THE ELECTROMAGNETIC FIELD

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The interpretation of field phenomena presents its own particular difficulties. The processes in space which are responsible for these phenomena are not directly observable and have not yet been formulated in detail. We can only infer these processes from observed effects. Fortunately, for most purposes we can dispense with a detailed interpretation, and deal directly with observed effects at the ordinary scale of magnitude. It is only when we seek to correlate field phenomena with small-scale processes that we need go further.

Field phenomena are concerned with processes in space outside the structure of matter, and with the interaction of these processes with the structure of matter. These phenomena form a sequence: (1) the interaction of the "source" entities in their region; (2) a pattern distributed in space and time; and (3) the interaction of the entity acted upon in its region. Steps (1) and (3) of this sequence may be considered symmetrical; i. e., we assign the same measure number to an entity whether it serves as a source or entity acted upon. These are taken as inverse aspects of the same process.

There is no requirement that quantities assigned for describing field phenomena shall correspond to the physical quantities used for other purposes. We do not have a one to one correspondence between electromagnetic quantities and the usual mechanical quantities, but we do need relationships which make it possible to derive observed results simply and directly. We are dealing with energy differences in space (force) or in time (power); but since observable field effects involve the interaction of a localized entity with a field associated with more distant sources, we do not split a system into the part acted upon and its surroundings in the same manner as is done with other types of phenomena. Instead, we split the usual mechanical quantities into two parts and assign one part to the entity acted upon and the other part to the field background. This is the basic procedure in defining electromagnetic quantities; it makes the product of two such assigned quantities yield a familiar physical quantity such as force or power. We should have no fear of fractional exponents here; they are introduced by definition for our convenience.

Our distribution pattern (step 2) reduces to one in space alone for stationary or quasi-stationary states. In such cases all field phenomena show the simple symmetrical distribution expressed by the inverse square law. For the interaction of two entities we need only assign values to the entities themselves, assuming simply a distribution law and the assignment of quantities characteristic of the interacting entities. Such an approach is familiarly applied to the electrostatic and magnetic fields; it is also applicable to the gravitational field. It expresses the over-all result of the three steps of our sequence, but contains *no* information whatever regarding the small-scale structure of the physical processes involved.

Thus, it is not necessary to resort to special definitions for each type of field structure; one mode of definition serves as a basis for our field units. The units may be derived initially for stationary states, and then applied more generally.

We will consider in some detail the relations which subsist between electromagnetic units and other familiar physical units. With these relations clearly and definitely stated, any needed electromagnetic units may be readily derived. Dimensional relations follow directly from the defining equations; they are in no sense arbitrary, but specific and unambiguous.

It has been well argued that dimensional status should be associated with concepts rather than with units. Unfortunately, electromagnetic concepts have never acquired the uniqueness of definition necessary to make such a course practicable. A concept defined in one specific way thereby acquires a unique, specific dimensional status; there can be no discrepancy or inconsistency in the physical relations which we seek to represent by dimensional methods. It is only when we are faced with more than one definition for any concept that we may have a dimensional status associated with each mode of definition. Such a situation has long existed in electromagnetic theory, and we can hardly hope to remedy it completely. Nevertheless, we can start out by choosing a simple, uniquely defined group of units for use in expressing the fundamental equations. The relationship of this group to other commonly used units is then readily traced. We may include any useful units; such an approach does not essentially restrict our choice. With our dimensional relations clarified, the electromagnetic field is portrayed as a dimensionally self-consistent structure.

The traditional three primary units are units of mass, length, and time. Two of these, length and time, apply to physical concepts which we cannot express or describe in terms of anything more fundamental. The concept of mass is not fundamental in this sense. Mass is generally interpreted as a measure of the intrinsic energy content of matter. Energy appears as the more fundamental physical concept.

Since we cannot readily establish and maintain a unit of energy, we need the traditional primary units to derive and specifiy our unit of energy, as well as the associated units of force (energy/length), and power (energy/time). Field phenomena are directly concerned with energy differences in space (force) and energy changes in time (power). But the preliminary process of deriving units of these quantities from the primary units lies definitely outside the scope of electromagnetic theory. We gain nothing by combining steps which should be kept separate and distinct. We, therefore, omit the preliminary process here, and take units of energy, length and time as initially specified.

We will consider first those units derivable from units of energy and length alone, without the direct introduction of a time unit. Such units obviously are determined initially from resultant forces in stationary states. We will refer to these units as a "basic" group.

We may define a basic erg·cm group, choosing the erg and centimeter as initial units. These units are familiar, being a subgroup of the usual cgs systems, so we will merely summarize them.

Electrostatic forces determine:

Electrostatic charge Q_s: cgse, $(\text{erg}\cdot\text{cm})^{1/2}$.
Scalar (electrostatic) potential P: cgse, $(\text{erg}/\text{cm})^{1/2} = (\text{dy})^{1/2}$.
Electrostatic field, E and D: cgse, $(\text{dy})^{1/2}/\text{cm}$.

Magnetic forces determine:

Current I: cgsm, $(\text{dy})^{1/2}$.
Vector (magnetic) potential A: cgsm, $(\text{dy})^{1/2}$.
Magnetic field, B and H: cgsm, $(\text{dy})^{1/2}/\text{cm}$.

These units, which have been traditionally assigned to two separate systems, are here used as a single system. As in the Gaussian system we need not distinguish E and D, or B and H, in free space; permittivity and permeability appear

only as dimensionless ratios applicable to regions occupied by matter. They play no part in the specification of our units. Our scalar and vector potentials correspond dimensionally and in magnitude. Electrostatic charge or flux has the dimensional status $(erg \cdot cm)^{1/2}$, the same as the corresponding unit of magnetic flux, the maxwell.

Expressing these relations requires only two dimensional symbols, since our definitions are based on only two units. Instead of using a symbol for energy directly, it is convenient to set $U = (force)^{1/2}$. This is the dimensional status of our potentials, scalar and vector. Current, the "source" of a vector potential, partakes of the same dimensional status U, while electrostatic charge has the status UL. The field vectors, being the gradient of the scalar potential and the curl of the vector potential, respectively, have the dimensions U/L. We derive at once the relations $U^2L =$ energy, $(U/L)^2 =$ energy density.

These definitions have been specified in terms of stationary state experiments; so, they make no direct use of a time unit. We can apply such a basic group of units more generally, including nonstationary states, without the introduction of additional units; but, with our definitions stated in terms of energy and length alone, we cannot bring in a time unit arbitrarily at this point. Instead, the electromagnetic field itself determines a relation between our space and time coordinates; it gives us a time unit corresponding to our unit of length. The relational factor is of course the familiar velocity, c.

With such a group of units the position of the factor, c, is not arbitrary; it appears wherever time is introduced and nowhere else. Time does not appear independently, but only in conjunction with the factor, c. Experimentally we find that this factor appears not only in obvious propagation processes but in other relations in the electromagnetic field as well, forming an intrinsic part of our field structure. An important such relation, historically the oldest, is the relation between charge and current. In basic units, electrostatic charge = current x time x c.

Unfortunately there exists a general misconception that "quantity of electricity" should have a unique dimensional status, independent of the defining experiments chosen. We do not fall into this error with more familiar matters. No one confuses "bushels of apples" with "pounds of apples," or confuses these measures with an actual counting of the apples. To avoid the somewhat mystical connotation assigned to electromagnetic phenomena, we need to recognize the relation between experimental procedure and dimensional status.

Apart from an actual counting process, which is not generally applicable, there are two principal ways of defining "quantity of electricity." If we state our definition in terms of the forces which quantities of electricity exert on each other, we have an electrostatic unit, with the dimensional status $(force)^{1/2}$ x length. If we first define current, in terms of the forces which currents exert on each other, and then define quantity of electricity as the time integral of a current, this cumulative or electromagnetic unit has the dimensional status $(force)^{1/2}$ x time. The relation between the two units, length/time, a velocity, is inherent in the definitions and can be determined from the defining experiments, without recourse to propagation processes. This fact has, of course, been familiar since Maxwell's day; it served as the basis for his interpretation of the electromagnetic field.

A second relation involving c is the expression for power. We have defined *both* potential and current directly in terms of force; their product appears in units of force (energy/length). To obtain units of power (energy/time), we must multiply by a velocity (length/time). Again the velocity is the familiar factor, c.

It is, thus, actually possible to determine the velocity, c, from the supply of power to a simple resistance, a dramatic illustration of the presence of this velocity as an intrinsic factor in field processes. Of course, in practice it is usually convenient to reverse the procedure, obtaining a direct expression for power by including c implicitly in either the current unit or the potential unit. The tradi-

tional cgs systems may be derived in this way. The position of the factor, c, at this point is vital in coordinating these groups of units.

The velocity, c, probably manifests its most direct significance in the equations of propagation, the familiar "wave equations." Scalar and vector potentials both satisfy such equations, as do their derivatives also. There is good reason to regard the equations of propagation as the fundamental equations of the electromagnetic field. If we make this assumption, the usual field equations are readily derived. We require only one additional condition. To assure conservation of the field energy associated with our potentials, we need a relation between scalar and vector potentials, an "equation of continuity," analogous to the relation between charge and current. These two conditions, propagation and continuity, suffice to determine the electromagnetic field equations (Holm, 1950).

The field equations for a basic group of units are identical with those of the Gaussian system, except that current appears in "electromagnetic" instead of the inconvenient "electrostatic" units; I here corresponds to I/c in the Gaussian system. The same equations are applicable to any basic group. As such a group is specified in terms of only *two* initial units, it is simpler dimensionally than the Gaussian system. It is also more flexible, we can apply the same fundamental equations to a group derived from any desired units of energy and length. Thus, by changing *one* unit in the equations of the Gaussian system, we obtain a group which is more generally useful in describing the structure of the electromagnetic field.

We can apply the same approach to the derivation of the practical electromagnetic units. These units are not derivable directly from the mks units; we will find it convenient to define first a basic erg·meter group, analogous to the above erg·cm group. We retain the same fundamental equations, but c now appears in meters per second instead of cm per second. The unit of force here is the erg per meter, the centidyne or 0.01 dyne. Units of potential, scalar and vector, and current are one-tenth the size of the corresponding cgs basic units. Their dimensional status, $U = (force)^{1/2}$, is here equal to $(erg/meter)^{1/2}$. This current unit is the ampere. The unit of electrostatic charge, UL, $(erg \cdot meter)^{1/2}$, equals 10 cgse.

With the fundamental equations stated for our basic groups, we may readily introduce additional units where convenient and derive equations using them. In particular, for dealing with power and electrodynamic processes, where time is directly involved, we may choose units which make use of the dimensional factor of time, T, in addition to U and L. There is probably only a limited need for cgs units outside the basic group. We shall be more concerned with the quite simple relations between the basic erg·meter group and the practical units which lie outside this group.

Among these units we have the coulomb, the time integral of current in amperes, with the dimensional status UT. The corresponding cgsm equals 10 coul. In either system, the factor c relates such a cumulative unit to the basic electrostatic unit of charge.

In the practical system we express power directly by defining an electrodynamic potential unit: V (volts) $= 10^{-7}$ cP; this is equivalent to volts = watts per ampere. The erg/joule ratio 10^{-7} is not a dimensional factor; it is introduced simply to adjust our order of magnitude. The volt thus has the dimensional status UL/T. The corresponding cgsm, using the product cP, is only 10^{-8} volt.

The ohm (volts per ampere) has the dimensions L/T, a velocity, a status which derives from the presence of the constant, c, in the definition of the volt. Resistance per se does not have this status; in basic units it is a dimensionless ratio. The dimensionless unit of resistance has the same value in any basic group; it is equal to 30 ohms.

The basic potential unit of the erg·meter group is similarly equal to 30 volts. The factor 10^{-7}c, approximately 30, which appears in these relations, is usually

referred to as a "field impedance" constant. Such a designation unfortunately fails to emphasize its derivation and significance. It is essentially a conversion factor relating units of the basic group, defined in terms of force, to units of the electrodynamic group, associated with power. More specifically, this constant represents the transition from force, in ergs per meter, to power, in joules per second. Its dimensional status remains that of c, and stems from this transition.

Those practical units which are decimally related to corresponding basic units retain the same dimensional status. The weber or volt·second is equal to 10^7 basic ampere·meter units; it has the dimensions UL. A magnetic field expressed in webers per meter^2 has the dimensional status U/L, the same as the curl of the vector potential expressed in amperes per meter, the same factor 10^7 relates these units.

Inductance and capacitance serve most generally as time constants, but they appear within the basic group expressed in units of length; such a group in effect uses a time unit corresponding to the unit of length. The practical unit of inductance, the henry or ohm·second, also has the dimensional status of length; it is equivalent to 10^7 meters. This corresponds to an inductive time constant of 1/30 sec. The farad (coulombs per volt) has the more complicated dimensional status T^2/L. The factor, c, appears twice in its derivation. The farad is equivalent to $c^2/10^7$ or 9 x 10^9 m, corresponding to a time constant of 30 sec.

If we express inductance and capacitance in time units, impedance will appear in basic dimensionless units. These quantities are all related to the corresponding practical units through the same factor, 30, so the same expression for impedance is applicable.

We see that units of the basic erg·meter group are related to other practical units through simple conversion factors, simpler than the field constants of the currently popular "mks" arrangement. But more importantly, we have gained the advantage of a single set of field equations containing no arbitrary constants.

In addition to the two systems discussed, we can make use of a basic group of units in other ways. We can, for example, readily translate our units to other orders of magnitude. We may choose to reduce our cgs units of energy and length ten decimal places, retaining the dyne for force, and the potential units of the cgs basic group. Such a system gives us simple values for some important constants. We can just as readily specify units suitable for dealing with astronomical magnitudes. We are freed from the restriction to one or two specific groups of units while we retain a single set of fundamental equations throughout their range of validity.

THE ELECTROMAGNETIC FIELD EQUATIONS

The equations are here stated for a basic group of units, which may be derived initially for stationary states, from units of energy and length, or force and length. Time is then introduced as an auxiliary quantity, related to the space coordinates through the velocity, c.

The principal relations in which c appears are:

The relation between charge and current:

$$Q_s = c \int I \, dt, \text{ or } \nabla \cdot \mathbf{i} = -\frac{1}{c}\frac{\partial q}{\partial t}.$$

A similar "equation of continuity" relating scalar and vector potentials:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial P}{\partial t}.$$

Equation of propagation, scalar potential:

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = -4\pi q.$$

Equation of propagation, vector potential:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = - 4\pi \mathbf{i}.$$

Power equation:

Power $= cPI$.

We define:

$$\mathbf{B} = \nabla \times \mathbf{A},$$

and:

$$\mathbf{E} = - \nabla P - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

From the equations of propagation and continuity, we derive:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla P - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = -\nabla^2 P + \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = 4\pi q.$$

$$\nabla \times \mathbf{E} = - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}.$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = - \frac{1}{c} \frac{\partial}{\partial t} (\nabla P) - \nabla^2 \mathbf{A}$$

$$= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{i}.$$

LITERATURE CITED

Holm, G. R. 1950. The dimensional structure of the electromagnetic field. Amer. Jour. Phys. 18: 509–514.