must be processed to extract the frequency
  – The signal must be sampled at the Nyquest frequency
  – Sample window size a free parameter that will be discussed later

• All the work done by a spectrogram function
  – Sliding STFT
    – Uses FFT
  – Filter Window
    – Matlab\Octave signal processing toolbox
Time Frequency Analysis

- **Signal**
  \[ d(t) = A(t)e^{i\phi(t)}; \quad D(\omega) = \frac{1}{\sqrt(2\pi)} \int d(t)e^{-i\omega t} \, dt \]

- **Power spectrum**
  \[ P^2(\omega) = D(\omega)D^*(\omega) \]

- **Moments**
  \[ \langle \omega^n \rangle = \int \omega^n P^2(\omega) \, d\omega = \int d^*(t) \frac{1}{i} \frac{d^n}{dt^n} d(t) \, dt \]

- **Local Mean Frequency**
  \[ \mu_\omega = \langle \omega^1 \rangle / \langle \omega^0 \rangle \]

- **Bandwidth**
  \[ \sigma^2_\omega = \langle \omega^2 \rangle / \langle \omega^0 \rangle - \mu^2_\omega \]

- **Skewness**
  \[ \gamma_\omega = \frac{\langle \omega^3 \rangle / \langle \omega^0 \rangle - 3 \mu_\omega \sigma^2_\omega / \sigma^3_\omega - \mu^3_\omega / \sigma^3_\omega}{\sigma^3_\omega} \]
Linear velocity

- Linear velocity + Gaussian window = Gaussian power spectrum
- Peak, mean, and velocities line up

\[ v(t) = v_0 + a t \]

\[ \varphi(t) = \frac{4\pi}{\lambda_0} \left( v_0 t + \frac{a_0}{2} t^2 + \phi_0 \right) \]

\[ P(\omega)^2 = \frac{\lambda_0}{8} \sqrt{\frac{\sigma^2}{2\sigma_v^2}} e^{-\frac{(\omega-\bar{\nu})^2}{2\sigma_v^2}} \]
Non-Linear Velocity

- Non-Linear Velocity → ?
- Spectrum is Skewed
- Peak, mean, and velocity no longer line up

\[ v(t) = v_0 + a_0 t + j_0 t^2 + s_0 t^3 + ... \]

\[ \phi(t) = \frac{4\pi}{\lambda_0} (v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{3} t^3 + \frac{s_0}{4} t^4 + ...) \]

\[ P(\omega) = \mathcal{I} \]

\[ \text{arg} (\max (P(\omega)^2)) = \mathcal{I} \]

\[ \mu_\omega = \phi(\bar{t}) + \frac{\sigma^2}{2} \ddot{\phi}(\bar{t}) + \frac{\sigma^4}{32} \sigma^{(5)} + ... \]
Moments

- **Zero'th**
  \[ \langle \omega_0 \rangle = |P|^2 = \int A^2(t) \, dt \]

- **First**
  \[ \langle \omega_1 \rangle = \int A^2(t) \dot{\phi}(t) \, dt \]

- **Second**
  \[ \langle \omega_2 \rangle = \int A^2(t) \dot{\phi}^2(t) - A(t) \ddot{A}(t) \, dt \]

- **Third**
  \[ \langle \omega_3 \rangle = \int A^2(t) \dot{\phi}^3(t) - A^3(t) \ddot{\phi}(t) - 3A(t) \dot{A}(t) \dddot{\phi}(t) - 3A(t) \ddot{A}(t) \dot{\phi}(t) \, dt \]
Expansions

- **Local Mean Frequency**
  \[ \mu_\omega = \ddot{\phi}(\bar{t}) + \frac{\sigma^2}{2} \dddot{\phi}(\bar{t}) + \frac{\sigma^4}{32} \sigma^{(5)} + \ldots \]

- **Bandwidth**
  \[ \sigma^2_\omega = \frac{1}{2} \sigma^2 + \frac{\sigma^2}{2} \dddot{\phi}(\bar{t})^2 + \frac{\sigma^4}{4} [ \ddot{\phi}(\bar{t}) \phi^{(4)}(\bar{t}) + \frac{1}{2} \dddot{\phi}(\bar{t})^2 ] + \ldots \]

- **Skewness**
  \[ \gamma_\omega = \frac{1}{\sigma^3_\omega} \left( \frac{3}{4} \dddot{\phi}(\bar{t})^2 \dddot{\phi}(\bar{t}) \sigma^4 + \frac{3}{4} \dddot{\phi}(\bar{t}) \dddot{\phi}(\bar{t}) \phi(\bar{t})^{(4)} \sigma^6 + \frac{1}{8} \dddot{\phi}(\bar{t})^3 - \frac{1}{4} \dddot{\phi}(\bar{t}) + \ldots \right) \]
Is Skewness a problem

- Recall

\[ \gamma_\omega = \frac{\langle \omega^3 \rangle}{\langle \omega^0 \rangle \sigma_\omega^3} - \frac{3 \mu_\omega \sigma_\omega^2}{\sigma_\omega^3} - \frac{\mu_3}{\sigma_\omega^3} = \frac{\mu_3}{\sigma_\omega^3} \]

- In general higher-order motion will skew spectrum creating an error between the peak, mean and true velocity at the center of the window

- This occurs when the 3\(^{rd}\) derivative or greater are non-zero

\[ \mu_3 = \frac{3}{4} \ddot{\phi}(\bar{t}) \ddot{\phi}(\bar{t}) \sigma^4 + \frac{3}{4} \dddot{\phi}(\bar{t}) \ddot{\phi}(\bar{t}) \phi(\bar{t}) \phi(\bar{t}) \sigma^6 + \frac{1}{8} \dddot{\phi}(\bar{t})^3 - \frac{1}{4} \dddot{\phi}(\bar{t}) + \ldots \]

- In statistics skewness is considered to be large when

\[ \left| \frac{\gamma_\omega}{SE} \right| \approx \left| \gamma_\omega \right| \sqrt{\frac{N}{6}} \leq 1.96 \]
How Big is the Error?

• Converges rapidly with small $\sigma$

$$\mu_\omega = \phi(\bar{t}) + \frac{\sigma^2}{2} \phi(\bar{t}) + \frac{\sigma^4}{32} \phi^{(5)} + ...$$

• Is small for small window sizes

$$\mu_\omega = \phi(\bar{t}) + \frac{T^2}{16 \alpha^2} \phi(\bar{t}) + \frac{T^4}{512 \alpha^4} \phi^{(5)}(\bar{t}) + ...; \text{ Let } \sigma = \frac{T}{2\alpha}$$

• What else can we say?
Optimal Window Size

• The minimum bandwidth window is

\[ T_{opt} = \sqrt{\frac{\lambda_0}{\pi a_1 \alpha}} \]

• Smaller windows will reduce error in accuracy, however the spectrum will be spread out and have a low resolution resulting in integration error

• Larger windows can increase resolution but risk accuracy
• Converges rapidly with small $\sigma$

$$\mu_\omega = \dot{\phi}(\bar{t}) + \frac{\sigma^2}{2} \ddot{\phi}(\bar{t}) + \frac{\sigma^4}{32} \phi^{(5)} + \ldots$$

• Is small for small window sizes

$$\mu_\omega = \dot{\phi}(\bar{t}) + \frac{T^2}{16 \alpha^2} \ddot{\phi}(\bar{t}) + \frac{T^4}{512 \alpha^4} \phi^{(5)}(\bar{t}) + \ldots; \text{ Let } \sigma = \frac{T}{2 \alpha}$$

• 1st order error in the velocity
  - Proportional to $j/a$!
  - Proportional to $\lambda_0$
  - Proportional to $n^2$ ($n$ is number of $T_{opt}$)

$$\mu_v = v_i + \frac{n^2 \lambda_0}{16 \pi} \frac{j_i}{a_i} + \frac{n^4 \lambda_0^2}{512 \pi^2} \frac{k_i}{a_i^2} + \ldots; \text{ Let } T = nT_{opt}; \phi^{(n)} = \frac{4 \pi}{\lambda_0} \chi^{(n)}$$
Noise

- Noise has multiple components
  - Shot noise $\rightarrow$ Poisson statistic
  - Thermal $\rightarrow$ Gauss statistic
- Gaussian is a good approximation and is standard practice in detector design.
Noise in the Spectrogram

• When processed by the STFT Gaussian distributed noise sample become Gaussian distributed Fourier coefficients
• The power spectrum follows the exponential distribution
• The mean value of the power spectrum is normally distributed

\[
\sigma_{P} = 4 \sigma_{G}^{4}
\]

\[
\mu_{P} = 2 \sigma_{G}^{2}
\]
Noise Analysis

- The expected values from mean value theorem

- Real moment can be corrected

\[ P_g^2 = 2 \sigma_G^2 + \frac{z_\alpha 2 \sigma_G^2}{\sqrt{(N)}} \]

\[ \langle \omega_n \rangle = \langle \omega_n \rangle_{\text{measured}} - 2 \sigma_G^2 \frac{\omega_N^{(n+1)}}{(n+1)} \pm \varepsilon_n \]

\[ \varepsilon_n = \frac{z_\alpha 2 \sigma_G^2 \omega_N^{(n+1)}}{\sqrt{(N)} (n+1)} \]

- Propagation of uncertainty

\[ \sigma_n^2 = \left( \frac{\varepsilon_n}{P_n^2} \right)^2 + \left( \frac{\langle \omega_n \rangle \varepsilon_0}{P_n^2} \right)^2 \]
Uncertainty in Local Mean Frequency

- Noise corrected local mean frequency with uncertainty
  \[ \bar{\omega} = \left\langle \omega \right\rangle_{\text{measured}} - \sigma_G^2 \omega_N^2 \left| P_m^2 \right| - 2 \sigma_G^2 \omega_N^2 \pm \sigma_1 \]

- Can be quite large when integrating over all \( w \)

In practice always threshold and integrate only over the peak
Uncertainty and Window Size

- At 10 dB for different window sizes
Uncertainty with Thresholding

- Performing the integration only over the peak reduces noise error

\[
\varepsilon_n = \frac{z_\alpha 2 \sigma_G^2}{\sqrt{(N)}} \sum_{\omega_n} (q_n \omega_n)
\]

\[
\mu_\omega = \frac{\langle \omega \rangle_{\text{measured}} - \sigma_G^2 \sum_{\omega_n} (q_n \omega_n)}{|P_m|^2 - 2 \sigma_G^2 \sum_{\omega_n} (q_n)} \pm \sigma_1
\]
Conclusion

• **Shown analytical treatment using time frequency analysis which demonstrates**
  – Errors are present between the peak, local mean, and true velocity for targets moving with non-linear acceleration
  – The optimal window which minimizes the bandwidth
  – Accuracy error

• **Analyzed the effect of noise in the captured signal**
  – Numerically Gaussian noise $\rightarrow$ exponential statistic in $P^2$
  – Use the mean value theorem to bound noise error
  – Shown the utility of thresholding to reduce uncertainty
Real Data

- Expanding ring at 3 different window sizes
Real Data

- Difference between mode and local mean frequency
Steps of Real Analysis

- Load data
- Take spectrogram
- Eliminate known bad data
Steps of Real Analysis

- **Iterate**
  - Calculate mode, $\mu$, $\sigma_v$ and $\sigma_m$ while narrowing to 3 $\sigma_v$ of peak,
  - Iteration stops
    
    $$iix = \text{mu2mode} < 0.5*\text{sigm};$$
    $$ikx = (p > pnoise).';$$
    $$(\text{mean}((\text{sigm2sigv}(iix&ikx)./\text{sigm}(iix&ikx))>zcrit);$$
    $$\text{count} > 3 \text{ and } \text{count} < 10$$
Steps of Real Data

- Calculate all values