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STOKES'S THEOREM APPLIED TO MICROLENSING OF FINITE SOURCES

ANDREW GOULD[1]
Department of Astronomy, Ohio State University, Columbus, OH 43210; gould@payne.mps.ohio-state.edu
AND
CÉDRIC GAUCHEREL
Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France; gauche@hep.saclay.cea.fr
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ABSTRACT

The computation of the magnification of a finite source by an arbitrary gravitational lens can be reduced from a two-dimensional to a one-dimensional integral using a generalization of Stokes's thereom. For a large source lensed by a planetary system whose planet lies at the position where one of the two images would be in the absence of a planet, the integral can be done analytically. If the planet lies at the position of the major (unperturbed) images, the excess flux is the same as it would be for an isolated planet. If the planet lies at the minor image, there is no excess flux.

Subject headings: gravitational lensing — methods: numerical — planets and satellites: general

1. INTRODUCTION

Four groups have detected more than 100 microlensing events toward the Large Magellanic Cloud and the Galactic bulge (Alcock et al. 1995, 1996a; Aubourg et al. 1995; Udalski et al. 1994a; Alard 1996). For most events, the source can be treated as a point of light. However, when the source comes sufficiently close to or crosses a caustic (locus of points of infinite magnification in the source plane), the finite size of the source affects the light curve. One may use these effects to infer the size of the Einstein ring relative to the angular size of the source. Since the latter is generally known from Stefan's law and the color and magnitude of the source, one can then determine the absolute size of the Einstein ring (Gould 1994; Nemiroff & Wickramasinghe 1994). This effect has already been observed for one point-mass lens (Pratt et al. 1996) and for two binary lenses (Udalski et al. 1994b; Alcock et al. 1996b, 1996c), and may ultimately be key to measuring the mass function of the lenses (Gould 1996).

For a point-mass lens, one may write the formula for the magnification of a finite source in closed form (Witt & Mao 1994), but for a binary lens, the evaluation is more complicated. In principle, one could compute the magnification at each point of the source and sum these to find the total flux of the images. However, because the magnification is divergent near the caustic, one must take special care in performing the integration in these regions. Since the caustics have a somewhat irregular structure, this form of numerical integration is often difficult.

The problem can be especially acute in the analysis of lensing events by planetary systems because the Einstein ring of a planet is generally of the same order as the size of the source. In order to simulate such events Bennett & Rhie (1996) developed an alternate approach: they applied the "ray-shooting" technique (Kayser, Refsdal, & Stabell 1986; Schneider & Weiss 1987) to examine the points in the image plane (rather than the source plane), calculated the source-plane position for each, and thereby identified all the image points originating in the source. For a source of uniform surface brightness, this method yields the ratio of the total area of the images to the area of the source which, since surface brightness is conserved (Liouville 1837; Misner, Thorne, & Wheeler 1973), is equal to the total magnification. The method is easily generalized to nonuniform sources by weighting each point of the image by the local flux of the corresponding point on the source. Lewis et al. (1993) and Witt (1993) have developed an efficient scheme for two-dimensional integration of light curves which, while designed primarily to deal with the complicated image structure arising from multiple point lenses, can be applied to the simpler cases of binary and planetary-system lenses as well.

While precise modeling of any observed light curve should take account of limb darkening, a good quantitative understanding of the various types of light curves that are possible for a given lens configuration can be gained by considering sources of uniform surface brightness. In this case, the magnification is given directly by the combined area of the images divided by the area of the source. By Stokes's theorem, these areas can be evaluated by one-dimensional integrals over their boundaries. One might then consider the following approach. First, find the locus of points that constitute the image of the source boundary. These form the boundaries of the images. Second, divide the image boundary into connected subsets, which will each be a closed loop. Third, determine the area Q_i inside each loop by contour integration. Some loops may be inside others, in which case their interiors are "holes" in the images. Therefore, fourth, for each loop determine the number of other loops that lie outside it, π_i. The total area of the image is then given by $\sum_i (-1)^{\pi_i} Q_i$. The algorithms required to carry out this procedure are not prohibitive, but they are cumbersome, and perhaps for this reason the approach has never been tried.

Here we show that by taking account of the image parities, one can dispense with all global and topological information about the image structure, and evaluate the boundary integrals by purely *local* integration. The boundary approach leads immediately to an analytic result for finite sources imaged by a Chang-Refsdal lens which has important implications for understanding planetary-system

[1] Alfred P. Sloan Foundation Fellow.

lensing events. We also present a prescription for numerical integration and discuss a generalization of the method to limb-darkened sources.

2. METHOD

Consider first a source that does not cross any caustics. The source will be imaged into m disjoint images. Let C be the boundary of the source and let C'_j be the boundary of the jth image. The parity of each image, $p_j = \pm 1$, is defined as the sign of its magnification tensor. As one moves counterclockwise around C, one moves counterclockwise around C'_j for $p_j = 1$ and clockwise for $p_j = -1$. By Stokes's theorem, the area of the source is $(1/2) \int_C \boldsymbol{r} \times \boldsymbol{dl}$ and the area of the jth image is $(1/2) p_j \int_{C'_j} \boldsymbol{r} \times \boldsymbol{dl}$, where $\boldsymbol{r}$ is the position on the contour and $\boldsymbol{dl}$ is the line element. Note that the direction of integration around the image contours is defined by counterclockwise motion around the source. The magnification is then

$$A = \sum_j p_j \int_{C'_j} \boldsymbol{r} \times \boldsymbol{dl} \bigg/ \int \boldsymbol{r} \times \boldsymbol{dl} , \tag{2.1}$$

where the two-dimensional cross products are to be regarded as signed scalars.

Equation (2.1) remains valid even when the source crosses one or several caustics. The proof is given in the Appendix. Note that the number of image contours C'_j changes as one crosses a caustic in the source plane. This part of the integration requires some care, as discussed in § 4.

3. APPLICATION TO PLANETARY SYSTEMS

Consider a planet of mass m orbiting a star of mass M, with $m \ll M$. If the planet were not there, the star would lens a background source into two images at positions $\pm y_\pm \theta_e$, where θ_e is the angular Einstein radius of the lensing star,

$$y_\pm \equiv \frac{(x^2 + 4)^{1/2} \pm x}{2} , \tag{3.1}$$

and $x\theta_e$ is the projected separation between the source and the lens. The magnification tensor is given by

$$\mathscr{M}_\pm = \begin{pmatrix} 1 + \gamma_\pm & 0 \\ 0 & 1 - \gamma_\pm \end{pmatrix}^{-1} , \quad \gamma_\pm = y_\mp^2 , \tag{3.2}$$

where the (1, 1) element represents the magnification along the source-lens axis. The magnification of each image is given by the absolute value of the determinant of this tensor, $A_\pm = |\mathscr{M}_\pm|$. Note that the shear $\gamma_+ < 1$ for the major image outside the Einstein ring ($y_+ > 1$) and that $\gamma_- = \gamma_+^{-1} > 1$ for the minor image inside the Einstein ring ($y_- < 1$).

Since $m \ll M$, the planet affects only a small region of the Einstein ring over which the shear changes very little. We therefore treat the shear as constant in the neigborhood of the planet. The planet plus shear is then a Chang-Refsdal lens (Chang & Refsdal 1979, 1984; Schneider, Ehlers, & Falco 1992; Gould & Loeb 1992).

We now suppose that the planet lies exactly at the position of one of the two *unperturbed* images of the center of the source. We adopt this position as the center of our coordinates and express all angular distances in units of the Einstein ring of the planet: $\theta_p = (m/M)^{1/2}\theta_e$. We denote positions within the source by $(\rho \cos \psi, \rho \sin \psi)$ and positions within the image by $(r \cos \phi, r \sin \phi)$. We evaluate equation (2.14) from Gould & Loeb (1992), noting that in their notation $(\rho \cos \psi, \rho \sin \psi) = -\epsilon^{-1/2}([1 + \gamma]\xi_p, [1 - \gamma]\eta_p)$ and $(r \cos \phi, r \sin \phi) = \epsilon^{-1/2}(\xi_i - \xi_p, \eta_i - \eta_p)$. We then find

$$\rho \cos \psi = \frac{\cos \phi}{r} [r^2(1 + \gamma) - 1] , \qquad \rho \sin \psi = \frac{\sin \phi}{r} [r^2(1 - \gamma) - 1] . \tag{3.3}$$

Squaring and adding these two equations yields a quadratic equation in r^2, the two solutions of which are

$$r_\pm^2 = \frac{b \pm (b^2 - 4a)^{1/2}}{2a} , \quad a \equiv 1 + \gamma^2 + 2\gamma \cos 2\phi , \quad b \equiv \rho^2 + 2 + 2\gamma \cos 2\phi . \tag{3.4}$$

Suppose that the source is large enough so that it covers all caustics (see, e.g., Fig. 3 from Gould & Loeb 1992). The boundary of the source will then always have two images, one at r_+ and one at r_-. Using equation (2.1), and assuming that the source has constant surface brightness, we find a magnification

$$A = \frac{1}{2\pi\rho^2} \int_0^{2\pi} d\phi (r_+^2 - r_-^2) = \frac{1}{|1 - \gamma^2|} + \frac{1 + \mathrm{sgn}\,(1 - \gamma)}{\rho^2} - \frac{q}{\rho^4} + \cdots , \tag{3.5}$$

where $\mathrm{sgn}\,(1 - \gamma_\pm) = \pm 1$ and $q = [(\frac{1}{2} + \rho^{-2})^2 - \gamma^2\rho^{-4}]^{-1/2}$. The first term is just the magnification of the source in the absence of a planet (see eq. [3.2]). For the major image, the second term is $2\rho^{-2}$. Thus, for a source of surface brightness S, the total excess flux is $2\pi\theta_p^2 S$, exactly the same as the result for an isolated planet. On the other hand, to this order there is no excess flux when a planet perturbs the minor image, a result already suggested by the numerical calculations of Bennett & Rhie (1996). Successive additional terms are each smaller by ρ^{-2}. (To find the total magnification, one must remember to add in the magnification of the unperturbed image on the opposite side of the lensing star, $|1 - \gamma^{-2}|^{-1}$.)

Equation (3.5) is exact for a Chang-Refsdal lens, but for a planetary system it is exact only in the limit of an infinitely small planet. Moreover, it is valid only when the planet is aligned with the unperturbed image. The main value of the result is therefore its use in understanding and classifying planetary-system light curves. Nevertheless, it is interesting to ask how accurate it is for realistic cases. For a planet of finite mass, the shear due to the lensing star will change by $\delta\gamma \sim O(\gamma\theta_*/\theta_e)$ over the source, where θ_* is the source radius. The error induced by this difference is $\lesssim (\delta\gamma/\gamma)^2 \pi\theta_*^2 S \sim \pi\theta_*^4 S/\theta_e^2$. This is smaller than the flux of the star by $\lesssim (\theta_*/\theta_e)^2$. For cases of practical interest in planetary searches, $\theta_* \lesssim 6$ μas, while $\theta_e \gtrsim 100$ μas, so the error is typically less than 1%.

4. NUMERICAL INTEGRATION

To translate equation (2.1) into a prescription for numerical integration, we first approximate the boundary of the source as a polygon of n (not necessarily equal) sides (see Fig. 1). We denote the (two-component) vertices in counter-

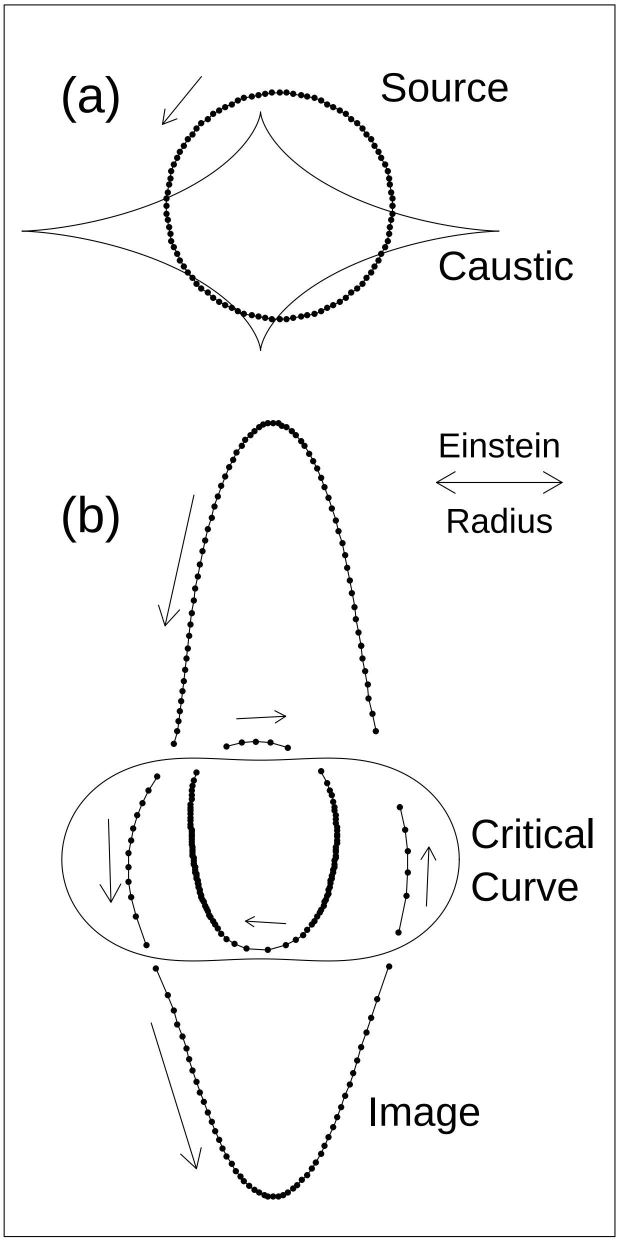

FIG. 1.—Illustration of integration technique. (*a*) Source plane for Chang-Refsdal lens with shear $\gamma = 0.6$ and source size $\rho = 0.9$ in units of the planet Einstein radius (*double arrow*). Filled circles on source boundary indicate 100 equally spaced positions of numerical integral (s_i) and arrow indicates direction of integration. Source boundary crosses caustic six times. (*b*) Image plane. Filled circles are multiple images ($u_{i,j}$, $j = 1, 2 \ldots$) of source position (s_i). Line segments between successive circles indicate terms in the numerical integration $p_j(u_{i-1,j} \times u_{i,j})$ as in eq. (4.1). Arrows indicate effective direction of integration for each connected curve: for $p = 1$ (outside the critical curve) the direction of increasing *i*, and for $p = -1$ (inside the critical curve) the direction of decreasing *i*. Image boundary crosses the critical curve six times, once each time the source crosses the caustic. Gaps of image boundary at critical-curve crossings correspond to the appearance or disappearance of two new images. Additional terms given by eq. (4.3) must be added to complete the contours across these jumps. Note that the sense of the arrows remains the same along an image boundary as it crosses the critical curve. The sense is opposite for the outer and inner contours, implying that the numerical integral will automatically subtract the area inside the inner contour from the area inside the outer contour. That is, the image is the region between these two contours. Note that with equal spacing along the source boundary [see (*a*)], some of the "jumps" across the critical curve are rather large which might lead to inaccurate integration. In practice, one would use a variable step size along the source boundary and would shorten the steps whenever the changes in the image position were large.

clockwise order by s_0, $s_1 \cdots s_n$, with $s_n = s_0$. For each source vertex s_i, there will be a variable number of image positions $u_{i,j}$. The vertex images should be ordered so that $u_{i-1,j}$ and $u_{i,j}$ lie on the same image curve. When the source contour crosses a caustic and two images disappear, these images should be replaced by "blanks." When a caustic is crossed and two new images appear, they should be entered into previously blank columns. With this ordering, the parities of the image vertices depend only on j: $p_{i,j} \to p_j$. For simplicity, we initially assume that if any caustics are crossed, one of the source vertices is chosen to lie right on the caustic.

The magnification is then given by

$$A = \sum_{i=1}^{n} \sum_{j'} p_j(u_{i-1,j} \times u_{i,j}) \Big/ \sum_{i=1}^{n} s_{i-1} \times s_i \, , \qquad (4.1)$$

where the prime in j' indicates that there is no summation for the first appearance of new images at a caustic (in which case there is, of course, no previous image position $u_{i-1,j}$).

In equation (4.1), we assumed that if the source boundary crossed a caustic (moving counterclockwise), thereby creating or destroying two images, then one of the vertices would be chosen to lie exactly on the caustic. We now show that if the first point does not lie on the caustic, there is a simple prescription which in effect replaces the two terms connecting the critical curve and the two images of the first point inside the caustic with a single term that connects the two image points directly. Let j and $j+1$ be two new images and let s_i be chosen to lie exactly on the caustic where they are created. The first term to be included in the sum for the j image would be $i+1$ and this term would include the boundary between the critical curve (at $u_{i,j}$) and the point at $u_{i+1,j}$. The situation is similar for image $j+1$. The two new images have opposite parities, $p_{j+1} = -p_j$. Because s_i lies on the caustic, $u_{i,j} = u_{i,j+1}$. The sum of the first terms for these two new images will then be

$$p_j(u_{i,j} \times u_{i+1,j}) + p_{j+1}(u_{i,j+1} \times u_{i+1,j+1}) = p_j u_{i,j} \times (u_{i+1,j} - u_{i+1,j+1}) \, . \qquad (4.2)$$

To a good approximation $u_{i,j} = (u_{i+1,j} + u_{i+1,j+1})/2$, so one may simply replace the two terms on the left-hand side of equation (4.2) with $p_j u_{i+1,j+1} \times u_{i+1,j}$. Now let $s_{i'}$ be the vertex on a caustic where the two images disappear. Using a similar argument, one can show that the two last terms for these images can be replaced by $-p_j u_{i'-1,j+1} \times u_{i'-1,j}$. Hence, it is not actually necessary to have vertices on the caustics. Suppose that there are k caustic crossings, $l = 1 \cdots k$ where two images j_l and $j_l + 1$ are created, and k other crossings were they are destroyed. Let the first point after the images have been created be i_l and last before they are destroyed be i'_l. If these first and last points do not lie on the caustic, then the numerator in equation (4.1) should be replaced by

$$\sum_{i=1}^{n} \sum_{j} p_j(u_{i-1,j} \times u_{i,j}) + \sum_{l=1}^{k} p_{j_l}[u_{i_l,j_l+1} \times u_{i_l,j_l} - u_{i'_l,j_l+1} \times u_{i'_l,j_l}] \, . \qquad (4.3)$$

5. DISCUSSION

Some of the most interesting applications of finite sources effects in microlensing involve the color changes due to differential limb darkening (Witt 1995). For example, this effect can be exploited to measure the Einstein ring size even when single-band photometric effects are undetectable, and it is especially useful in understanding planetary events (Loeb & Sasselov 1995; Gould & Welch 1996). The method given above cannot be directly applied to limb-darkened stars since constant surface brightness was assumed. However, one could model the source star as being composed of rings of constant surface brightness, and each ring could be evaluated by taking the difference of fluxes due to sources contained within two successive rings.

At first sight, this solution for the limb-darkened case seems to reintroduce a second dimension to the integration, and therefore appears to render the method no more efficient than traditional approaches. In fact, systematic investigations of finite sources effects require that one determine the magnification for many different source sizes. For the case of constant surface brightness, one must evaluate the magnification separately for, say, 20 different values of ρ. For the limb-darkened case, one could determine magnifications for all source sizes simultaneously by first evaluating the fluxes from disks of constant surface brightness $\rho = 0.1, 0.2, \ldots 5.0$, then taking differences to determine the fluxes in rings, and finally weighting these rings appropriately for limb-darkened sources of various radii. Thus, in practice, the computation for limb-darkened stars is no more than a factor $\sim O(2)$ more time consuming than for stars of constant surface brightness. See Gaudi & Gould (1996) for more details.

The method given here is simpler than that of Bennett & Rhie (1996) in that it requires only a one-dimensional integral, but it is more complicated in that one must find the individual image positions corresponding to the source boundary. (One must also find the parity and hence the magnification, but this need be done only once for each image contour.) The method of choice therefore depends on the lens system. For planetary system lenses, it is often possible to treat the effect of the planet as a perturbation on the background shear generated by its parent star. In these cases, the lens equation can be reduced to a quartic equation (Gould & Loeb 1992) which can be solved analytically. The method given here is therefore far more efficient. In general, binary lenses require solution of a fifth-order equation (Witt 1990; Witt & Mao 1995). These are straightforward to solve numerically, although there is no analytic solution. However, in general, an n-mass lens system requires solution of an $(n^2 + 1)$-order equation so that for sufficiently complicated lenses, two-dimensional integration over the image plane may be preferable.

Finally, we remark on an interesting aspect of the relationship between Chang-Refsdal lenses and binary lenses. Witt & Mao (1995) have shown that the minimum magnification of a point source inside a binary-lens caustic is $A_{\min} = 3$. Planetary-system lenses are special cases of binaries, so the theorem should hold rigorously for these also. In the limit of a very small planet mass, the Chang-Refsdal approximation is exact. Using the formalism of Schneider et al. (1992), it is easy to show that for the case of $\gamma < 1$, the minimum magnification inside the caustic is $A_{c,\min}(\gamma) = [\gamma(1-\gamma^2)]^{-1}$, i.e., a factor γ^{-1} higher than the magnification of the unperturbed image. This achieves an overall minimum at $\gamma = 3^{-1/2}$, where $A_{c,\min}(3^{-1/2}) = (27/4)^{1/2} \sim 2.6 < 3$. However, to find the total magnification of the binary lens, one must add in the magnification of the other image, $A'(\gamma) = (\gamma^{-2} - 1)^{-1}$. Hence, the minimum magnification of the whole binary lens is

$$A_{\min}(\gamma) = A_{c,\min}(\gamma) + A'(\gamma) = \frac{\gamma^3 + 1}{\gamma(1-\gamma^2)} . \tag{5.1}$$

This takes on its minimum value at $\gamma = \frac{1}{2}$ for which $A_{\min}(\frac{1}{2}) = 3$, and so saturates the limit of Witt & Mao (1995).

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APPENDIX

PROOF FOR CAUSTIC CROSSING SOURCES

To see why equation (2.1) remains valid even when the source crosses a caustic, divide the source into subsources each of which lies entirely inside or entirely outside of caustics. For definiteness, take the case of a binary lens for which the source can be divided into two subsources, one lying inside a caustic and having five images and the other lying outside and having three images. The magnification is then given by the sum of two integrals of the form of equation (2.1), one integral for each subsource. The difference between this sum and equation (2.1) applied directly to the whole source is eight additional line integrals, five for the image contours mapped from motion in one direction along the inside of the caustic segment, and three for the image contours mapped from motion in the opposite direction along the outside of the caustic segment. We now show that the sum of these eight contour integrals is identically zero and that it is therefore not necessary to break the source into subsources. Consider first the two images that are present inside but not outside the caustic. These have opposite parities and, for points along the caustic, are mapped into exactly the same points along the critical curve in the image plane. Hence the two line integrals from these images make equal contributions of opposite sign. Now consider the remaining three images. These are unaffected by the presence of the caustic and therefore the contours just inside and just outside the caustic are mapped to the same contours in the image plane. However, since the directions of integration are opposite, the two line integrals cancel for each image. It is therefore not necessary to break the source up into subsources. Equation (2.1) yields the correct magnification when applied directly to the source as a whole.

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