SPRINGS OF MINIMUM INERTIA.

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In certain forms of apparatus where springs are employed to indicate the instantaneous value of a varying force it is of the utmost importance that they should be as light as possible in order that the inertia should be reduced to a minimum. In designing such a piece of apparatus the writer found it necessary to investigate mathematically the conditions which must be fulfilled in order that the spring might be of minimum weight. In the course of this investigation certain unexpected and very interesting results were reached which not only are of general application, but also lead to simple formulae of design. Such springs may be grouped into two classes, namely, those whose action depends upon bending and those whose action depends upon torsion. Evidently, in either kind, the maximum allowable fiber stress and the maximum deflection to which the beam will ever be subjected must be reached simultaneously with the maximum load for, were it otherwise, the spring could have some metal removed and still be strong enough to resist the stress. We will take, therefore, as the quantities entering into this investigation the following:

f = the maximum allowable fiber stress.
P = the maximum load.
Δ = the maximum deflection.
Z = the modulus of the section.
I = the moment of the inertia.
E = Young's Modulus.
w = the weight per cubic inch.
W = the total weight.
b† = the breadth.
d = the depth.
L = the length.

†In beams of constant strength, and depth constant, b is breadth of base.

SPRINGS DUE TO BENDING.

All springs of this type take the form of some sort of a beam and we will then have the following formulae from the fundamental principles of strength of materials.

$$(1)\quad f=\frac{PL}{ZN} \qquad\qquad (2)\quad \Delta=\frac{PL^3}{KEI}$$

In the above equations N and K are constants which determine the method of loading and the manner in which the beam is supported.

BEAMS OF RECTANGULAR SECTION.

In this case—

$$Z=\frac{bd^2}{6,} \qquad I=\frac{bd^3}{12,} \qquad \text{and} \qquad W=wbdL$$

These give when substituted in equations (1) and (2),

$$(3)\quad f=\frac{6PL}{Nbd^2} \qquad\qquad (4)\quad \Delta=\frac{12PL^3}{KEbd^3}$$

Square equation (3) and divide the result by equation (4) and we have,

$$(5)\quad \frac{f^2}{\Delta}=\frac{3PKE}{N^2bdL} \qquad \text{whence} \qquad (6)\quad bdL=\frac{3PKE\,\Delta}{N^2f^2}$$

In equation (6) bdL is simply the volume of the spring and hence if we multiply this by its weight per cubic inch, we will have its total weight, whence

$$(7)\quad W=wbdL=3w\left(\frac{K}{N^2}\right)\left(\frac{E}{f^2}\right)P\,\Delta$$

In either of the cases of a beam fixed at one end and loaded at the other, supported or fixed at both ends and loaded in the middle, the quantity $(K/N^2)=3$, whence the weight will be given by

$$(8)\quad W=9\left(\frac{wE}{f^2}\right)P\,\Delta$$

It will be noticed that the dimensions of the beam have entirely disappeared from equation (8) or in other words the weight is entirely independent of the section of the beam. To put this in other words a spring in the form of a ruler laid flat wise would weigh exactly as much as one in the form of a ruler laid edgewise provided only that in both cases the same load produced the same deflection with the same fiber stress. At first sight this seems almost absurd and to contradict our experience, but a moment's consideration shows that the reason of our doubt is that in applying the formulas for strength and flexture, the length of our beam is almost invariably given in advance and we forget that a short flatwise ruler will be as stiff as a long narrow one edgewise.

In an exactly similar manner we may investigate the case of a beam of constant strength and of constant depth, or one of constant strength and constant breadth and we will find that the expression for the weight of the beam is identical in both cases. In the case of the hollow cylinder there are two limiting cases, first, where the walls are very thin, and second, where the cylinder became a solid rod. The following table gives the results.

Beam of constant strength loaded at one end fixed at the other	$W = 3\left(\frac{wE}{f^2}\right) P\Delta$
Hollow cylinder walls very thin	$W = 6\left(\frac{wE}{f^2}\right) P\Delta$
Rectangular Beam	$W = 9\left(\frac{wE}{f^2}\right) P\Delta$
Round rod	$W = 12\left(\frac{wE}{f^2}\right) P\Delta$

It is to be noted that in the above expressions for the weight of the several types of beams that if we take the weight of the beam of constant strength equal to one the weight of the others will be 2, 3, and 4. Attention should be called to the great superiority of the beam of constant strength when the depth is made constant, the plan, in that case, being a triangle. In this case not only is the total weight the least of the cases considered, but the weight is nearly all concentrated at the fixed end, while the free end which moves is the vertex of the triangle and consequently of almost no weight.

We will now give the formulas for design for a few cases which will sufficiently illustrate the method so that the designer can readily apply the same process to any case that he might have in hand. Introducing the values of K, N, I, and Z in equations 1 and 2 we find after reduction the following:

BEAM OF CONSTANT STRENGTH, DEPTH CONSTANT, FIXED AT ONE END.

(a) $bdL = 6(E/f^2)P\Delta$
(b) $bd^2 = 6PL/f$
(c) $\Delta = 6PL^3/(bd^3E)$ Check

RECTANGULAR BEAMS FIXED AT ONE END.

(a′) $bdL = 9(E/f^2)P\Delta$
(b′) $bd^2 = 6PL/(f)$
(c′) $\Delta = 4PL^3/(bd^3E)$ Check

SOLID ROUND ROD, FIXED AT ONE END.

(a″) $\pi r^2 L = 12(E/f^2)P\Delta$
(b″) $r^3 = 4PL/(\pi f)$
(c″) $\Delta = 4PL^3/(3\pi E r^4)$ Check

At first sight it might seem that the three equations in each of the above cases were independent of one another, but that is not the case as the expression for the volume was obtained from a combination of the other two. The equation for the flexture is introduced to serve to check the computation. In the first two cases there are three unknowns, b, d, and L; so that any one of them may be assumed and the other two computed which gives great flexibility in design; in the third case, however, there are but two unknowns and hence but a single solution of the problem. As an illustration, let us assume—

$P = 100$ lbs.
$\Delta = 0.1''$
$f = 40,000$ lbs. per square inch.
$E = 29,000,000$ lbs. per sq. in.

First Case. From (a) $bdL = 1.0872$, from (b) $bd^2 = 0.015L$. Let us assume $d = \frac{1}{4}''$ and we will then have $L = 4.26''$, $b = 1.02''$. Second Case. From (a′) $bdL = 1.631$, from (b′) $bd^2 = 0.015L$ and as above assume $d = \frac{1}{4}''$ and we will have $L = 5.21''$ and $b = 1.25''$. Third Case. From (a″) $r^2L = 0.6922$, from (b″) $r^3 = 0.003181L$, whence $r = 0.2942$ and $L = 8.002$.

SPRINGS DEPENDING UPON TORSION (COILED SPRINGS).

Under this class we will consider those springs which are wound in the form of a helix and in which the action is such as to stretch or compress the spring along the axis of the helix. In his "Mechanics applied to Engineering," Goodman, neglecting the angularity of the helix, gives for the maximum fiber stress and the total deflection

$$(9)\ f=\frac{8PD}{\pi d^3} \qquad (10)\ \Delta=\frac{8nPD^3}{Gd^4}$$

In the above D equals the mean diameter of the coil, d the diameter of the wire, n the number of turns, G the coefficient of rigidity, and the other quantities as before. A moment's consideration shows that we will have for the length of the wire and the total weight

$$L=\pi nD \text{ and } W=(1/4)\pi^2 nwd^2D$$

From Equations 9 and 10 we find

$$\Delta/f^2=\pi^2 nd^2D/(8PG)=W/(2wPG)$$

From this it follows that

$$W=2wPG(\Delta/f^2)$$

From this it follows that the weight of the spring depends only upon the material, the applied load, the coefficient of rigidity, the maximum deflection and the allowable fiber stress in shear; and is entirely independent of the size of wire, diameter of the helix and number of turns. The formulas for design may now be easily derived:

$$(a''')\ nd^2D=8PG\Delta/(\pi^2f^2)$$
$$(b''')\ d^3/D=8P/(\pi f)$$
$$(c''')\ \Delta=8nPD^3/Gd^4)$$

As before, we have two independent equations containing the three unknowns, n, d, and D, so that we may assign to any one of them an arbitrary value and then compute the others and use the third equation as a check.

As an illustration, let us assume $P=100$ lbs., $G=12{,}000{,}000$, $f=50{,}000$ and $\Delta=\frac{1}{2}''$. We will then find that $nd^2D=0.1945$, and $d^3/D=0.00509$. If then we assume that the spring is to have twenty turns of wire, we find from the above that $D=0.513''$, $d=0.1377''$, and if the spring is to be wound close, its length will be $2.8''$. The following formula is useful in computing the length of the helix (L), $L'\doteq n\ (d+s)$ where s is the width of the open space, which is small in the springs considered.

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