# Detecting the Elusive Larkin-Ovchinnikov Modulated Superfluid Phases for Imbalanced Fermi Gases in Optical Lattices

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(Received 4 September 2009; revised manuscript received 28 February 2010; published 22 April 2010)

A system with unequal populations of up and down fermions may exhibit a Larkin-Ovchinnikov (LO) phase consisting of a periodic arrangement of domain walls where the order parameter changes sign and the excess polarization is localized. We find that the LO phase has a much larger range of stability in a lattice compared to the continuum; in a harmonic trap, the LO phase may involve 80% of the atoms in the trap, and can exist up to an entropy $s \sim 0.5k_B$ per fermion. We discuss detection of the LO phase (i) in real space by phase-contrast imaging of the periodic excess polarization; (ii) in $\mathbf{k}$ space by time-of-flight imaging of the single-particle and pair-momentum distributions; (iii) in energy space from the excess density of states within the gap arising from Andreev bound states in the domain walls.

DOI: 10.1103/PhysRevLett.104.165302 PACS numbers: 67.85.−d, 03.75.Ss, 37.10.Jk, 74.20.−z

An imbalanced population of fermions with two hyperfine states and interacting via attractive interactions offers the exciting possibility of observing superfluidity with a spatially modulated order parameter. For a small imbalance, the ground state is a BCS/BEC superfluid state with paired fermions, but for a large imbalance, the ground state is a polarized Fermi liquid [1,2]. At intermediate polarizations, mean-field theories predict Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states with a spatially modulated superfluid order parameter [3,4] that is a compromise between pairing and polarization. There is controversy over whether the FFLO state exists. One-dimensional (1D) systems only allow a quasi-long-range-ordered version of FFLO, whereas in 2D and 3D continua, FFLO only occupies a tiny sliver of the phase diagram and is vulnerable to fluctuations. So far, ordered FFLO has not been observed except in some reports on layered organic and heavy-fermion superconductors [5]. Both BCS and polarized states have been observed in imbalanced cold fermionic gases [6,7], but the LO phase has so far remained elusive.

In this Letter, we study the full phase diagram of the cubic lattice Hubbard model. We use approaches based on variational mean-field theory (MFT) in six channels, which includes Bogoliubov–de Gennes (BdG) and Hartree corrections. Our main results are the following: (1) ''Larkin-Ovchinnikov'' (LO) states, which break translational symmetry, are much more stable than ''Fulde-Ferrell'' (FF) states, which break time-reversal symmetry. (2) LO phases occupy a large region of the phase diagram between the BCS superfluid and polarized Fermi liquid phases. (3) With increasing field (or imbalance), the fully paired state becomes unstable to an LO phase consisting of domain walls at which the order parameter changes sign. The polarization is confined to these domain walls. At higher fields, the domain wall structure evolves into a sinusoidal variation of the order parameter accompanied by a polarization variation at twice the wave vector. We suggest that the most promising way to detect the LO phase is to focus on this spatial variation of the polarization. (4) The momentum distribution functions $n_\sigma(\mathbf{k})$ in the LO phase show features that break the lattice symmetry, such as Fermi arcs, Fermi pockets, and blocking regions, unlike in the fully paired state. (5) The LO phase has additional states within the $2\Delta$ gap arising from the weakly bound Andreev states localized at the domain walls. As the domain walls come closer and the bound states delocalize, a narrow band of states close to zero energy forms. Since $\Delta \sim U$ for a strong coupling superfluid, there is a clear separation between the low energy excess states and the gap of the paired superfluid, suggesting that the excess states may be detectable in spectroscopic measurements. (6) Depending on parameters, the LO phase can exist below an entropy $s \sim 0.5k_B$. (7) In an optical lattice in a shallow trap with appropriate parameters, LDA predicts that most (>80%) of the atoms participate in the LO phase (Fig. 1).

*Model and methods.*—The Hubbard Hamiltonian is given by

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \left( n_{\mathbf{k}\sigma} - \frac{1}{2} \right) - \sum_{\mathbf{r}\sigma} \mu_\sigma \left( n_{\mathbf{r}\sigma} - \frac{1}{2} \right) + U \sum_{\mathbf{r}} \left( n_{\mathbf{r}\uparrow} - \frac{1}{2} \right) \left( n_{\mathbf{r}\downarrow} - \frac{1}{2} \right) \qquad (1)$$

where $\xi_{\mathbf{k}} = -2(\cos k_x + \cos k_y + \cos k_z)$ is the dispersion relation on the cubic lattice for nearest-neighbor hopping, $\sigma = \pm 1$ labels (hyperfine) spin states, $n_{\mathbf{r}\sigma} = c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}\sigma}$ are number operators, $\mu_\sigma = \mu + \sigma h$ are the chemical potentials for the two spin species, and $U$ is the local pairwise Hubbard interaction. The hopping $t$ is the unit of energy. We use the convention that repulsive $U$ is positive. We find it convenient to work in terms of the average chemical potential $\mu$ and the Zeeman field $h = \frac{1}{2}(\mu_\uparrow - \mu_\downarrow)$. The observables of interest are the density $n_{\mathbf{r}} = \frac{1}{2}(n_{\mathbf{r}\uparrow} + n_{\mathbf{r}\downarrow})$, imbalance $m_{\mathbf{r}} = \frac{1}{2}(n_{\mathbf{r}\uparrow} - n_{\mathbf{r}\downarrow})$, and pairing density

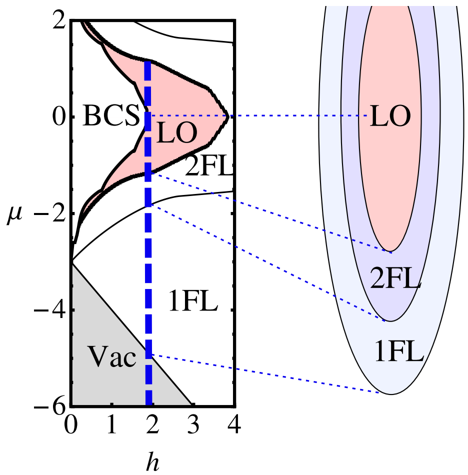



FIG. 1 (color online). The left panel shows the mean-field phase diagram of the cubic lattice Hubbard model at $U = -6t$, $T = 0$ in the $(\mu, h)$ plane. The thick dashed line indicates a slice through the phase diagram corresponding to suitable parameters (polarization fraction $P = 0.37$). The right panel is a schematic of the corresponding shell structure in a harmonic trap within the local density approximation, i.e., $\mu(r) = \mu(0) - Ar^2$. About 80% of the atoms are in the LO phase.

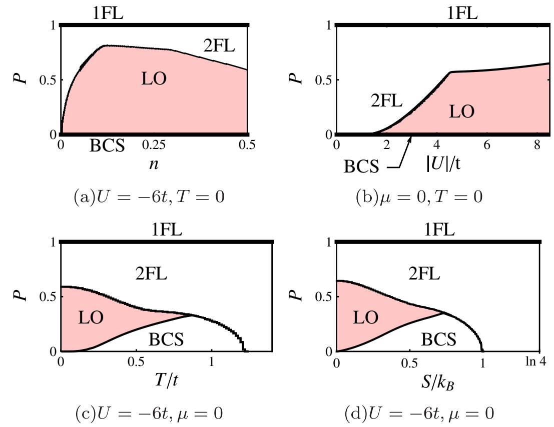



FIG. 2 (color online). Critical polarization of the cubic lattice Hubbard model as a function of (a) average density $n = \frac{1}{2}(n_\uparrow + n_\downarrow)$, (b) attraction $U$, (c) temperature $T$, and (d) entropy $S$. Energy scales are measured in units of the hopping $t$. BCS and LO represent superfluid states, 2FL represents a two-component Fermi liquid, and 1FL represents a fully polarized Fermi liquid (half-metal).

$F_{\mathbf{r}} = \langle c_{\mathbf{r}\downarrow} c_{\mathbf{r}\uparrow} \rangle = \frac{\Delta_{\mathbf{r}}}{-U}$. The "Lieb-Mattis" transformation (LMT) relates the repulsive and attractive Hubbard models [8]. Our calculations are based on $\mathrm{Tr}\rho \ln\rho$ variational mean-field theory. The Hubbard $U$ is approximated by $6N$ potentials, where $N$ is the number of sites: the local Hartree chemical potentials $\mu_{\mathbf{r}}^{\mathrm{int}}$, Zeeman fields $\mathbf{h}_{\mathbf{r}}^{\mathrm{int}}$, and complex pairing potentials $\Delta_{\mathbf{r}}$ [8].

*Phase diagrams.*—Figure 2 shows various slices through the phase diagram in $(n, P, U, T)$ parameter space. For symmetries of the phase diagram and a comparison with the free-fermion phase diagram, see Ref. [8]. Note that the hopping bandwidth is $12t$ and that $U_u = -7.91355t$ is the coupling where two fermions on a lattice first form a bound state (the analog of unitarity in the continuum). The main encouraging observation is that between the BCS state at $h = 0$ and the polarized FL state at moderate $h$, there is a sizeable region where the ground state is an LO state. This LO phase, which shows microscale phase separation, is a lower energy state compared to both the FF and "macroscopic phase separation" regions in Ref. [9]. Also, the LO region in a lattice is much bigger than the tiny sliver in the continuum [10]. The enhancement of the LO region by the cubic lattice can be traced to a combination of Hartree corrections, nesting, and domain wall formation as discussed below; for our parameters, all three effects are of similar importance.

*Real-space fingerprints of the LO state.*—As the field is increased from zero, the BCS ground state eventually becomes unstable towards penetration by domain walls containing excess polarization [11–14]. At higher fields, the domain walls become more closely spaced and eventually form a weak LO state characterized by a sinusoidal order parameter with a $\mathbf{q}$ vector related to the difference between the Fermi wave vectors of the majority and minority components. A variety of patterns are possible (vertical stripes, diagonal stripes, 2D modulations), but for weak couplings, modulations along the [100] direction are favored. Figure 3 shows examples of strong and weak LO ground states in real space.

Our variational calculations find that FFLO ground states are always LO states with a real order parameter that breaks translational symmetry. In contrast, Ref. [9] studied FF states with a complex order parameter $\Delta \sim e^{i\mathbf{q}\cdot\mathbf{r}}$ that break time-reversal symmetry; we find that LO states have a pairing energy that can be 50 times larger than such FF states. Qualitatively, this is because in LO states, the $\pm\mathbf{q}$ pairing opens gaps on both sides of each Fermi surface,

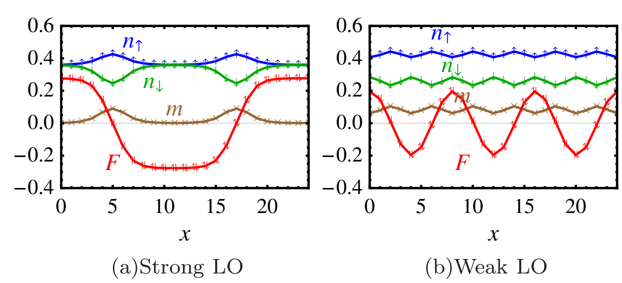



FIG. 3 (color online). Results of BdG on a $24 \times 20 \times 29$ cubic lattice with modulation along [100], with parameters $T = 0.01t$, $U = -4t$, $\mu = -0.5t$. (a) Strong LO state at $h = 0.85t$, just above the critical field for domain wall penetration. At each domain wall, the order parameter changes sign, and the polarization is finite due to occupation of Andreev bound states. (b) Weak LO state at $h = t$, with sinusoidal pairing density $F_{\mathbf{r}} \sim \cos\mathbf{q}\cdot\mathbf{r}$ accompanied by density oscillations at wave vector $2\mathbf{q}$.

taking advantage of the available phase space for pairing, whereas FF states only open gaps on one side of each Fermi surface. According to the LMT, an LO state maps to a coplanar spin texture (spins in $xz$- plane), whereas an FF state maps to a noncoplanar "helical" texture. Thus, for the *repulsive* Hubbard model, coplanar textures are more favorable.

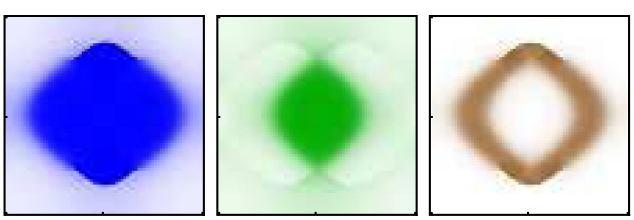
*Momentum-space fingerprints of the LO state.*—The pairing of up and down fermions belonging to unequal-sized Fermi surfaces leads to complicated features in the momentum distribution function $n_\sigma(\mathbf{k})$, such as Fermi arcs, Fermi pockets, and blocking regions (see Fig. 4 for an example). The most robust feature is the breaking of the lattice symmetry. In experiments, this effect may be complicated by twinning due to trap geometry.

*Energy-domain fingerprints of the LO state.*—In a superfluid at $h = 0$, there is a finite gap in the density of states of size $2\Delta_0$. As $h$ is increased, the up- and down-spin densities of states shift in opposite directions by $\pm h$ [Fig. 5(a)]. Eventually, the system enters an LO phase, which has excess density of states within the gap arising from Andreev bound states in the domain walls [Fig. 5(b)]. At intermediate coupling, $\Delta_0$ is of the order of $U$, so the states within the gap may be detectable by spectroscopic methods [16].

*Shell structure in a trap.*—We now consider optical lattices in traps within the local density approximation (LDA), which is applicable to shallow traps with many fermions. In LDA, the local phase is assumed to be determined by the local chemical potential, $\mu(r) = \mu_0 - V_{\text{trap}}(r)$. This predicts shell structures corresponding to vertical slices through the phase diagram, as depicted in Fig. 1. Spherical traps may cause twinning between LO states of different orientations, whereas a cigar-shaped trap helps align domain walls perpendicular to the long axis. The boundaries of the LO phase are marked by kinks in the density profiles $n_\uparrow(\mathbf{r})$ and $n_\downarrow(\mathbf{r})$ with appropriate critical exponents; however, whether these kinks are observable depends on parameters and experimental resolution.

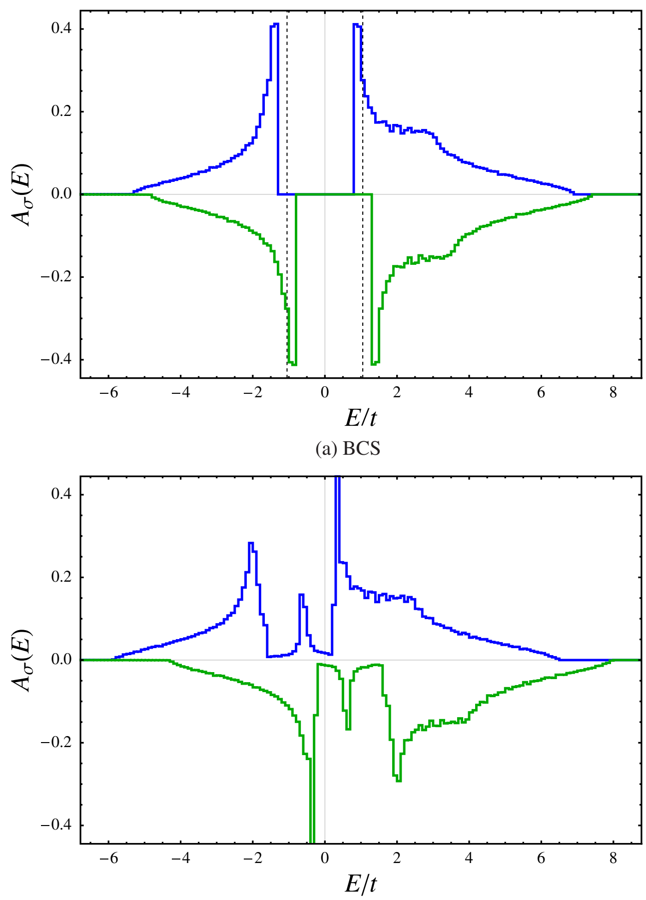
*Entropy for observing LO states.*—We predict that LO phases should be possible to observe at temperatures (or entropies) that are accessible to experiments. Within MFT, we have found LO states up to $T_{\text{LO}} = 0.6t$ ($s_{\text{LO}} = 0.5k_B$), for the parameters $U = -6t$, $\mu = 0.25t$, $h = 2.25t$. This is not much lower than the critical temperature for the BCS phase, $T_{\text{BCS}} \approx 1.1t$ ($s_{\text{BCS}} \approx 0.8k_B$), at $U = -6t$, $\mu = 0.25t$, $h = 0$. [See Fig. 2(d).]

*Fluctuation effects.*—Our results are at medium coupling, $|U| = 6t$, where fluctuations in the BCS phase only reduce $T_{\text{BCS}}$ and $s_{\text{BCS}}$ by a factor of the order of 1.2–2 [17,18]. In the strong LO phase, the orientation and position of the domain walls are pinned by the lattice, causing the rotational and translational Goldstone modes to become gapped. Therefore, the only gapless degrees of freedom are the U(1) superfluid phase modes, similar to those in the BCS phase. In the case of incommensurate LO order, translational modes are gapless, but rotational modes will still be gapped. Hence, fluctuation effects on the lattice will be less severe than in the continuum, where fluctuations of the LO "smectic" preclude long-range order at



(a)$n_\uparrow(k_x, k_y)$ (b)$n_\downarrow(k_x, k_y)$ (c)$m(k_x, k_y)$

FIG. 4 (color online). Momentum distributions of up and down fermions, and their difference $m(\mathbf{k}) = n_\uparrow(\mathbf{k}) - n_\downarrow(\mathbf{k})$, along the slice $k_z = \frac{\pi}{2}$, for a weak LO state with $q = (\frac{\pi}{3}, 0, 0)$. The distributions break fourfold symmetry but preserve inversion symmetry, in contrast to the results for FF states in Ref. [9].




(a) BCS

(b) Strong LO

FIG. 5 (color online). (a) Total densities of states $A_\uparrow(E)$ (top, blue line) and $A_\downarrow(E)$ (bottom, green line) for a fully paired BCS state in a small Zeeman field ($T = 0.01t$, $U = -4t$, $\mu = -0.5t$, $h = 0.25t$). The dashed lines indicate $E = \pm\Delta_0$. (b) DOS's for the strong LO state depicted in Fig. 3(a), showing midgap states arising from domain walls.

finite temperatures even in 3D. If quantum fluctuations are strong enough to destroy LO order (as might happen for $U \gtrsim U_u \approx 8t$), we expect that this would lead to a homogeneous mixture of strongly bound pairs coexisting with excess fermions.

*Conclusions.*—We find that for the cubic lattice within fully self-consistent mean-field theory, LO states occur over an enhanced region of the phase diagram, as compared to the continuum. This suggests that imbalanced ultracold fermion systems in optical lattices should readily exhibit LO ground states, which could be detectable by virtue of the accompanying polarization oscillations. Based on our calculations, we find that for $N \sim 10^5$ fermions with an overall polarization $P \sim 0.37$ at coupling $U = -6t$, about 83% of the atoms are in the LO phase. The polarization in each domain wall $P_{\rm DW} = \sum_{\mathbf{r}\in {\rm DW}} \frac{n_\uparrow(\mathbf{r}) - n_\downarrow(\mathbf{r})}{n_\uparrow(\mathbf{r}) + n_\downarrow(\mathbf{r})}$ for a strong LO state such as in Fig. 3(a) is about 30%; the polarization between domain walls is practically zero, giving a large contrast. The spacing between domain walls can be of order $10a$, where $a$ is the optical lattice constant. Typically, $a \sim 0.5\ \mu$m, which implies a domain wall spacing of about 5 $\mu$m.

The relations between effective Hubbard parameters and experimental variables are well known [19]. For example, $^{40}$K on the BCS side of the Feshbach resonance at about $B = 228$ G, where the scattering length is $a_s = -280a_0$, with an optical lattice spacing $a = \lambda/2 = 390$ nm and amplitude $V/E_R = 6$, corresponds to $U/t \approx -6$.

There is a connection between LO phases in the attractive Hubbard model and spin textures in the repulsive Hubbard model via the Lieb-Mattis transformation (a mapping that does not exist in the continuum) [8]. Therefore, by changing the sign of $U$, experiments in traps can effectively measure slices through the $(\mu, h)$ phase diagram in both vertical and horizontal directions. We also point out that since the repulsive Hubbard model has a tendency towards $d$-wave pairing, the *attractive* Hubbard model at half-filling and weak imbalance will have a tendency towards exotic $d$-wave magnetism described by order parameters such as $\langle c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\downarrow}\rangle \sim \cos k_x - \cos k_y$.

We expect that adding anisotropy to the cubic lattice will improve nesting and further enhance the LO region. Other authors have studied 2D arrays of 1D tubes [15,20]; it remains to be shown whether anisotropic lattices or coupled tubes are more favorable for LO. It will be important to include quantum and thermal phase fluctuations in reduced dimensions to get accurate estimates of phase boundaries.

We thank Randy Hulet, David Huse, and Leo Radzihovsky for helpful discussions. We acknowledge support from ARO and DARPA Grant No. W911NF-08-1-0338.

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