

All of Mathematics Is Easy Once You Know How to Do It

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Editor's note: This essay was originally delivered as an invited lecture at the Academy of Teaching's "Mini-Conference on Scholarship, Teaching, and Best Practice," on Friday, April 25, 2008. Some of the marks of the occasion have been retained in order to preserve the full spirit of the lecture.

Thank you for the invitation to speak here. I am now nearing the end of my 37th year as a teacher of mathematics and my 33rd year as a faculty member at Ohio State. But this is the first time I have been forced to write a teaching statement or to articulate a philosophy of teaching. I must say that my only personal rules are: love the material that you are teaching; and care about the students you are teaching. The year for which I received a teaching award, I combined these rules with a huge investment of time. I divided my class of roughly 25 students into five groups and I required each group to come to my office for an hour each week. So I was putting in 8 contact hours per week for a 3-credit hour course. It worked, but I'm sure there must be an easier way to be an effective teacher. But it is hard to replace time and personal contact.

I stumbled into the title of my lecture at the urging of Dr. Yousif. The quote "All of mathematics is easy once you know how to do it" is from my thesis advisor, Walter Feit. I don't believe this statement myself. Indeed, if I were asked to give evidence to the contrary, I would put forward as Exhibit A, Walter's own magnum opus, *The Odd Order Theorem*, a 255-page tour de force of mathematics, done jointly with John G. Thompson. It has been simplified somewhat. It has been studied carefully. It still doesn't seem the least bit easy to me.

But Walter's statement "All of mathematics is easy, once you know how to do it" does contain enough truth to be the source of one paradox: If you don't understand a topic in mathematics deeply, then you can't help someone else to understand it. On the other hand, if you understand

a topic in mathematics deeply, then you have probably forgotten (if you ever consciously knew) why anyone would ever have any trouble understanding it. Often mathematical understanding—the magical “ah ha!”—is like the moment when the optical illusion of the ugly hag turns into the beautiful young woman. Sometimes you can’t make the step back to see the picture as a hag again—and even more significantly, you certainly can’t explain to someone else how to see the beautiful woman.

This problem is certainly not unique to mathematics. I experienced a similar transformational moment with regard to Beethoven’s late string quartets. While in college, I discovered the beauty of classical music, thanks to a required two-semester music appreciation course. Let me digress to put in a plug for a core set of courses required of all undergraduates. I am old-fashioned enough—our former Provost Barbara Snyder called me “paternalistic”—to believe that college is not just trade school and that we elders really do have a responsibility to pass on the fruits of our accumulated wisdom concerning the collective culture of the human race to our children. Evidence suggests that this may be the principal quality distinguishing us from chimpanzees. Resuming—I discovered the beauty of classical music in a required college course, more specifically, thanks to a magical encounter there with Mozart’s great *G Minor Symphony*. It was an experience like St. Paul’s epiphany on the Road to Damascus. Since I had some disposable cash and LPs were cheap, I hastened to purchase, ear unheard, quite a few classical records, including the complete Beethoven string quartets performed by the Budapest String Quartet. I quickly loved most of them, but when it came to the last five, I found myself sharing the opinion of Ludwig Spohr, as quoted on the record jacket, that, alas, Beethoven was deaf and did not know what he was doing. Nonetheless I persevered in listening to them, because after all, it was Beethoven. Suddenly one day, on perhaps the fifth or sixth listening, the set of ugly hags turned into a collection of five of the most beautiful women I had ever seen. I cannot explain it. And I wouldn’t know what to tell someone else, except: Don’t give up. Keep on listening.

So, one big challenge for us as teachers is to persuade our students to persevere. Trust me. This makes sense. You can understand it, and when you do you will wonder what it was that ever baffled you. Indeed, all of mathematics is easy, once you understand it. But the journey may be arduous, and you have to believe it is worth the effort. (This is a valuable white lie. Most students will never reach the level of mathematics where

this is demonstrably false. And if they do, they will thank us for giving them the confidence to get that far.) When I first came to OSU, my office neighbor was a Japanese mathematician, Eiichi Bannai. He and I and my wife would have lunch together in my office. When he wasn't showing off his phenomenal skill with chopsticks, Eiichi shared his view that all people are capable of learning mathematics. Apparently this is a fairly general view in the Orient: work diligently, and you will master mathematics. Here in America, the view that math is too difficult for most people seems to prevail. Interesting how self-fulfilling both prophecies seem to be. (No Child Left Behind would seem to move America towards the Oriental paradigm. But I have a very uneasy feeling that, as generally implemented, it is rather No Child Moved Ahead. And the new paradigm seems to be: The Child as Glorified Parrot.)

But there is no Royal Road to Geometry. As I said, the journey will be arduous, and the student has to believe it is worth the effort. It takes patience to see the old hag turn into the beautiful woman. It takes patience to hear the mishmash of incoherent notes turn into Beethoven's sublime *Cavatina*. And it takes a lot of patience and hard work to really understand x the unknown and functions and limits and so on. Persuading students that they need both patience and diligence has become all the more challenging in the modern era of instant gratification and short attention spans.

“Ontogeny recapitulates phylogeny.” This is the theory that the evolution of the human fetus retraces the evolution of life on Earth from one-celled amoeba to fish to amphibian to reptile to mammal. An educational version of this theory suggests that the development of each person's understanding of concepts retraces the development of this understanding in the human species. I think this is a reasonable working hypothesis, one that argues in favor of a historical development in the teaching of mathematics. One obstacle is that much of the evolution of mathematics is pre-historic. Anthropology can no doubt shed considerable light on this pre-historic era through the study of primitive tribes, where, for example, the entire concept of number amounts to little more than one, two, and many. Somehow in the primary grades, we have to rush children through thousands of evolutionary years to the extremely sophisticated decimal place value representation of numbers. Fortunately, it is not my responsibility to attempt this portion of mathematical instruction. Nevertheless, as college teachers, I think we need to be aware of certain crucial conceptual leaps: fractions, the

“unknown” x , functions, groups, categories.

I would like to focus on my favorite mathematical subject, which is one where a historical perspective is of great value to the teacher. My favorite course to teach is the course we usually call “Abstract Algebra.” The subject material of this course evolved primarily during the period 1750–1832 in the work of a sequence of brilliant mathematical thinkers,

primarily Euler, Lagrange, Gauss, Abel, and Galois. Each knew and reacted to the work of his predecessors, and it is possible to see very clearly and dramatically how some fundamental concepts were present from the very beginning, but densely shrouded in a mist of mental resistance and confusion—a mist that affected some of the greatest geniuses who ever lived. Indeed, Lagrange was one of the greatest of all mathematical geniuses. But his proof in 1771 of what has come to be called “Lagrange’s Theorem” is totally garbled and would get 0 points if submitted today as part of an undergraduate homework assignment. Finally these concepts achieved a fairly high level of clarity in the work of Evariste Galois, one of the most remarkable disturbed teenage geniuses who ever lived and the greatest of mathematical romantics. He was an unsung genius, a political revolutionary, and a frustrated lover, killed in a duel at the age of 20 on the eve of the great battle on the barricades immortalized by Victor Hugo in *Les Miserables*. To quote Hermann Weyl, one of the greatest mathematicians of the 20th century:

Galois’ ideas, which for several decades remained a book with seven seals but later exerted a more and more profound influence upon the whole development of mathematics, are contained in a farewell letter written to a friend on the eve of his death, which he met in a silly duel...This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind.

As Weyl implies, many great mathematicians spent the next century staring at Galois’ old hag. Finally she was turned into a beautiful

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woman around 1930 by Emmy Noether and Emil Artin. Unfortunately, they did such a good job that everyone teaches this material following Artin's notes. Why unfortunately? Because now it all looks perfect and polished—and dead. If you only saw the beautiful woman from the start, you would miss the wonderful “ah ha” moment and the excitement of experiencing the transformation. The power of Galois' ideas is that it took 100 years to kill them. A similar statement can be made about other great mathematical discoveries, like calculus.

Our job as educators is to bring these discoveries back to life. A great educator, Uri Treisman, said: “It doesn't bother me that students finish a calculus class and are confused, don't understand. What bothers me is that they finish a calculus class and aren't awe-struck.” The great ideas of mathematics are among the deepest and most beautiful ideas of the human mind. If we don't convey this to our students, then we have done them a gross disservice.

Just last Saturday evening was the first Passover Seder. A couple of years ago, my wife and I were at a Passover seder at the home of my sister-in-law. She had hired a woman to help with the serving and clean-up, and I was chatting a bit with this lady as she loaded the dishwasher. I told her that I was a mathematician. I have gotten used to the reply: “Oh, I was never any good at mathematics.” What I didn't expect was her follow-up remark: “Well, I guess different people's minds work differently. Some people are creative and some can do math.” Having always thought of mathematics as a quintessentially creative activity, I was taken aback. But the more I thought about it, the more I realized that the blame falls on our educational system which has turned mathematics, for most people, from a sequence of creative insights into a set of tedious algorithms—boring recipes for plugging some numbers into a formula, turning a crank, and getting another number out the other end. Sausage making without the smells. Let me here acknowledge my high school geometry and calculus teacher, Blossom Backal, who helped me see for the first time all that is beautiful in mathematics. And why in God's name have we stopped teaching Euclidean geometry with proofs to our children? No Child Moved Ahead indeed! The great French mathematician, Rene Thom, was furious about the disappearance of geometry from the K-12 curriculum and its replacement with algebra—and though I wouldn't describe myself as furious, I agree with him. Thom's comment, vaguely reminiscent of Feit's, was:

Everything in algebra is either trivial or impossible. Whereas there are many, many beautiful facts in geometry which aren't at all obvious, but can be discovered with some imagination. Geometry depends on mathematical creativity, that apparent oxymoron. If we fail to share these geometrical gems with our children, we are condemning them to the equivalent of twelve years of piano finger exercises, while forbidding them to ever listen to a sonata by Mozart or a prelude by Chopin.

In conclusion, we need to meet the challenge of finding ways to make mathematics exciting for our children, of inspiring them to do the hard work necessary to achieve the “ah ha” moments. The ideal teaching moment would be the act of moving from confusion to clarity in unison with your students. Sometimes this can happen. It argues for being prepared, but not too well prepared, for class. But it is a delicate trick to pull off. And as with the performance of all such tricks, you have to be willing to fall flat on your face. And, even if you do it perfectly, only a few of your students will be in the perfect place to receive the value of it. The rest will just hear it as a muddled lecture.

Returning to the theme of Passover, the Talmudic rabbis tell the story of a man who, when following Moses between the parted waters of the Red Sea, grumbled the whole time because his sandals were getting muddy. Of course, we can use this story to let ourselves off the hook: there are some people whom even God can't impress; how can we hope to impress them? But letting ourselves off the hook in this way would mean giving up the “ah ha” moments and failing to show the next generation of students the complex truth of the maxim that “all of mathematics is easy once you know how to do it.”